

# Solutions to Exercise 1. Control in Matlab

```
1.1 >> A = [0 1; 1 0];
>> B = [1 0]';
>> C = [0 1];
>> D = 0;
>> eig(A)
```

```
ans =

    -1
     1
```

```
1.2 >> sys = ss(A,B,C,D);
>> tf(sys)
```

```
Transfer function:
      1
-----
s^2 - 1
```

```
1.3 >> zero(sys)
```

```
ans =

Empty matrix: 0-by-1
```

```
>> pole(sys)
```

```
ans =

    -1
     1
```

```
>> dcgain(sys)
```

```
ans =

    -1
```

```
1.4 >> s = tf('s');
>> P = 1/(s^2+0.6*s+1)
```

```
Transfer function:
      1
-----
s^2 + 0.6 s + 1
```

```
>> P.InputDelay = 1.5
```

```
Transfer function:
              1
exp(-1.5*s) * -----
              s^2 + 0.6 s + 1
```

```
>> bode(P)
```

## Solutions to Exercise 1

```
>> grid
>> nyquist(P)
>> pzmap(P)
>> step(P)
```

As seen in the pole-zero map, the *open-loop* system is stable, as also indicated by the step response. The Bode and Nyquist plots show that the *closed-loop* system will be unstable.

**1.5** >> Wc = ctrb(A,B);  
>> rank(Wc)

```
ans =
```

```
2
```

Since the controllability matrix has full rank, the system is controllable.

```
>> p=[1 1.4 1];
>> L=place(A,B,roots(p))
```

```
L =
```

```
1.4000 2.0000
```

**1.6** P = 1/(s<sup>2</sup>+0.6\*s+1);  
>> C = 0.5\*(1+4\*s);  
>> margin(C\*P)

The amplitude margin is infinite, whereas the phase margin is 101°.

**1.7** >> CLSYS = feedback(C\*P,1)

```
Transfer function:
```

```
2 s + 0.5
```

```
-----
s2 + 2.6 s + 1.5
```

```
>> CLSYS = minreal(C*P/(1+C*P))
```

```
Transfer function:
```

```
2 s + 0.5
```

```
-----
s2 + 2.6 s + 1.5
```

**1.8** >> step(CLSYS)  
>> dcgain(CLSYS)

```
ans =
```

```
0.3333
```

**1.9** >> A=[-1 1 0 -1/2 0; 4 -1 0 -25 8; 0 1 0 0 0; 0 0 0 -20 0; 0 0 0 0 -20];  
>> B=[0 0; 3/2 1/2; 0 0; 20 0; 0 20];  
>> C=[0 1 0 0 0; 0 0 1 0 0];  
>> jas=ss(A,B,C,[0 0; 0 0]);  
>> pole(jas)  
ans =

```
0
```

```

    1.0000
   -3.0000
  -20.0000
  -20.0000
>> rank(ctrb(jas))
ans =
     5
>> rank(observ(jas))
ans =
     4

```

We see that the system is unstable. This means that without some type of control, the plane will crash. Fortunately, the system is controllable, which means that it is possible to stabilise the aircraft with the given actuators. However, since we do not have observability, we need to have some other combination of sensors if we to use feedback from observed states.

To get the transfer function, we use

```

>> G=tf(jas)

Transfer function from input 1 to output...
    1.5 s^2 - 468.5 s - 510
#1:  -----
    s^3 + 22 s^2 + 37 s - 60

    1.5 s^2 - 468.5 s - 510
#2:  -----
    s^4 + 22 s^3 + 37 s^2 - 60 s

Transfer function from input 2 to output...
    0.5 s^2 + 170.5 s + 170
#1:  -----
    s^3 + 22 s^2 + 37 s - 60

    0.5 s^2 + 170.5 s + 170
#2:  -----
    s^4 + 22 s^3 + 37 s^2 - 60 s

>> G(1,2) % To output 1 from input 2 (note the order of indexing)

Transfer function:
    0.5 s^2 + 170.5 s + 170
-----
    s^3 + 22 s^2 + 37 s - 60

```

**1.10** >> G1 = 1/(s+1)^3

```

Transfer function:
     1
-----
    s^3 + 3 s^2 + 3 s + 1

>> pole(G1)

ans =

```

## Solutions to Exercise 1

```
-1.0000
-1.0000 + 0.0000i
-1.0000 - 0.0000i

>> G2 = zpk(1/(s+1)^3)

Zero/pole/gain:
  1
  -----
 (s+1)^3

>> pole(G2)

ans =

-1.0000
-1.0000 + 0.0000i
-1.0000 - 0.0000i

>> G3 = 1/(s^3+2.99*s^2+3*s+1);
>> pole(G3)

ans =

-1.0888 + 0.2131i
-1.0888 - 0.2131i
-0.8124

>> G4 = 1/(s+0.99)^3;
>> pole(G4)

ans =

-0.9900 + 0.0000i
-0.9900 - 0.0000i
-0.9900
```

We see that the same small modification in a parameter, causes larger changes in the dynamics when the system is represented as  $G_3$ . The transfer function format of  $G_4$  (three poles in the same spot as for  $G_2$ ), which can be kept with the `zpk` command, is in general better numerically compared to the format in which  $G_3$  is represented (the same form as the command `tf` gives).

**1.11** >> `Wo = obsv(A,C)`

```
ans =

  3   4
 -3  -4

>> rank(Wo)

ans =
```

```

1
>> rank(ctrb(A,B))

ans =

1

```

Since neither the observability matrix nor the controllability matrix has full rank, the system is neither observable nor controllable. It can be seen directly from the state equations, where we have two states that are completely decoupled from each other and have the same eigenvalue. This means that evolution of the states will look exactly the same for any control signal  $u(t)$  (assuming that the initial state is at the origin). Therefore we will never be able to control these states arbitrarily. We will only be able to control them along some controllable subspace. The same goes for the observability.

**1.12** The transfer function for the mass-spring system will be

```

>> zpk(ss(A,B,C,D))

Zero/pole/gain:
      2
-----
(s^2 + 20)

```

The transfer function of a PID controller is

$$R = \frac{K(sT_i + 1 + s^2T_dT_i)}{sT_i}$$

and the closed loop transfer function is

$$G_{cl}(s) = \frac{R(s)P(s)}{1 + R(s)P(s)} = \frac{2K(s^2T_d + s + 1/T_i)}{s^3 + s^22KT_d + s(20 + 2K) + 2K/T_i}$$

The closed loop characteristic equation is then

$$s^3 + s^2(2KT_d) + s(20 + 2K) + 2K/T_i = 0$$

Identify the coefficients and solve for  $K$ ,  $T_i$  and  $T_d$  as functions of  $\omega$  and  $\zeta$ :

$$K = 0.5(\omega^2 + 2\zeta\omega^2 - 20)$$

$$T_i = \frac{2K}{\omega^3}$$

$$T_d = \frac{2\zeta\omega + \omega}{2K}$$

The closed loop system is then

```

>> G_cl = feedback(R*P,1);
>> step(G_cl)

```

## Solutions to Exercise 1

The specification is met for many different choices of  $\omega$  and  $\zeta$ . One choice can be  $\omega = 6$  and  $\zeta = 0.7$ .

```
1.13 >> s = tf('s');
>> P = (3-s)/((s+1)*(s+2));
>> [A,B,C,D] = ssdata(P);
>> rank(ctrb(A,B))
```

```
ans =
```

```
2
```

```
>> p = [1 5.6 16];
>> L = place(A,B,roots(p));
```

The system is controllable, since the controllability matrix has full rank. With the control law  $u(t) = -Lx+r$ , the closed loop system get the following appearance

```
>> A_cl = A-B*L;
>> B_cl = B;
>> C_cl = C;
>> D_cl = 0;
>> G_cl = ss(A_cl,B_cl,C_cl,D_cl);
>> step(G_cl)
>> dcgain(G_cl)
```

```
ans =
```

```
0.1875
```

The system is non-minimum phase, which we can see directly since the process has a zero in the right half plane.

