Summary of Last Lecture

- Look at all transfer functions the closed-loop system! (Gang of Four / Gang of six)
- Stochastic disturbances
- From state realization to output spectrum
- From output spectrum to transfer function

From state realization to output spectrum

Consider the linear system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

The transfer function from v to x is

$$G(s) = (sI - A)^{-1}B$$

and the spectrum for x will be

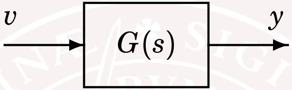
$$\Phi_x(\omega) = (i\omega I - A)^{-1} BR \underbrace{B^*(-i\omega I - A)^{-T}}_{G(i\omega)^*}$$

Covariance matrix for state x:

$$\Pi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)d\omega$$

can be computed by solving $A\Pi_x + \Pi_x A^T + BRB^T = 0$.

From output spectrum to transfer function



Find a filter $G(s)$ such that a process y generated by filtering unit intensity white noise through G will give

$$\phi_y(\omega) = \frac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9},$$

Solution. We have

$$\phi_y(\omega) = \frac{\omega^2 + 4}{(\omega^2 + 1)(\omega^2 + 9)} = \left| \frac{i\omega + 2}{(i\omega + 1)(i\omega + 3)} \right|^2$$

so $G(s) = \frac{s+2}{(s+1)(s+3)}$ works. So does $G(s) = \frac{s-2}{(s+1)(s+3)}$.

Lecture 4: Loop shaping design

Continuing from lecture 3...

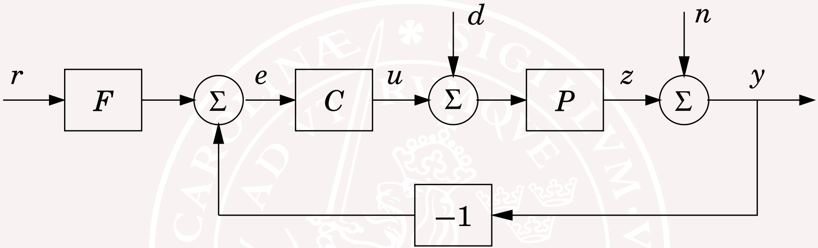
- The closed-loop system
  - Look at all transfer functions in the loop! (Gang of Four / Gang of six)
  - Robustness

New today

- Loop shaping

[Glad & Ljung] Ch. 6.4–6.6, 8.1–8.2 + AK

Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Key Issues

Find a controller that

A: Reduces effects of load disturbances

B: Does not inject to much measurement noise into the system

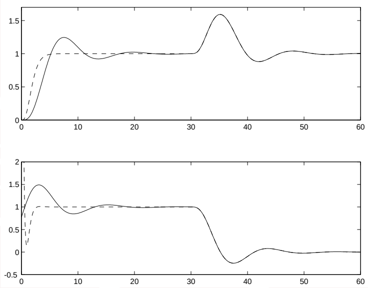
C: Makes the closed loop insensitive to variations in the process

D: Makes output follow command signals

Convenient to use a controller with two degrees of freedom, i.e. separate signal transmission from y to u and from r to u. This gives a complete separation of the problem: Use feedback to deal with A, B, and C. Use feedforward to deal with D!

Time domain specifications

- Step response (w.r.t reference and/or load disturbance)
  - rise-time T_r
  - overshoot
  - settling time T_s
  - static error e_0
- ...



step responce load disturbance

Frequency domain specifications

Closed loops specs.

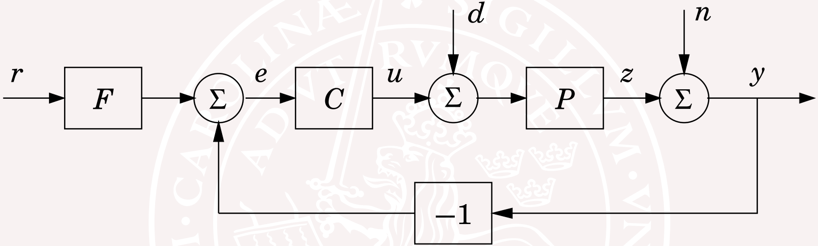
- resonace peak M_p
- bandwidth ω_B

Open-loop measures

- M_S and M_T-circles
- Amplitude margin A_m, phase margin ϕ_m
- cross-over frequency ω_c
- ...

