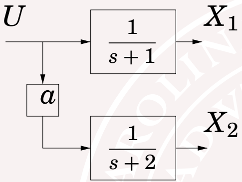
Limitations: Controllability [from lec 6]



System $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ a \end{bmatrix} u$

State x_2 is *uncontrollable* for $a = 0$ and "hard to control" for small values of a.

Controllability matrix

$$W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ a & -2a \end{bmatrix}$$

Controllability gramian S

$$AS + SA^T + BB^T = 0 \Longrightarrow$$

$$S = ... = \begin{bmatrix} \frac{1}{2} & \frac{1}{3}a \\ \frac{1}{3}a & \frac{1}{4}a^2 \end{bmatrix}$$

Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot S^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$ corresponds to what states we can reach by $\int_0^{t_1} |u(t)|^2 dt = 1$.

Lecture 7: Fundamental Limitations

- Limitations from unstable poles and zeros: Intuition
- A back-wheel steered bicyle?
- Limitations from unstable poles and zeros: Hard proofs
- Bode's integral formula and Bode's relations

See lecture notes and [G&L Ch. 7]

Unstable poles — “intuitive reasoning”

An unstable pole p makes the output signal for a bounded input grow exponentially as $\sim e^{pt}$. To stabilize this system, one has to act fast, on a time scale proportional to $\sim 1/p$.

Intuitive conclusion: *Unstable poles give a lower bound on the speed of the closed loop.*

Systems with time-delay

Assume that the plant contains a time-delay T. This means e.g. that a load disturbance is not visible in the output signal until after at least T time units. Of course, this puts a hard constraint on how quickly a feedback controller can reject the disturbance!

Intuitive conclusion: *Time delays give an upper bound on the speed of the closed loop.*

Unstable zeros — "intuitive reasoning"

The step response of a system with a process *zero in the right half plane* (i.e, with positive real part) goes initially in the "wrong direction".

Intuitive conclusion: *Unstable zeros give an upper bound on the speed of the closed loop.*

Why the wrong direction? Let z be a process zero in the right half plane. If we look at the step response, call it $y(t)$, and its Laplace transform we get

$$0 = Y(z) = \int_0^\infty y(t) \underbrace{e^{-zt}}_{>0} dt$$

Hence, $y(t)$ must take both positive and negative values!

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Rehearsal: Step response for non min phase system

How did we do in the basic course (Reglerteknik AK) to show that the step response went in the “wrong direction” for systems with one zero in the RHP?

Use the “Initial value theorem” (see collection of formulae)

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

and apply it to the output derivative $\dot{y}(t)$.

(That is, look at sign of $\dot{y}(0+)$ and compare it to sign of final value $\lim_{t\to\infty} y(t)$)

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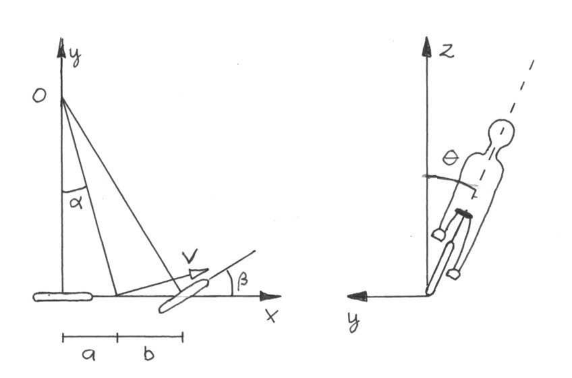
Mini-problems

1. Give examples of systems that initially respond in the "wrong" direction.
2. Which of the intuitive arguments can be applied to
   - an inverted pendulum?
   - a rear wheel steered bicycle?