## Solutions to Exercise 2. System Representations and Stability

**2.1** A state-space representation of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

**2.2** Laplace transformation of the differential equation gives

$$Y(s) = \frac{(b_{11}s + b_{12})}{(s^2 + a_1s + a_2)} U_1(s) + \frac{(b_{21}s + b_{22})}{(s^2 + a_1s + a_2)} U_2(s)$$

The transfer matrix becomes

$$\left( \begin{array}{cc} \frac{b_{11}s+b_{12}}{s^2+a_1s+a_2} & \frac{b_{21}s+b_{22}}{s^2+a_1s+a_2} \end{array} \right)$$

**2.3**

**a.** The equation can be written as

$$y = g * u \tag{2.1}$$

where  $g(t) = te^{-2t}$ ,  $t \geq 0$ . Taking the Laplace transform of (2.1) gives with  $u = r - y$

$$Y(s) = \frac{1}{(s+2)^2} (R(s) - Y(s))$$

$$Y(s) = \frac{1}{s^2 + 4s + 5} R(s)$$

**b.** The transfer function has poles in

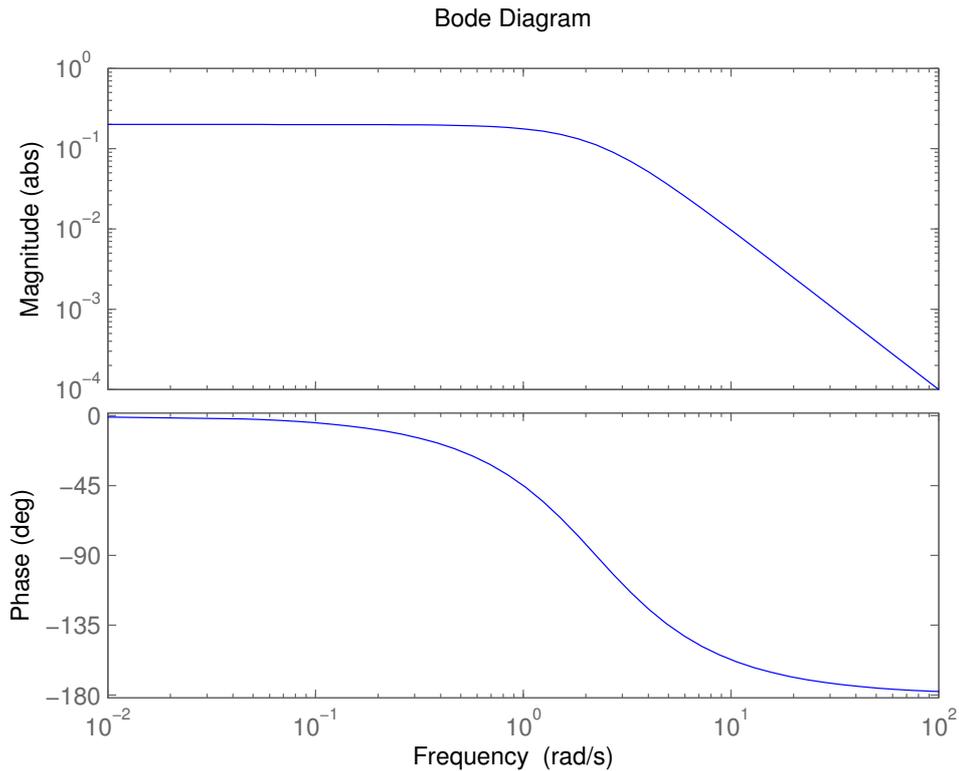
$$s_1 = -2 + i$$

$$s_2 = -2 - i$$

Since all poles have negative part the system is input-output stable.

Another way of checking stability of a second order system with characteristic equation  $s^2 + a_1s + a_2$  is that  $a_1, a_2 > 0$ .

**c.** Since the system is stable, the  $L_2$ -gain is given by the supremum of the transfer function gain, so we want to find the peak of the Bode amplitude plot.



**Figure 2.1** Bode diagram for Problem 2.3(c).

```
>> s = tf('s');
>> G = 1 / (s^2 + 4*s + 5);
>> bode(G)
```

Alternatively, one can find the frequency that maximizes the gain by the following reasoning: Since it is a second order system, it can be written as

$$G_c(s) = \frac{K}{s^2 + 2\zeta\omega s + \omega^2}$$

In our case  $\zeta = 2/\sqrt{5} \approx 0.9$ . This means that the system is well damped and that it does not have a resonance peak in the gain curve. Since the gain is decreasing with frequency, the maximum gain can thus be found at  $\omega = 0$ .

$$|G_c(i \cdot 0)| = \frac{1}{5}$$

## 2.4

a.

$$\begin{aligned} S(s) &= \frac{1}{1 + CP} = \frac{s^3 + 2s^2 + s}{s^3 + 2s^2 + 2.4s + 1.4} = \\ &= \frac{(s+1)(s^2+s)}{(s+1)(s^2+s+1.4)} = \frac{s^2+s}{s^2+s+1.4} \end{aligned}$$

Remark: Notice that we have a 3rd order system (with a 1st order controller and 2nd order plant), but the transfer functions  $S(s)$  is only of 2nd order! Looking at the block-diagram of the system one can clearly see the pole-zero cancellation of the term  $(s+1)$  for  $P \cdot C$ . These kind of pole-zero cancellations imply loss of either observability or loss of controllability, which will be studied later in the course.

$$T(s) = \frac{CP}{1 + CP} = \frac{1.4(s+1)}{s^3 + 2s^2 + 2.4s + 1.4} = \frac{1.4}{s^2 + s + 1.4}$$

Remark: Also for  $T$  there has been a pole-zero cancellation of  $(s+1)$ , but a corresponding cancellation **does not** appear in for instance  $G_{d \rightarrow z} = \frac{P}{1+PC}$ . The transfer function from  $n$  to  $z$  is  $T(s)$  (the minus sign can be ignored since we could just as well say that the unknown noise is given by  $-n$ ). This means that the reference and the measurement noise have the same effect on the output.

- b. We know that  $S(s)$  is the transfer function from load disturbance to output. Since the control system should remove the effects of load disturbances, which often are of low frequency character, it would seem reasonable if the curve representing  $S(s)$  decreases as we move to the left. This corresponds to the upper curve.

We could also look at the function  $S(s)$  that we just determined. We see that

$$\lim_{s \rightarrow 0} S(s) = 0$$

Comparing with the upper curve, which has a gain that goes to zero for low frequencies, we conclude that this represents the sensitivity function.

- c. In order to have good tracking of the reference value, we want the gain from reference to output to be close to one. Looking at the gain curve of the *complimentary transfer function*  $T$  we see that for  $\omega < 1$ , we have  $T \approx 1$ , resulting in good tracking of the reference value.

**Additional comments:** At the same time, we want to be insensitive to process noise and measurement noise, i.e. we want the gain to be as small as possible for these two signals.

The transfer function from process noise to output is  $S$ , while  $T$  is the transfer function of both reference values and measurement noise to the output.  $S$  and  $T$  can not be small at the same frequencies, due to the fact that

$$S(s) + T(s) = \frac{1}{1 + C(s)P(s)} + \frac{C(s)P(s)}{1 + C(s)P(s)} = 1$$

Thus, we need to think about the frequency character of these signals, and compare with the shapes of the transfer functions: Process noise and reference signals are often of low frequency, so we want to have  $S \approx 0$  and  $T \approx 1$  at low frequencies. Measurement noise is most often of high frequency, so we want to have  $T \approx 0$  at high frequencies.

- d. At  $\omega > 1$   $T$  is small, resulting in good attenuation of measurement noise. (Do you see how the “speed” of control relates to the impact of measurement noise?)

## 2.5

- a. The sensitivity function is given by  $S = \frac{1}{1+PC}$ , so  $S$  is small at frequencies where  $PC$  is large. The stationary gain of  $P$  is finite.  $C_2$  and  $C_3$  both have integral action and infinite stationary gain. Thus, for these controllers,  $S$  will go to zero as  $\omega \rightarrow 0$ .  $C_1$ , being a pure P-controller, has a finite stationary gain.  $S$  will then also have a finite stationary gain.

$C_2$  and  $C_3$  are PI-controllers, but  $C_3$  has a delay which will introduce extra phase loss. This decreases the phase margin and therefore introduces a higher sensitivity peak. Thus, we have:  $C_1 \rightarrow A$ ,  $C_2 \rightarrow C$ , and  $C_3 \rightarrow B$ .

- b. Since controller  $C_1$  does not have integral action, we will get a stationary error in the response to a constant load disturbance,  $d$ . The response using the delayed controller  $C_3$  will be less damped than the response using the PI-controller because of the smaller phase margin,  $C_2$ . This gives:  $C_1 \rightarrow II$ ,  $C_2 \rightarrow I$ , and  $C_3 \rightarrow III$ .

## 2.6

- a.

$$y = \alpha h_2, \quad f = \beta(h_1 - h_2)$$

$$\dot{h}_1 = \frac{1}{A_1}(u_1 - f), \quad \dot{h}_2 = \frac{1}{A_2}(u_2 + f - y)$$

$$\dot{h} = \begin{pmatrix} -\frac{1}{A_1}\beta & \frac{1}{A_1}\beta \\ \frac{1}{A_2}\beta & -\frac{1}{A_2}(\beta + \alpha) \end{pmatrix} h + \begin{pmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{pmatrix} u$$

$$y = (0 \quad \alpha) h$$

- b.

$$G(s) = \frac{1}{s^2 + (2\beta + \alpha)s + \alpha\beta} (\alpha\beta \quad \alpha(s + \beta))$$

- c. Since the system is stable, the  $L_2$ -gain can be computed in Matlab as:

```
>> s = tf('s');
>> G = 1/(s^2+3*s+1)*[1 s+1];
>> P = norm(G, inf)
```

The  $L_2$ -gain is  $\sqrt{2}$ .