Note: Often the design is made in Bode/Nyquist/Nichols diagrams for loop-gain $L = PC$ (open loop system)

Mini-problem

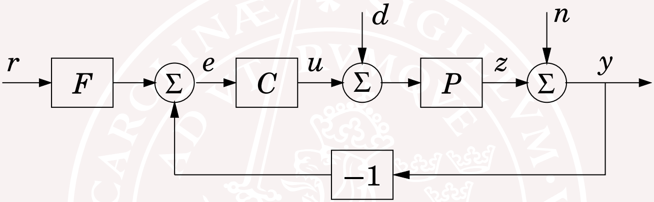


Give an expression for

$$\mathbf{E}\left(y^2 + u^2\right)$$

when $r = n = 0$ and d is white noise.

Specifications on closed loop system



Would like:

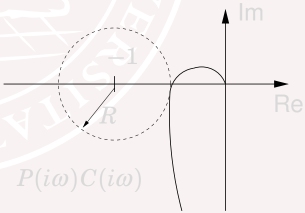
- Small influence of low-frequency disturbance d on z
- Limited amplification of high-frequency noise n in control u
- Robust stability despite high-frequency uncertainty

[Lecture 2]:

Different interpretations of the *Sensitivity function* $S = \dfrac{1}{1+PC}$

1. $S = G_{n\to y}(s) = G_{r\to e}(s)$ [See previous slide]
   - Note: $S = G_{r\to e}(s)$; Want low gain for low fq's...
2. $S = \dfrac{d(\log T)}{d(\log P)} = \dfrac{dT/T}{dP/P}$
   - ("How sensitive is the closed loop T wrt process variations")
3. S measures the distance from the Nyquist plot to $(-1+0i)$.

$$R^{-1} = \sup_{\omega} \left| \frac{1}{1+P(i\omega)C(i\omega)} \right|$$



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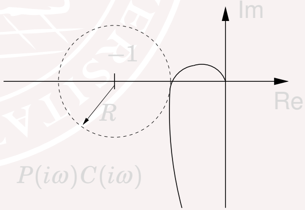

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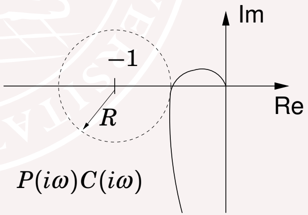

Frequency domain specs.

Closed-loop:
Find specifications W_T and W_S for closed-loops transfer functions s.t

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$
$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$

(Magnitude transfers to singular values for MIMO-systems)

Examples:

- $|S(i\omega)| < 1.5$ for $\omega < 5$ Hz
- $|S| < |W_S^{-1}|$ = s/(s+10)
- $|T| < |W_T^{-1}|$ = 10/(s+10)
- "The closed loop system should have a bandwidth of at least ... rad/s"
-

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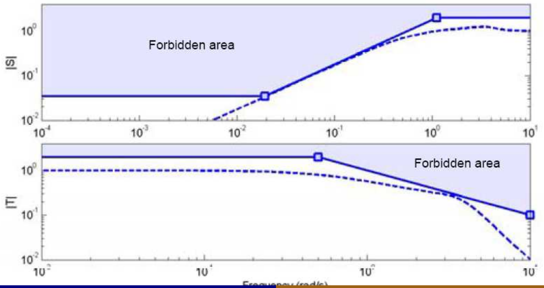


These specifications can not be chosen independently of each other.

$$S + T = 1$$

Limiting factors:

- Fundamental limitations [Lecture 7/Ch 7]:
  - RHP zero at $z \Longrightarrow \omega_{BS} \leq z/2$
  - Time delay $T \Longrightarrow \omega_{BS} \leq 1/T$
  - RHP pole at $p \Longrightarrow \omega_{OT} \geq 2p$
- Bode's integral theorem
  - The "waterbed effect"
- Bode's relation
  - good phase margin requires certain distance between ω_{BS} and ω_{0T}
- Model uncertainty:
  - Robust stability gives new "forbidden area"
  - Robust performance somewhat more complicated

Design: Consider open loop system

Try to look at ***loop-gain*** $L = PC$ for design and to translate specifications of S & T into specs of L

$$S = \frac{1}{1+L} \approx 1/L \qquad \text{if } L \text{ is Large}$$

$$T = \frac{L}{1+L} \approx L \qquad \text{if } L \text{ is small}$$

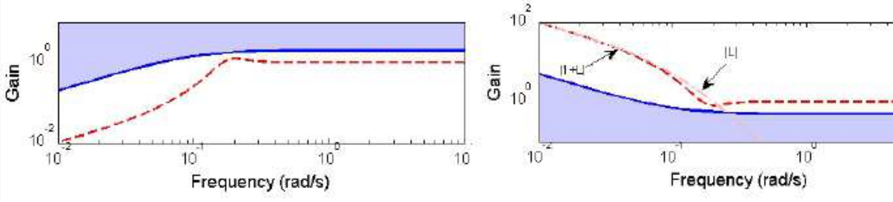
Classical loop shaping:

- design C so that $L = PC$ satisfies constraints on S and T
- how are the specifications related?
- what to do with the regions around cross-over frequency ω_c (where $|L| = 1$)?