Bike example

A (linearized) torque balance can be approximated as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Bike example, cont'd

$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

where the physical parameters have typical values as follows:

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed:	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g = 9.81$ ms^{-2}

The transfer function from β to θ is

$$P(s) = \frac{mV_0\ell}{b}\frac{as + V_0}{Js^2 - mg\ell}$$

Bike example, cont'd

The system has an unstable pole p with time-constant

$$p^{-1} = \sqrt{\frac{J}{mg\ell}} \approx 0.4 \text{ s}$$

The closed loop system must be at least as fast as this. Moreover, the transfer function has a zero z with

$$z^{-1} = -\frac{a}{V_0} \approx -\frac{0.3\text{m}}{V_0}$$

For the back-wheel steered bike we have the same poles but different sign of V_0 and the zero will thus the be unstable!

An unstable pole-zero cancellation occurs for $V_0 \approx 0.75$m/s.

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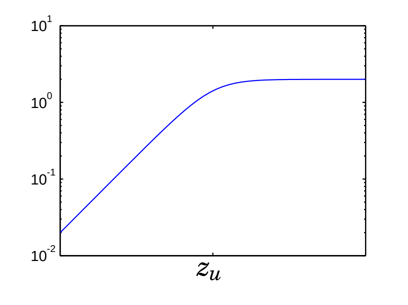
Hard limitations from unstable zeros

If the plant has an unstable zero z_u, then the specification

$$\left|\frac{1}{1+P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{1+z_u^2/\omega^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.



Hard limitations from unstable poles

If the plant has an unstable pole p_u, then the specification

$$\left|\frac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{1+\omega^2/p_u^2}} \qquad \text{for all } \omega$$

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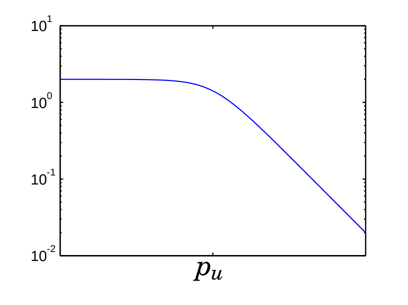

The Maximum Modulus Theorem

The proofs will be based on the following theorem:

Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\max_{\text{Re } s \geq 0} |G(s)| = \max_{\omega \in \mathbf{R}} |G(i\omega)|$$

Sensitivity bounds from unstable zeros

It is easy to see that the sensitivity function must be equal to one at a righ-half-plane zero $s = z_u$ of the transfer function:

$$P(z_u) = 0 \qquad \Rightarrow \qquad S(z_u) := \frac{1}{1 + \underbrace{P(z_u)}_{0} C(z_u)} = 1$$

Notice that the unstable zero in the plant can not be cancelled by an unstable pole in the controller, since this would give an unstable transfer function $C/(1 + PC)$ from measurement noise to control input.

Sensitivity bounds from unstable poles

Similarly, the complimentary sensitivity must be one at an unstable pole p_u:

$$P(p_u) = \infty \qquad \Rightarrow \qquad T(p_u) := \frac{P(p_u)C(p_u)}{1 + P(p_u)C(p_u)} = 1$$

In this case, cancellation by an unstable zero in the controller would give an unstable transfer function $P/(1 + PC)$ from input disturbance to plant output.

Corollary of the Maximum Modulus Theorem

Suppose that the plant $P(s)$ has unstable zeros z_i and unstable poles p_j. Then the specifications

$$\sup_{\omega} |W_a(i\omega)S(i\omega)| < 1 \qquad \sup_{\omega} \left| W^b(i\omega)T(i\omega) \right| < 1$$

are impossible to meet with a stabilizing controller unless $\|W_a(z_i)\| < 1$ for every unstable zero z_i and $\|W^b(p_j)\| < 1$ for every unstable pole p_j.

In particular, if $W_a = (s+a)/(2s)$ and $W^b(s) = (s+b)/(2b)$, it is necessary that $a < \min_i z_i$ and $b < \max_j p_j$. This proves the statements on slide 12 & 13.