- d.** *Short explanation:* The gain definition uses 2-norm to measure signals. The error of thinking is a result of forgetting this and instead assuming that the size of the input is equal to the sum of the inflows.

*Longer explanation:* The  $L_2$ -gain is the largest output signal that can be achieved (in 2-norm) by an input signal which is not larger than one (in 2-norm). More formally, if we let the vector  $u = (u_1 \ u_2)^T$ , the gain is given by

$$\|G\|_\infty = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\|u\|_2 \leq 1} \|Gu\|_2 = \sqrt{2}.$$

Obviously, the amount of water going into the tank is given by  $u_1 + u_2$ . But the gain definition does not mention the sum of the elements of  $u$ . It does however say that  $1 \geq |u| = \sqrt{u_1^2 + u_2^2}$ . If, e.g., we let  $u_1 = u_2 = \sqrt{2}/2$ , then the maximum output is achieved, while  $|u| = 1$ . But still, the output is equal to the sum of the inputs.

## Solutions to Exercise 3. Disturbance Models and Robustness

3.1 Let

$$G(s) = \frac{1}{s+1}.$$

- a. The closed loop is guaranteed to be stable according to the small gain theorem if  $\|G\| \cdot \|\Delta\| < 1$ . We see that  $\|G\| = 1$  (use  $\text{norm}(G, \text{inf})$  or look at the Bode Plot), so the system is guaranteed to be stable if  $\|\Delta\| < 1$ .
- b. The loop gain is given by

$$L(s) = G(s)K = \frac{K}{s+1},$$

and the sensitivity function by

$$S = \frac{1}{1+L} = \frac{1}{1+\frac{K}{s+1}} = \frac{s+1}{s+1+K}.$$

We see that there is one closed loop pole in  $s = -(1+K)$ , so the system is stable exactly when  $K > -1$ . We can compare this to the result in **a**, which guarantees that the system is stable when  $|K| < 1$ .

The different results arise from the fact that the small gain theorem is conservative in nature, i.e. it gives a *sufficient* condition on stability, but that condition may not be *necessary*. The main reason of such a conservatism, is that there is no *a priori* assumptions on  $\Delta$ .  $\Delta$  in **a** can be a transfer function of an arbitrary order, not just an unknown scalar as in **b**. Looking at the closed loop poles, on the other hand, shows exactly when the system is stable.

3.2 a. Block diagrams of the original and the rewritten closed-loop system are shown in Figure 3.1. We have

$$C(s) = \frac{2s+2}{s} \quad P(s) = \frac{1}{(s+1)^2} \quad W(s) = \frac{s}{s+2}$$

$$G_{vn}(s) = -\frac{C(s)W(s)}{1+P(s)C(s)} = -\frac{2s^4 + 6s^3 + 6s^2 + 2s}{s^4 + 4s^3 + 7s^2 + 8s + 4} = -\frac{2s^3 + 4s^2 + 2s}{s^3 + 3s^2 + 4s + 4}$$

Matlab commands:

```
>> s = tf('s');
>> C = 2*(s+1)/s
>> P = 1/(s+1)^2
>> W = s/(s+2);
>> Gvn = -(C(s)W(s))/(1+P(s)C(s));
```

- b. The  $L_2$ -gain of  $G_{vn}$  is equal to 2.63. This corresponds to the peak magnitude in the Bode Diagram of Figure 3.2.

- c. The small gain theorem shows stability for all perturbations,  $\Delta$ , satisfying

$$\|\Delta\| \cdot \|G_{vn}\| < 1$$

The closed loop system is therefore stable for all perturbations  $\Delta$  with

$$\|\Delta\|_{\infty} < 1/\|G_{vn}\|_{\infty} = 0.38$$

Matlab commands:

```
>> norm(Gvn, inf)
>> 1/ans
```

- d. Process models are often better (that is, they match the real process more closely) in the low frequency range. As frequency increases there is usually excitation of higher-order dynamics and non-linearities in the real process, which is not covered by the model.

Since we know that there is more uncertainty for high frequencies, this can be used to get some *structure* on the uncertainty block. This structure is given by the extra factor (such factors are usually called *weighting functions*), which effectively makes the uncertainty smaller for low frequencies (approximately when  $\omega < 2$ ). Without this factor, the analysis would assume equal uncertainty for all frequencies, yielding a lower bound on the  $L_2$ -gain of  $\Delta$ . In other words, the system would appear less robust.

### 3.3 a. Matlab commands:

```
>> s = tf('s');
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s)
```

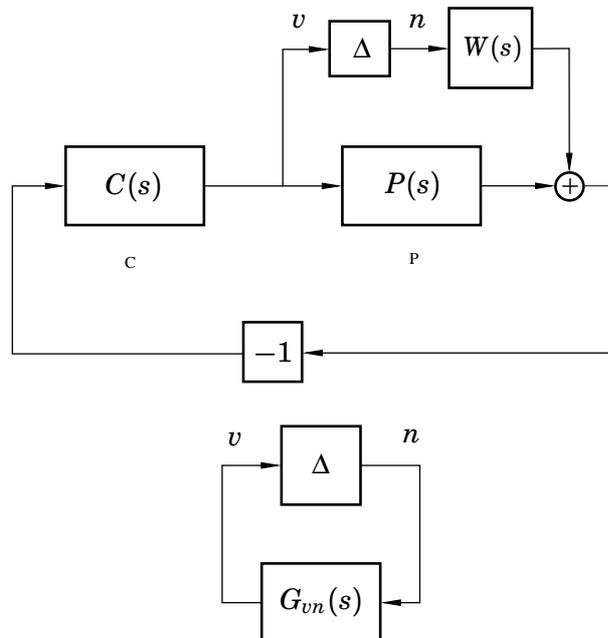


Figure 3.1 Systems for Problem 3.2.

Solutions to Exercise 3

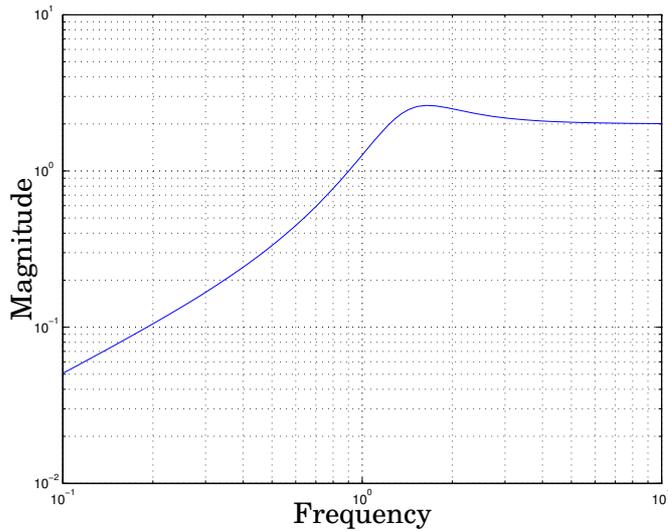


Figure 3.2 Bode magnitude diagram for  $G_{vn}(s)$ .

```
>> G = feedback(C*P,1);
>> pole(G)

ans =

-2.8000 + 2.8566i
-2.8000 - 2.8566i
```

b. The transfer functions are

$$Z = \frac{1}{1+CP}V + \frac{P}{1+CP}D - \frac{CP}{1+CP}N + \frac{CP}{1+CP}R$$

$$Z = SV + SPD + T(R - N)$$

$$Z(s) = \begin{pmatrix} S(s) & S(s)P(s) & T(s) & -T(s) \end{pmatrix} \begin{pmatrix} V(s) \\ D(s) \\ R(s) \\ N(s) \end{pmatrix}$$

Matlab commands:

```
>> T = feedback(C*P,1);
>> S = 1-T;
>> bode(T)
>> hold on
>> bode(S)
>> S
```