Sensitivity vs Loop Gain

$$S = \frac{1}{1+L}$$

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| \iff |1 + L(i\omega)| > |W_S(i\omega)|$$



small frequencies, W_S large $\Longrightarrow$ $1 + L$ large, and $|L| \approx |1 + L|$.

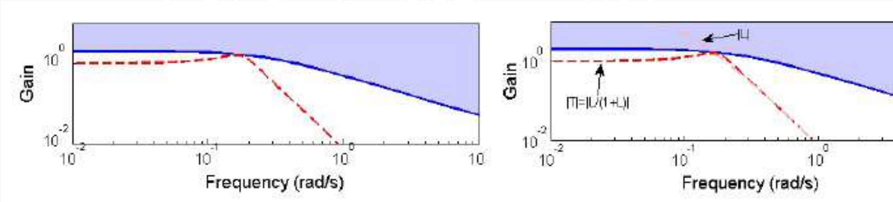
$$|L(i\omega)| \geq |W_S(i\omega)| \quad (approx.)$$

(typically valid for $\omega < \omega_{BS}$)

Complementary Sensitivity vs Loop Gain

$$T = \frac{L}{1+L}$$

$$|T(i\omega)| \leq |W_T^{-1}(i\omega)| \iff \frac{|L(i\omega)|}{|1+L(i\omega)|} \leq |W_T^{-1}(i\omega)|$$

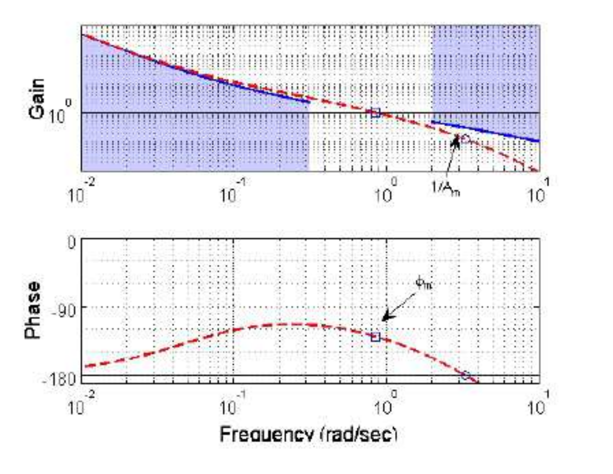


large frequencies, W_T^{-1} small $\Longrightarrow |T| \approx |L|$

$$|L(i\omega)| \leq |W_T^{-1}(i\omega)| \quad (approx.)$$

(typically valid for $\omega > \omega_{OT}$)

Resulting constraints on loop-gain L:

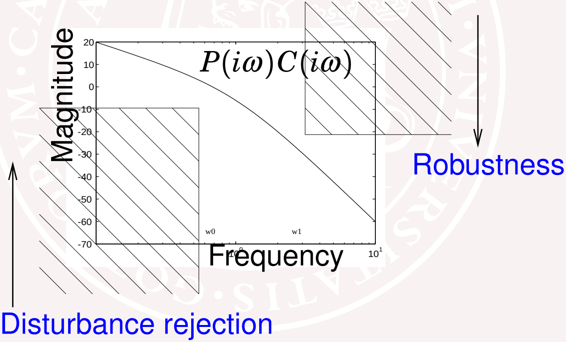


Remark: approximations inexact around cross-over frequency ω_c. In this region, focus is on stability margins A_m, ϕ_m.

These requirements is to say that the *loop transfer matrix*

$$L = P(i\omega)C(i\omega)$$

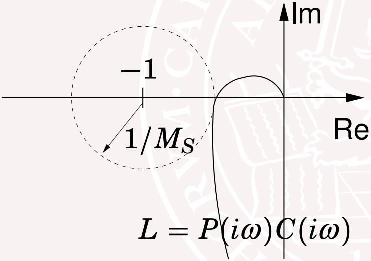
should have small norm $\|P(i\omega)C(i\omega)\|$ at high frequencies, while at low the frequencies instead $\|[P(i\omega)C(i\omega)]^{-1}\|$ should be small.