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Bode's Integral Formula ("The water bed effect")

For a system with loop gain $L = PC$ which has a relative degree ≥ 2 and unstable poles $p_1, \ldots, p_M$, the following *conservation law* for the sensitivity function $S = \dfrac{1}{1+L}$ holds.

$$\int_0^{+\infty} \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i)$$

See [G&L Theorem 7.3] for details/asumptions.

G. Stein: "Conservation of “dirt!”"

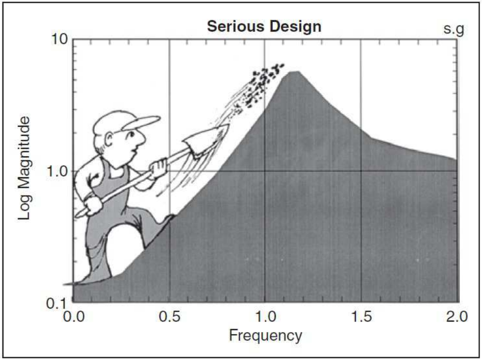
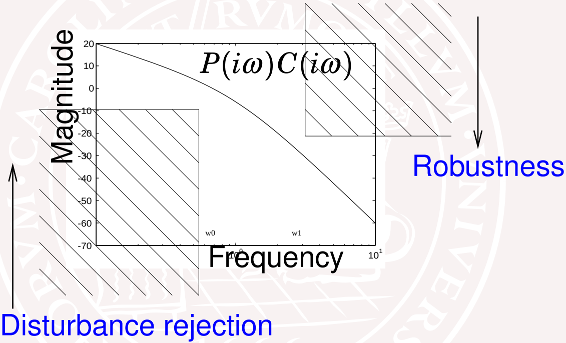


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Picture from Gunter Steins Bode Lecture (1985) “Respect the unstable”. Reprint in [IEEE Control Systems Magazine (Aug 2003)]

Recall that the loop transfer matrix should have small norm $\|P(i\omega)C(i\omega)\|$ at high frequencies, while at low the frequencies instead $\|[P(i\omega)C(i\omega)]^{-1}\|$ should be small.



How quickly can we make the transition from high to low gain?

Bode's Relation — Approximate version

If $G(s)$ is stable with no unstable zeros (*minimum-phase*), then

$$\arg G(i\omega_0) \approx \frac{\pi}{2} \left. \frac{d \log |G(i\omega)|}{d \log \omega} \right|_{\omega=\omega_0}$$

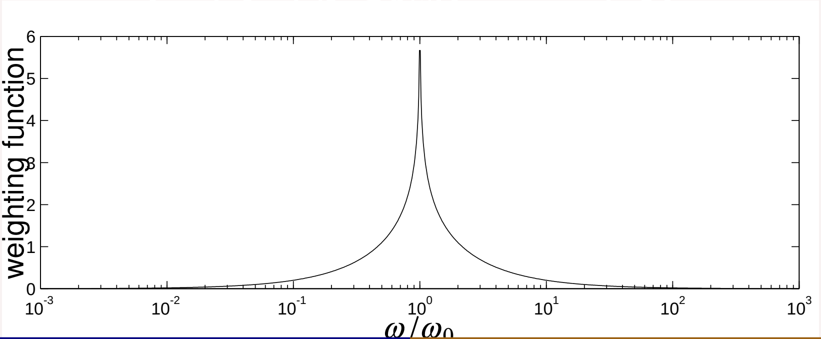
Otherwise the argument is even smaller.

As a consequence, the decay rate of the magnitude curve must be less than 2 at the cross-over frequency.

Bode's Relation — Exact version

If $G(s)$ is stable with no unstable zeros (*minimum-phase*), then

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi}\int_0^\infty \frac{\log|G(i\omega)| - \log|G(i\omega_0)|}{\omega^2 - \omega_0^2}d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \frac{d\log|G(i\omega)|}{d\log\omega}\underbrace{\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|}_{\text{weighting function}}d\log\omega$$



Summary: Fundamental Limitations

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