Transfer function:

$$\frac{s^2 + 2s}{s^2 + 5.6s + 16}$$

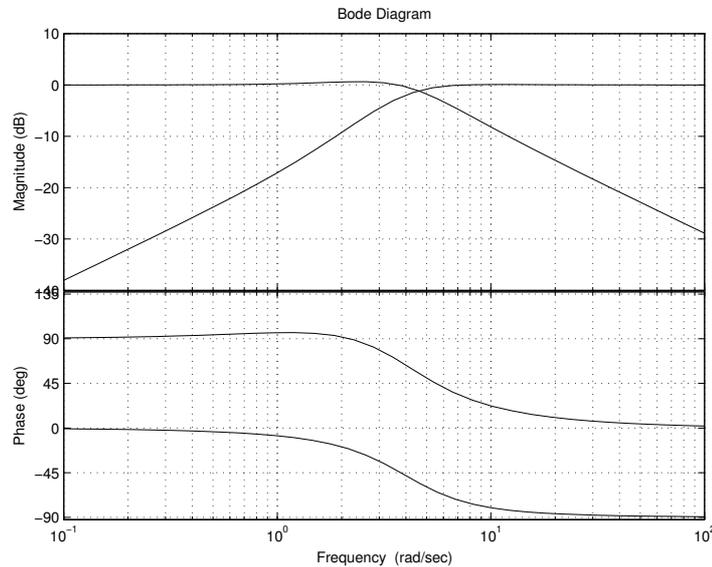
```
>> T
```

Transfer function:

$$3.6 s + 16$$

---


$$s^2 + 5.6 s + 16$$



**Figure 3.3** Bode diagrams of S and T in Problem 3.3.

- c.** In the bode plot of the sensitivity function, we see that  $\|S(i0.5)\| = -23.8 \text{ dB} = 10^{(-23.8/20)} = 0.0646$

Matlab commands:

```
>> abs(freqresp(S,0.5))
```

```
ans =
```

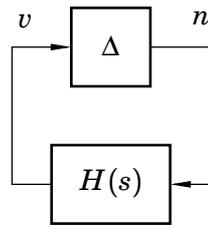
```
0.0644
```

- d.** We convert  $\omega = 2\pi 50 \text{ Hz} = 314.16 \text{ rad/s}$ . In the bode plot of the complementary sensitivity function, we see that  $\|T(i314.16)\| = -38.8 \text{ dB} = 10^{(-38.8/20)} = 0.0115$

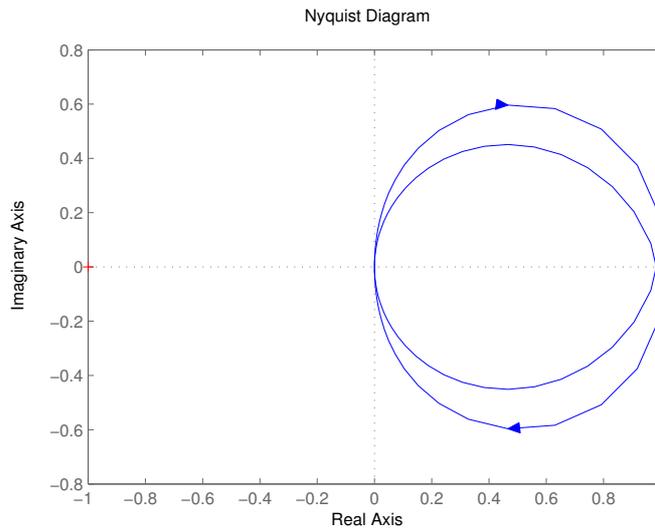
We have very good attenuation of both load disturbances and measurement noise.

- 3.4 a.** The transfer function from  $n$  to  $v$  as seen in Figure 3.4 can be written as  $H = \frac{-PCW}{1+PC}$  according to the following Matlab commands:

```
>> s = tf('s');
>> W = s/(s+1);
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s);
>> H = -feedback(P*C,1)*W;
>> norm(H,inf)
ans =
```



**Figure 3.4** Rewritten closed-loop system for Problem 3.4(a).



**Figure 3.5** Nyquist plot of  $-H(s)$  in problem 3.4(b).

```

1.0065
>> lower_bound = 1/ans
ans =
0.9935

```

- b.** We know that  $\Delta(s) = \delta$ , a real number. Looking at Figure 3.4 we see that we can apply the Nyquist Criterion to analyze the closed loop stability.

From the Nyquist Plot of  $-H(s)$  in Figure 3.5, we see that the closed loop is stable for all  $\delta \geq 0$ . For negative  $\delta$ 's, the closed loop will become unstable once the bubble formed by the Nyquist Curve has grown so large that  $-1$  is no longer on its outside. We find this value to be  $-\delta = 1.0119$  from the gain margin of  $-H$ . Thus, the system is stable when  $\delta > -1.0119$ .

The small gain theorem is easy to use, but it can be conservative, since there is no prior assumptions on structure of uncertainty. With more information about the uncertainty, the bounds can be less conservative and we can allow all positive values of  $\delta$  as well.

Matlab code:

```

>> nyquist(-H)
>> allmargin(H)
ans =

```

```

GainMargin: 1.0119
GMFrequency: 2.3256
PhaseMargin: [-4.4628 -18.9141]
PMFrequency: [2.5211 3.1922]
DelayMargin: [2.4614 1.8649]
DMFrequency: [2.5211 3.1922]
Stable: 1

```

- c. In the referenced figure the uncertainty is *added* to the process. This is called an *additive uncertainty*, ie.  $P + \Delta$ . Here, the uncertainty is multiplied to the output signal, giving a *multiplicative uncertainty*, ie.  $P(1 + \Delta)$ . In this type of model, the uncertainty is proportional to the process gain.

**3.5**  $\Phi_u(\omega)$  is an even, scalar, non-negative function. Thus we can divide it into

$$\Phi_u(\omega) = G(i\omega)G(-i\omega)\Phi_e(\omega)$$

where  $G(s)$  has its poles and zeroes in the left half-plane and  $\Phi_e = 1$  (white noise).

a.

$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}\Phi_e(\omega) = \frac{a}{i\omega + |a|} \cdot \frac{a}{-i\omega + |a|}$$

So the linear filter is

$$G(s) = \frac{a}{s + |a|}$$

b. In the same way, we get

$$\begin{aligned} \Phi_u(\omega) &= \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}\Phi_e(\omega) \\ &= \frac{ab}{(i\omega + |a|)(i\omega + |b|)} \cdot \frac{ab}{(-i\omega + |a|)(-i\omega + |b|)} \\ \Rightarrow G(s) &= \frac{ab}{(s + |a|)(s + |b|)} \end{aligned}$$

**3.6 a.** To make a state-space description, we let  $x_1 = z$ ,  $x_2 = \dot{z} \implies$

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{m}(u - k_1 x_2 - v). \end{aligned}$$

In matrix form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k_1}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} v,$$

$$z = (1 \quad 0) x.$$

- b. We want to find a filter  $H$  such that

$$\Phi_v(\omega) = |H(i\omega)|^2 \Phi_e(\omega)$$

Thus  $H(s) = \frac{\sqrt{k_0}}{s+|a|}$ , which is equivalent to  $\dot{v} + |a|v = \sqrt{k_0}e$ .

Adding a new state  $x_3 = v$  to the state-space description, gives

$$\dot{x}_3 = -|a|x_3 + \sqrt{k_0}e$$

and

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{m} & -\frac{1}{m} \\ 0 & 0 & -|a| \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{pmatrix} e$$

$$z = (1 \ 0 \ 0)x, \quad \Phi_e(\omega) = 1$$

### 3.7

- a. With  $\{A, B, C, N\}$  according to the solution of problem 3.6, we have

$$\begin{aligned} \dot{x} &= Ax + Bu + Ne \\ y &= Cx + n \end{aligned}$$

where  $n$  has spectral density  $\Phi_n \equiv 0.1$ .

- b. A noise signal with the specified spectral density is given by the output of a linear system with white noise input that has spectral density  $\Phi_{w_n} = 0.1$ . The transfer function of the system is

$$G_n(s) = \frac{s}{s+|b|} = \frac{s+|b|-|b|}{s+|b|} = 1 - \frac{|b|}{s+|b|}$$

In state-space form this can be expressed as

$$\begin{aligned} \dot{x}_4 &= -|b|x_4 + |b|w_n \\ n &= -x_4 + w_n \end{aligned}$$

Combining the noise model with our original system gives the expanded state-space description:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & |b| \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & -1 \end{pmatrix} x + w_n, \quad \Phi_{\omega_n} = 0.1 \end{aligned}$$

Note that the disturbance can be described using a transfer function and white noise of any spectral density. For instance, it is often convenient to assume white noise with a spectral density of 1. In this case, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}s}{s+|b|}$$

The expanded state space description would then need to be adjusted to account for this.