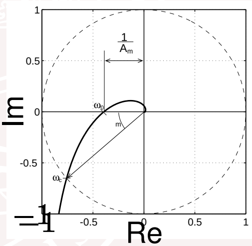


M_S and M_T and stability margins

Specifying $|T(i\omega)| \leq M_T$ and $|S(i\omega)| \leq M_S$ gives bounds for the amplitude and phase margins (but not the other way round!)

$$|S(i\omega)| \leq M_S \quad \Longrightarrow \quad A_m > \frac{M_S}{M_S - 1}, \quad \phi_m > 2 \arcsin \frac{1}{M_S}$$





Q: Why does not A_m and ϕ_m give bounds on M_T and M_S?

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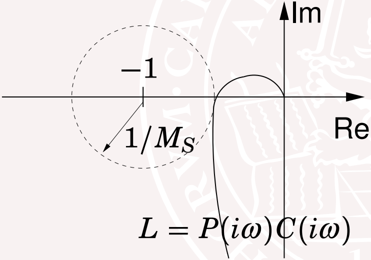

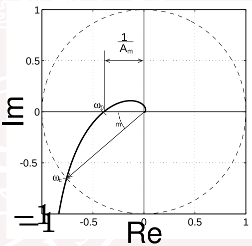

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Classical loop shaping

Map specifications on requirements on loop gain L.

- Low-frequency specifications from W_S
- High-frequency specifications from W_T^{-1}
- Around cross-over frequency, mapping is crude
  - Position cross-over frequency (constrained by W_S, W_T)
  - Adjust phase margin (e.g. from M_S, M_T specifications)

Lead-lag compensation

Shape loop gain $L = PC$ using a compensator C composed of

- Lag (phase retarding) elements

$$C_{lag} = \frac{s + a}{s + a/M}, \quad M > 1$$

- Lead (phase advancing) elements

$$C_{lead} = N \frac{s + b}{s + bN}, \quad N > 1$$

- Gain

$$K$$

Typically

$$C = K \frac{s + a}{s + a/M} \cdot N \frac{s + b}{s + bN}$$

Properties of leads-lag elements

- Lag (phase retarding) elements
  - Reduces static error
  - Reduces stability margin
- Lead (phase advancing) elements
  - Increased speed by increased ω_c
  - Increased phase
    $\Longrightarrow$ May improve stability
- Gain
  - Translates magnitude curve
  - Does not change phase curve

See "Collection of Formulae" for lead-lag link diagrams

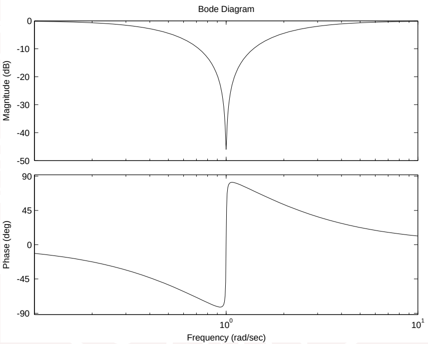
Iterative lead-lag design

- Step 1: Lag (phase retarding) element
  - Add phase retarding element to get low-frequency asymptote right
- Step 2: Phase advancing element
  - Use phase advancing element to obtain correct phase margin
- Step 3: Adjust gain
  - Usually need to "lift up" or "push down" amplitude curve to obtain the desired cross-over frequency.

Adjusting the gain in Step 3 leaves the phase unaffected, but may ruin low-frequency asymptote (need to revise lag element) $\Longrightarrow$ An iterative method!

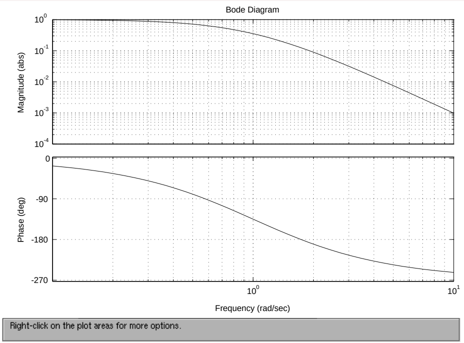
Example of other compensation-link:

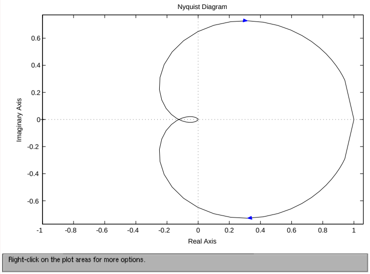
Notch-filter $\dfrac{s^2 + 0.01s + 1}{s^2 + 2s + 1}$