- c. Now, the transfer function of the noise model is  $G_n(s) = \frac{1}{s+|b|}$ . In state-space form, this is

$$\dot{x}_4 + |b|x_4 = w_n.$$

The expanded system becomes

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= (C \ 1)x, \quad \Phi_{w_n} = 0.1 \end{aligned}$$

As in subproblem b, the disturbance can be described using a transfer function and white noise of any spectral density. Assuming white noise with a spectral density of 1, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}}{s + |b|}$$

### 3.8

- a. The spectrum of the wind is of low-pass character with cut-off frequency  $\alpha$ . When  $\alpha$  is increased,  $v(t)$  becomes more similar to white noise, i.e. there is more high-frequency content in the signal. Thus, higher  $\alpha$  means more wind gusts.

Alternatively, one could look at the covariance function:

$$R_v(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_v(\omega) e^{i\omega\tau} d\omega = e^{-\alpha|\tau|}, \quad \alpha > 0.$$

The covariance function has a sharper peak when  $\alpha$  is large. That is, the correlation between  $v(t)$  and  $v(t+\tau)$  is small, meaning that the wind changes more often.

- b. Using spectral factorization, the influence of wind can be described as white noise  $e(t)$  with intensity 1 filtered through a linear system with transfer function

$$H(s) = \frac{\sqrt{2/\alpha}}{1 + s/\alpha}$$

Thus  $Y(s) = G(s)KH(s)E(s)$ , where

$$G(s)KH(s) = \frac{K\sqrt{2\alpha}}{(\alpha + s)(s^2 + s + 1)} = \frac{K\sqrt{2\alpha}}{s^3 + (1 + \alpha)s^2 + (1 + \alpha)s + \alpha}.$$

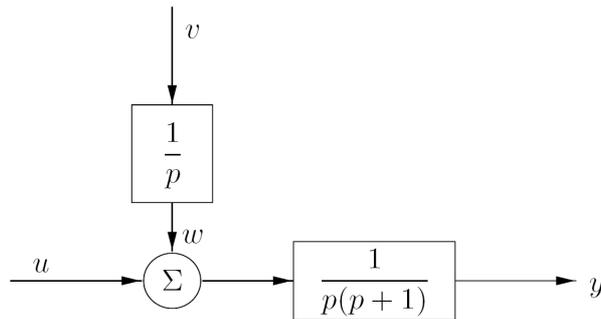
The variance of the output is

$$\begin{aligned} \text{Var}(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)KH(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{K\sqrt{2\alpha}}{(i\omega)^3 + (1 + \alpha)(i\omega)^2 + (1 + \alpha)i\omega + \alpha} \right|^2 d\omega \\ &= \frac{K^2(1 + \alpha)}{1 + \alpha + \alpha^2}. \end{aligned}$$

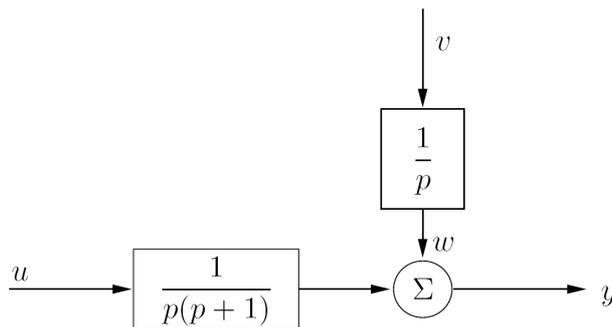
Apparently, the variance increases with wind strength, which is no surprise. However, the variance decreases with the amount of wind gusts. The reason is that a low amount of gusts means that there are longer periods of almost constant wind force, where the swing is displaced far from the origin. A lot of gusts, on the other hand, results in the wind force changing sign frequently, more or less cancelling its own effect a lot of the time.

3.9

a. (i)



(ii)



$v(t)$  is a unit disturbance

b. (i)

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_B u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{(1 \ 0 \ 0)}_C x.$$

(ii)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^A x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^B u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{(1 \ 0 \ 1)}_C x.$$

- c.** (i)  $w(t)$  could be an offset current on the input to the motor, and/or a step disturbance in the load.
- (ii) In this case  $w(t)$  is a measurement disturbance, i.e. an additive error (constant) in the angle measurement.

## Solutions to Exercise 4. Loop Shaping

**4.1** *This is a preparatory exercise for the practical session. This is not a complete solution, just some helpful tips.*

- a.** Since we cannot change the phase of the system using a P-controller, higher gain will lead to lower phase margin (as the phase approaches  $-180$  for high frequencies).

Higher gain will also decrease stationary errors, but increase the maximum peak in the sensitivity function (making the system very sensitive to measurement noise).

```
>> figure(1)
>> step(P/(1+0.1*P),P/(1+1*P),P/(1+5*P),P/(1+10*P));
>> title('Step responses')
>> figure(2)
>> bode(P/(1+0.1*P),P/(1+1*P),P/(1+5*P),P/(1+10*P));
>> title('Transfer functions from load disturbance');
>> figure(3)
>> bode(1/(1+0.1*P),1/(1+1*P),1/(1+5*P),1/(1+10*P));
>> title('Sensitivity functions');
```

- b.** It is not possible to achieve good behavior with a PI-controller, but try to get it as good as possible:

```
>> figure(1)
>> K= ... ; Ti = ... ;
>> C = tf(K*[1 1/Ti],[1 0]);
>> step(P/(1+C*P);
```

**4.2** *This is a preparatory exercise for the practical session. This is not a complete solution, just some helpful tips.*

- a.** From the basic course: We calculate the gain  $\|C(i\omega)\| = 1/\sqrt{\omega^2/a^2 + 1}$  and use log scale. Then

$$\log |C(i\omega)| = -0.5 \log(\omega^2/a^2 + 1) \approx \begin{cases} 0 & \omega \ll a \\ \log(a) - \log(\omega) & \omega \gg a \end{cases}$$

and the two lines meet where  $\omega = a$  (the breakpoint). Also, the phase is at  $-45^\circ$  at  $\omega = a$ , starts at  $0^\circ$  and ends at  $-90^\circ$ .

We can add a pole to the controller if we want to decrease gain for higher frequencies, e.g. to limit the cut-off frequency  $\omega_c$ . It is often the case that we want to increase the gain at low frequencies, but keep it low at high frequencies. We can then use a controller of the type  $C(s) = K/(s/a + 1)$  with a pole to limit high frequency gain and a static gain larger than one to increase the low frequency gain.

```
>> C01 = tf([1],[1/0.1 1]);
>> C1 = tf([1],[1/1 1]);
>> C5 = tf([1],[1/5 1]);
>> bode(C01, C1, C5);
```

- b.** The same as in (a), except that a zero breaks the gain **up** at  $b$ .

$$\log |C(i\omega)| = 0.5 \log(\omega^2/b^2 + 1) \approx \begin{cases} 0 & \omega \ll b \\ \log(\omega) - \log(b) & \omega \gg b \end{cases}$$

We can add a zero to the controller to increase gain at high frequencies in order to increase the cut-off frequency  $\omega_c$ . Also, since the phase of the zero goes to  $+90^\circ$ , we increase the phase margin by adding a zero.

- 4.3** *This is a preparatory exercise for the practical session. This is not a complete solution, just some helpful tips.*

The following Matlab code shows some relevant plots for a design:

```
>> s = tf('s');
>> C = ...;           % Make up your own design

>> figure(1)
>> margin(C*P)       % Plot open loop frequency response

>> figure(2)
>> % Plot step responses from load disturbance and reference signal to output signal
>> subplot(2,1,1)
>> step(P/(1+P*C));
>> title('Load step response');
>>
>> subplot(2,1,2)
>> step(P*C/(1+P*C))
>> title('Reference step response');
```

- 4.4** *This is a preparatory exercise for the practical session. This is not a complete solution, just some helpful tips.*

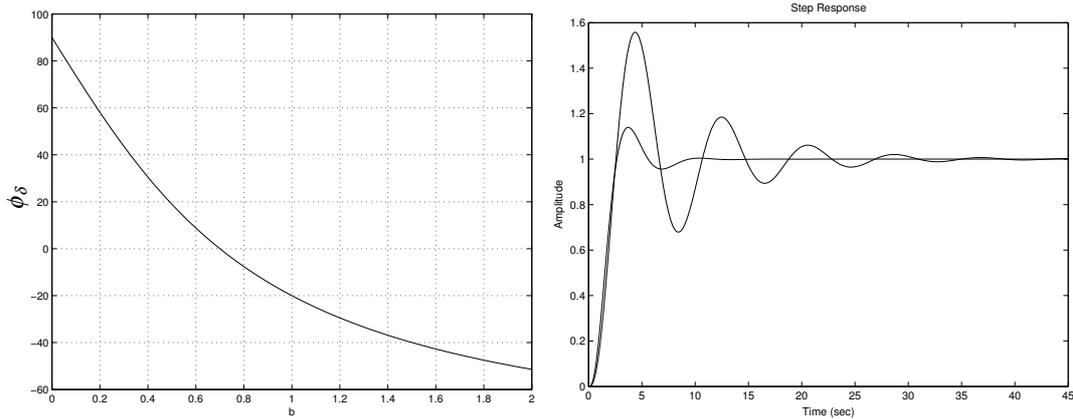
- a.** The ideal frequency response is  $G_{yr} \equiv 1$ . Then we would always have  $y = r$ . However, achieving something close to this would require very aggressive control, so that is not a good idea. (The controller would need to invert the process dynamics, resulting in second-order derivative action on the control error).
- b.** We want to shape  $F(s)$  so that the constraints on the control signal are respected, for a step change in the reference. This may be achieved by reducing the bandwidth.

- 4.5** Plot the Bode diagram for  $G_o(s)$  in Matlab or use the command

```
>> [Gm,Pm,Wcg,Wcp] = margin(G_o)
```

to calculate the cut-off frequency  $\omega_c = 0.73$  and the phase margin  $\phi_m = 20.7^\circ$ . To reach the aim of a  $\phi_{m,desired} = 50^\circ$ , the controller has to increase the phase at the cut-off frequency with approx  $30^\circ$ . We use the lead compensation given by

$$G_k(s) = KN \frac{s + b}{s + bN}$$



**Figure 4.1** To the left: Plot of  $\phi_\delta$  against  $b$ . To the right: Step response from the original system and the compensated system in Problem 4.5.

with the phase

$$\phi = \arctan\left(\frac{s}{b}\right) - \arctan\left(\frac{s}{bN}\right)$$

The maximum of the phase compensation for the compensator is at the frequency  $b\sqrt{N}$ , which preferably should coincide with  $\omega_c$ , hence  $N = (\omega_c/b)^2$ . Plot the phase addition of the compensator given by

$$\phi_\delta = \arctan\left(\frac{\omega_c}{b}\right) - \arctan\left(\frac{b}{\omega_c}\right)$$

and determine that the factor  $b \approx 0.4$  for  $\phi_\delta = 30^\circ$  (see Figure 4.1). To keep the cut-off frequency invariant the gain of the compensator has to be calculated from  $|G_k(i\omega_c)G_o(i\omega_c)| = K\sqrt{N} \cdot 1$  gives  $K = \frac{1}{\sqrt{N}} = 0.55$ . Plot the step response by the commands:

```
>> G_1=tf(K*N*[1 b],[1 b*N])
>> step(G_o*G_1/(1+(G_o*G_1)))
```

The stationary error:

$$E(s) = \frac{1}{1 + G_k G_o} U(s) = \frac{s(s + 0.5)(s + 3)(s + bN)}{s(s + 0.5)(s + 3)(s + bN) + 2KN(s + b)} U(s)$$

The Laplace transform of a ramp function is  $U(s) = 1/s^2$  and the error is

$$\lim_{s \rightarrow 0} sE(s) = \frac{1.5}{2K} = 1.37$$

which fulfills the specification.

**4.6 a.** The transfer function from  $d$  to  $y$  is given by

$$G_{yd}(s) = \frac{P}{1 + PC}$$

For frequencies  $\omega \leq 0.5$  (approximately), it can be seen in the Bode diagram that both  $|P(i\omega)| \gg 1$  and  $|P(i\omega)C(i\omega)| \gg 1$ . Therefore  $G_{yd}(s) \approx \frac{1}{C}$ , and  $|C(i\omega)|$  becomes larger than 1 for frequencies  $\omega \leq 0.02$ .

The magnitude of  $G_{yd}(s)$  is thus smaller than 1 in a frequency range of approximately  $[0, 0.02]$ , thus  $\omega_b = 0.02$  rad/s.

This can also be seen as the frequency point where  $|PC|$  becomes larger than  $|P|$  in the bode diagram.

- b.** To increase  $\omega_b$ , we would like to increase the gain of  $C(i\omega)$  for frequencies  $\omega > 0.02$ . This is done by moving the zero in  $C(s)$  (the break-point in the Bode diagram) from 0.02 to some higher frequency.

Choose e.g.  $a = 0.1$ . Motivation:

- As  $G_{yd}(s) \approx \frac{1}{C}$ , and  $|C(i\omega)|$  now becomes larger than 1 for frequencies  $\omega \leq 0.1$ ,  $\omega_b$  has been increased to about 0.1.
- The cut-off frequency for  $a = 0.02$  is  $\omega_c \approx 0.8$ . As this frequency is higher than the new break-point 0.1,  $C(i\omega_c) \approx 1$  still holds  $\Rightarrow$  the cut-off frequency stays the same.

## Solutions to Exercise 5. Multivariable Zeros, Singular Values and Controllability/Observability

5.1 a. In Matlab, we may derive the controllability- and observability matrices using

```
>> Wc = ctrb(A,B)
```

```
Wc =
```

```
    1    -1    1
    1    -2    4
    0     0    0
```

```
>> rank(Wc)
```

```
ans =
```

```
    2
```

```
>> Wo = obsv(A,C)
```

```
Wo =
```

```
    1     0     1
   -1     0    -3
    1     0     9
```

```
>> rank(Wo)
```

```
ans =
```

```
    2
```

Since the system is on diagonal form we can see, using theorem 3.1 in the course book (Glad&Ljung), that the uncontrollable mode corresponds to the third state (as that row in the  $B$  matrix is 0). By theorem 3.2 in the course book, the unobservable mode is determined to be the second state in a similar fashion (the column of  $C$  equal to 0).

The system is illustrated in the block diagram in Figure 5.1. We can see that the state  $x_2$  will not influence  $y$ , and is therefore not observable. We can also see that the control signal  $u$  will not affect the state  $x_3$ , and therefore this state is not controllable.

b. The transfer function is simply

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s + 1}$$

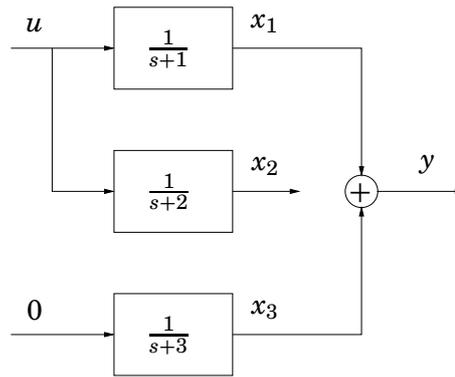


Figure 5.1

and the system can thus be represented as a minimal realization in state space form of order 1. Note that this corresponds to the first subsystem in Figure 5.1 which is both observable and controllable.

If we are only interested in the relationship between  $u$  and  $y$ , we can use the resulting first order transfer function  $G(s)$ . However, the original third order state space model contains additional information, as seen in Figure 5.1. The second and third subsystems in this model may represent physical entities of the plant that must be taken into account. If we need to influence  $x_3$  or monitor  $x_2$ , additional sensors or actuators are needed.

## 5.2

- a.** First of all, define the system in Matlab

```
>> A = [-0.21 0.2; 0.2 -0.21];
>> B = 0.01*eye(2);
>> C = eye(2);
>> D = 0;
>> sys = ss(A,B,C,D);
```

The controllability Gramian is calculated using

```
>> W = gram(sys, 'c')
```

W =

```
0.0026    0.0024
0.0024    0.0026
```

- b.** Recall the formula from the lecture notes:

$$\int_0^{\infty} u^2(s) ds \geq x^T(\infty) W^{-1} x(\infty)$$

Therefore to identify the hardest to control state direction, we calculate the eigenvalues and the eigenvectors of  $W^{-1}$ :

### Solutions to Exercise 5

```
[T L] = eig(inv(W))
```

```
T =
```

```
    -0.7071    -0.7071  
    -0.7071     0.7071
```

```
L =
```

```
1.0e+03 *  
    0.2000     0  
     0     8.2000
```

Apparently one eigenvalue of the inverse of the Gramian is almost 40 times larger than the other. Hence one state direction is poorly controllable.

Inspection of the corresponding eigenvectors, i.e. the columns of T, shows that the small eigenvalue corresponds to a state direction where both temperatures move in the same way, while the poorly controllable state direction corresponds to temperatures moving in opposite directions.

#### 5.3 a. Continuing the code we get

```
>> syms c1 c2 c3 c4 c5  
>> C = [c1 c2 c3 c4 c5];  
>> Wo = [C;C*A;C*A^2;C*A^3;C*A^4];  
>> det(Wo)
```

```
ans =
```

```
0
```

```
>> rank(Wo)
```

```
ans =
```

```
4
```

Since the system does not have full rank (5) we see that no matter how we choose C (when it's a vector), the system can never be made observable. This means that we need information from more than just one signal to make the system observable.