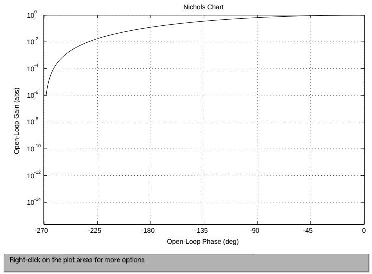


Bode, Nyquist and Nichols diagrams

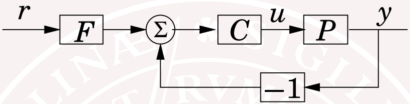
$$\log|PC| = \log|P| + \log|C|$$
$$\arg\{PC\} = \arg\{P\} + \arg\{C\}$$

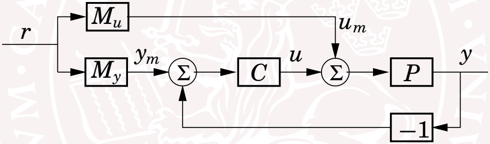






Feedforward design





The reference signal r specifies the desired value of y.

Ideally

$$\frac{P(s)C(s)}{1+P(s)C(s)}F(s) \approx 1$$

Equivalently

$$F(s) \approx \frac{1 + P(s)C(s)}{P(s)C(s)}$$

Exact equality is generally impossible because of pole excess in P.

The simplest and most common approximation is to use a constant gain

$$F = \frac{1 + P(0)C(0)}{P(0)C(0)}$$

A more advanced option is

$$F(s) = \frac{1 + P(s)C(s)}{P(s)C(s)(sT+1)^d}$$

for some suitable time constant T and d large enough to make F proper and implementable.

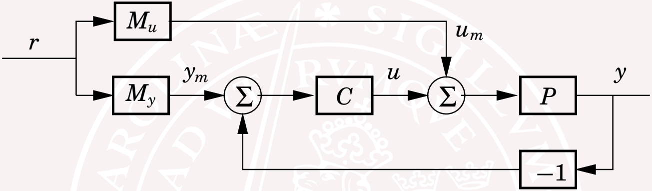
Example

$$P(s) = \frac{1}{(s+1)^4} \qquad F(s) = \frac{1 + P(s)C(s)}{P(s)C(s)(sT+1)^d}$$

The closed loop transfer function from r to u then becomes

$$\frac{C(s)}{1 + P(s)C(s)} F(s) = \frac{(s+1)^4}{(sT+1)^4}$$

which has low-fq gain 1, but gain $1/T^4$ for $\omega \longrightarrow \infty$.



Notice that M_u and M_y can be viewed as generators of the desired output y_m and the inputs u_m which corresponds to y_m.

Design of Feedforward revisited

The transfer function from r to $e = y_m - y$ is $(M_y - PM_u)S$

Ideally, M_u should satisfy $M_u = M_y/P$. This condition does not depend on C!

Since $M_u = M_y/P$ should be stable, causal and not include derivatives we find that

- Unstable process zeros must be zeros of M_y
- Time delays of the process must be time delays of M_y
- The pole excess of M_y must be greater than the pole excess of P

Take process limitations into account!

Example of Feedforward Design revisited

If

$$P(s) = \frac{1}{(s+1)^4} \qquad M_y(s) = \frac{1}{(sT+1)^4}$$

then

$$M_u(s) = \frac{M_y(s)}{P(s)} = \frac{(s+1)^4}{(sT+1)^4} \qquad \frac{M_u(\infty)}{M_u(0)} = \frac{1}{T^4}$$

Fast response (T small) requires high gain of M_u.

Bounds on the control signal limit how fast response we can obtain.

Summary

Frequency design;

- Good mapping between S,T and $L = PC$ at low and high frequencies (mapping around cross-over frequency less clear)
- Simple relation between C and $L \Longrightarrow$ easy to shape L!
- Lead-lag control: iterative adjustment procedure
- What if closed-loop specifications are not satisfied?
  - we made a poor design (did not iterate enough), or
  - the specifications are not feasible (fundamental limitations in Lecture 7)
- Later in the course::
  - Use optimization to find stabilizing controller that satisfies constraints, if such a controller exists

Feedforward design

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

1. Introduction and system representations
2. Stability and robustness
3. Specifications and disturbance models
4. Control synthesis in frequency domain
5. Case study

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Next lecture

Case study DVD-player

- Use **loop-shaping techniques from this lecture** for focus control design in DVD-player
- track following (modelling of disturbances, control)