#### b. Determine the eigenvectors of the system

```
>> [V,D]=eig(A)
```

```
V =
```

```
     0     0.3333     0.4286    -0.0261     0.0206  
     0     0.6667    -0.8571     0.7973    -0.3916  
  1.0000     0.6667     0.2857    -0.0399     0.0196  
     0         0         0         0.6017         0
```

```

0      0      0      0      0.9197
...

```

Rewrite the system on diagonal form using the variable change  $x(t) = Vz(t)$

$$\begin{aligned} \dot{x}(t) &= V\dot{z}(t) = AVz(t) + Bu(t) \Rightarrow \\ \dot{z}(t) &= V^{-1}AVz(t) + V^{-1}Bu(t) = \Lambda z(t) + V^{-1}Bu(t) \\ y(t) &= CVz(t) \end{aligned}$$

where  $\Lambda$  is a diagonal matrix with the eigenvalues in the diagonal. Now that we have the system on the wanted form, we can determine if there are any columns in  $CV$  that are zero

```
>> C*V
```

```
ans =
```

```

0      0.3333      0.4286      -0.0261      0.0206
0      0.6667      -0.8571      0.7973      -0.3916

```

The first state in  $z$  therefore corresponds to the unobservable mode. In the original variables this is the third state:

```
>> V*[1;0;0;0;0]
```

```
ans =
```

```

0
0
1
0
0

```

So, the third state is the unobservable mode.

- 5.4** System A depicts the observable system. Obviously the problem is in the pole  $p_0 = -3$ . We control directly the plant  $P_1$ , and we observe the output of plant  $P_2$ . It means that we observe the effect of the pole  $p_0 = -3$ , but due to pole-zero cancellation, we cannot control it.

Similarly for system B, we control directly the plant  $P_2$ , and the pole  $p_0 = -3$  is controllable, but the effect of that pole is cancelled by the zero in  $P_1$  and we do not observe it. Hence the whole system is not observable.

- 5.5 a.** The largest subdeterminant of the transfer function matrix is

$$\frac{(s+1)}{(s+2)^2} + \frac{1}{(s+2)^2} = \frac{1}{(s+2)}$$

Furthermore, the matrix elements in themselves are subdeterminants. The pole polynomial, i.e. the least common denominator of all subdeterminants, is then

$$p(s) = (s+2)$$

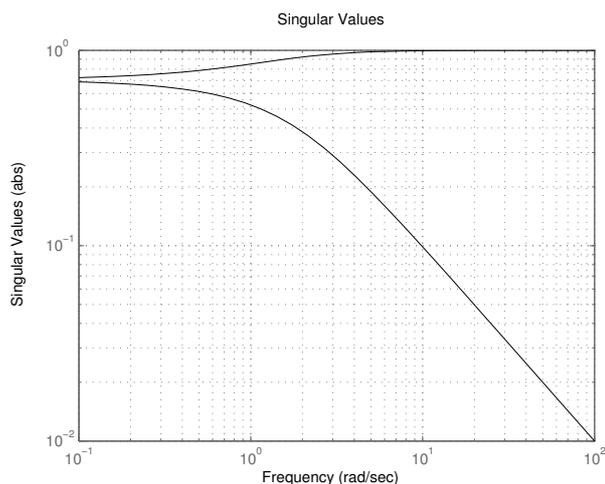


Figure 5.2

This means that the system has a pole in  $s = -2$ . The system can thus be realized on state space form of order 1.

The largest possible subdeterminant was

$$\frac{1}{(s + 2)}$$

The zero polynomial is thus just a constant and the system does therefore not have any zeros.

Note, that we basically calculated the determinant of the transfer matrix  $\det(G(s))$  and took its denominator as the pole polynomial and numerator as the zero polynomial.

**b.**

$$\begin{aligned} G(s) &= \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & 1 - \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{s+2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

A state-space realization can now be written as

$$\begin{aligned} \frac{dx}{dt} &= -2x + \begin{pmatrix} 1 & -1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u \end{aligned}$$

- c.** The singular value plot (see Figure 5.2) is drawn using the command `sigma`. The  $\mathcal{L}_2$  gain  $\|G\|_\infty$  is the largest singular value of  $G(i\omega)$  across all frequencies  $\omega$ , from the figure we can see that  $\|G\|_\infty = 1$  in this case. We also see that the largest gain of this system is achieved at high frequencies.

- 5.6 a.** To determine the frequency response at a certain frequency  $\omega$ , it's handy to use the Matlab command `freqresp`. To calculate the singular values together with the  $U$  and  $V$  matrices, use the function `svd`. The Matlab code can look like this:

```
>> s = tf('s');
>> G = [1 1/s];
>> [U,S,V] = svd(freqresp(G,1))
```

U =

1

S =

1.4142          0

V =

0.7071          0 + 0.7071i  
0 + 0.7071i    0.7071

The maximum gain, corresponding to the highest singular value, is obtained as the first element in  $S$  and is  $\bar{\sigma} = 1.4142$ . The first column of  $V$ ,  $v_1 = (0.7071 \ 0.7071i)^T$ , corresponds to the input direction that gives the maximum gain  $\bar{\sigma}$ . Since the system has two inputs and only one output, there will always be an input direction that gives zero output (where the inputs cancel each other). The second column of  $V$  gives us this direction,  $v_2 = (0.7071i \ 0.7071)^T$ .

- b.** If the input signal is a sinusoid with frequency  $\omega = 1$  rad/s, the complex numbers will correspond to a phase shift of this sinusoid. The input direction giving the highest gain is  $v_1 = [0.7071 \ 0.7071i]^T$ , meaning that the second input has  $90^\circ$  of phase lead before the first.

The first input comes through the system unchanged; the second goes through an integrator, causing a phase lag of  $90^\circ$ . Thus the input direction  $v_1 = [0.7071 \ 0.7071i]^T$  will cause the two sinusoids that sum up at the output to be in phase; resulting in maximal gain.

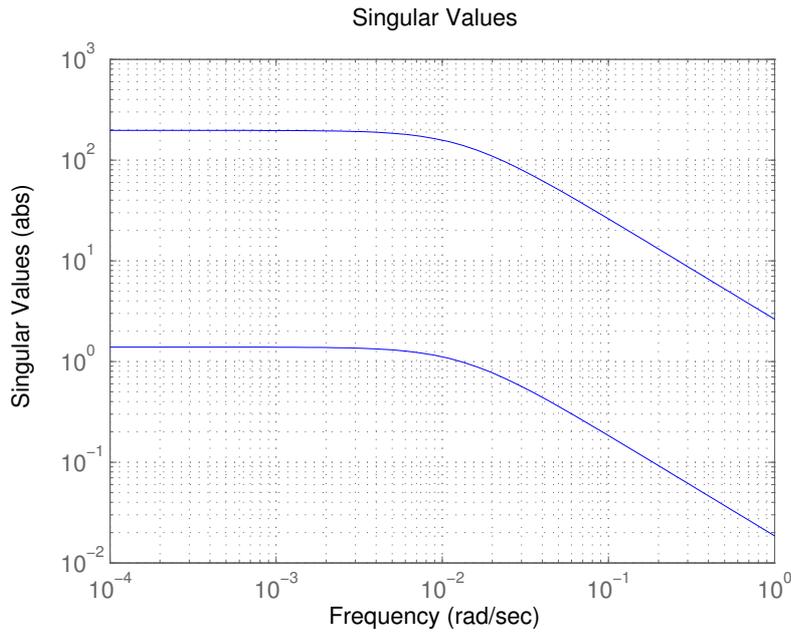
If we instead use the lowest gain input direction  $v_2 = [0.7071i \ 0.7071]^T$ , the second input will have a phase lag of  $90^\circ$ , causing a  $180^\circ$  phase lag at the output. The two signals will cancel at the output, resulting in minimal gain.

- 5.7 a.**

```
>> s = tf('s');
>> G = 1/(75*s+1)*[87.8 -86.4;108.2 -109.6];
>> sigma(G)
>> grid
```

See Figure 5.3 for the Matlab plot.

- b.** Calculate the frequency responses at the given frequencies



**Figure 5.3** Singular value plot for Problem 5.7

```
>> Gfr1 = freqresp(G,0)
```

```
Gfr1 =
```

```
87.8000 -86.4000
108.2000 -109.6000
```

```
>> Gfr2 = freqresp(G,0.1)
```

```
Gfr2 =
```

```
1.5336-11.5022i -1.5092+11.3188i
1.8900-14.1747i -1.9144+14.3581i
```

The gain of a transfer matrix at a particular frequency,  $\omega$ , is computed as  $\sup_{d \neq 0} \frac{\|G(i\omega)d\|_2}{\|d\|_2}$ . If we choose a particular direction  $d_0$  then the supremum disappears and the gain is given by  $\frac{\|G(i\omega)d_0\|_2}{\|d_0\|_2}$ .

Thus the gains are given by

$$\frac{\|G(0)d_1\|_2}{\|d_1\|_2} = \frac{\|[-5.1 \quad -8.6]^T\|_2}{\|[0.6713 \quad 0.7412]^T\|_2} = \frac{\sqrt{(-5.1)^2 + (-8.6)^2}}{\sqrt{(0.6713)^2 + (0.7412)^2}} = \frac{10.0}{1}$$

$$\frac{\|G(0)d_2\|_2}{\|d_2\|_2} = 139.3$$

$$\frac{\|G(0.1i)d_1\|_2}{\|d_1\|_2} = 1.3$$

$$\frac{\|G(0.1i)d_2\|_2}{\|d_2\|_2} = 18.4$$

They can also be calculated in Matlab using

```
>> d1 = [0.6713;0.7412];
>> d2 = [1;0];
>> norm(Gfr1*d1),norm(Gfr1*d2),norm(Gfr2*d1),norm(Gfr2*d2)
```

ans =

9.9990

ans =

139.3416

ans =

1.3215

ans =

18.4159

**c. Using Matlab:**

```
>> [U,S,V] = svd(Gfr1)
```

U =

```
-0.6246  -0.7809
-0.7809   0.6246
```

S =

```
197.2087   0
   0      1.3914
```

V =

```
-0.7066  -0.7077
 0.7077  -0.7066
```

*Solutions to Exercise 5*

The maximum gain is  $\bar{\sigma} = 197.2$  and the minimum gain is  $\underline{\sigma} = 1.39$ . The input direction associated with the maximum gain is  $v_1 = [-0.7066 \quad 0.7077]^T$ . The input direction giving the least gain is  $v_2 = [-0.7077 \quad -0.7066]^T$ . These directions are constant for all frequencies. The reason is that the denominators of all matrix elements are the same, which gives

$$G(i\omega) = \frac{1}{75i\omega + 1} G(0).$$

Let  $G(0) = U\Sigma V^*$ . We then have  $G(i\omega) = U \left( \frac{1}{75i\omega + 1} \Sigma \right) V^*$ , and we can see that  $\omega$  will only change the singular value matrix  $\Sigma$ , not the direction matrices  $U$  and  $V$ .

## Solutions to Exercise 6. Fundamental Limitations

6.1 a. The transfer function of the process  $P(s)$  is given by

$$P(s) = \frac{s}{s^2 + 2s + 1}$$

and the zero is located in the origin.

- b. The sensitivity function is given by  $S(s) = \frac{1}{1 + P(s)C(s)}$  and it will be one at low frequencies since  $P(0) = 0$ . Note that you can not cancel the process zero in  $s = 0$  with your controller since you then would not have an asymptotically stable system.
- c. The error  $e(t)$  is given by  $r(t) - \theta(t)$  and the static error is then given by the final value theorem, which can be used if all poles of  $sE(s)$  have a strictly negative real-part.

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Here the transfer function from  $r$  to  $e$  is given by:

$$G_{re}(s) = \frac{1}{1 + P(s)C(s)}$$

The following result is obtained if  $r(t)$  is assumed to be a step,  $R(s) = a/s$ .

$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} sG_{re}(s)R(s) = a$$

since  $P(0) = 0$  (and thereby  $G_{re}(0) = 1$ ). This means that the ball will not follow a reference trajectory that changes step-wise; there will be a static error equal to  $a$ . Hence, no matter the reference value, the ball will end up in the bottom of the cylinder.

An alternative explanation is that the sensitivity function  $S$  is 1 at  $s = 0$ , therefore

$$T(0) = \frac{P(0)C(0)}{1 + P(0)C(0)} = 1 - S(0) = 0$$

and then  $y(t)$  does not follow  $r(t)$ .

- d. The transfer function for the open loop with a PI controller is given by:

$$P(s)C(s) = \frac{s}{s^2 + 2s + 1} K \frac{s + 1/T_i}{s} = K \frac{s + 1/T_i}{s^2 + 2s + 1}$$

Here the process zero is canceled by the controller.

If we would have no stationary error the control signal from a PI controller would be constant. But if we have a constant control signal  $\omega$  that would imply that  $\dot{\omega} = 0$  which would give  $\theta = 0$  and we would get a stationary error (if  $r(t) \neq 0$ ). This is contradictory and therefore we must have a stationary error. And with a PI controller a stationary error would give a linearly increasing control signal.

**6.2** The sensitivity function is given by:

$$S(s) = \frac{1}{1 + P(s)C(s)}$$

From this it follows that

$$S(i\omega) = \frac{1}{1 + P(i\omega)C(i\omega)}$$

**a.** In the case of a purely imaginary process *pole* in  $i\omega_p$  we have

$$P(i\omega_p) = \infty$$

and consequently

$$S(i\omega_p) = 0$$

**b.** A measurement disturbance  $n$  with frequency  $\omega_p$  will have a vanishing effect on  $y$  and  $e$ , since

$$S(i\omega_p) = 0.$$

Note that this implies that  $n$  will have a big impact on  $z$  since  $z(t) = y(t) - n(t)$ .

**c.** No stabilizing controller can change the fact that  $S(i\omega_p) = 0$ . Cancellations of poles on the imaginary axis should always be avoided.

**d.** In the case of a purely imaginary process *zero* in  $i\omega_z$  we have

$$P(i\omega_z) = 0$$

and consequently

$$S(i\omega_z) = 1$$

**e.** The transfer function from  $n$  to  $z$  is given by  $-T(s)$ , where  $T$  is the complementary sensitivity function. Since  $S(i\omega_z) = 1$  and  $S + T = 1$  it must hold that  $T(i\omega_z) = 0$ , i.e. an output disturbance with frequency  $\omega_z$  will have no effect on  $z$ .

**6.3 a.** The transfer function from  $n$  to  $z$  is given by

$$G_{n \rightarrow z}(s) = -\frac{P(s)C(s)}{1 + P(s)C(s)}$$

We want to determine  $C(s)$  such that  $G_{n \rightarrow z}(s) = 5/(s + 5)$ . This gives the equation

$$-\frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{5}{s + 5} \implies C(s) = -\frac{\frac{5}{s+5}}{P(s) \cdot (1 + \frac{5}{s+5})}$$

Inserting  $P(s) = (3 - s)/(s + 1)^2$ , we obtain

$$C(s) = -\frac{5 \cdot (s + 1)^2}{(3 - s)(s + 10)}$$

However, this is not a stabilizing controller. For example, the transfer function from  $n$  to  $u$ ,  $G_{n \rightarrow u} = -\frac{C(s)}{1 + P(s)C(s)}$ , will be unstable because of the cancellation of the unstable zero in  $P(s)$ .

**b.** The specification

$$|S(i\omega)| \leq \frac{2\omega}{\sqrt{\omega^2 + 36}} \quad \omega \in \mathbb{R}^+$$

is equivalent to

$$\sup_{\omega} \left| \frac{\sqrt{\omega^2 + 36}}{2\omega} S(i\omega) \right| \leq 1$$

However,  $W_a(i\omega) = \frac{i\omega+a}{2i\omega}$  gives  $|W_a(i\omega)| = \frac{\sqrt{\omega^2+36}}{2\omega}$ , for  $a = 6$  so the specification can equivalently be written

$$\sup_{\omega} |W_a(i\omega)S(i\omega)| \leq 1$$

There is a right half plane zero in  $z = 3$ . According to Theorem 7.4 in [Glad&Ljung] this makes the specification impossible to satisfy unless  $|W_a(z)| \leq 1$ . We see here that  $|W_a(3)| = \left| \frac{3+6}{2 \cdot 3} \right| = \frac{3}{2} > 1$ , so the specification is impossible to satisfy for  $a = 6$ .

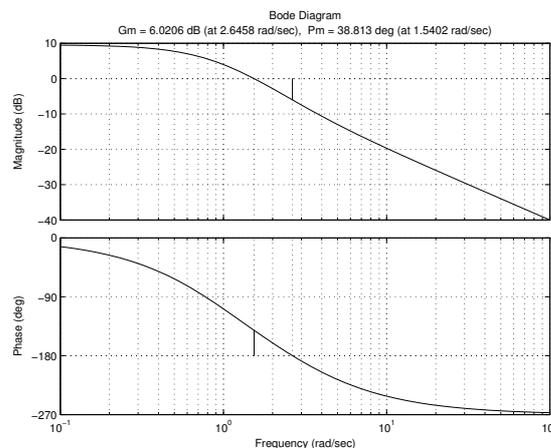
- c.** The Bode plot of  $P(s)$  is given in Figure 6.1 and the sensitivity function when  $C(s) = 1$  is given in Figure 6.2 together with the specification. Since the specification  $\frac{2\omega}{\sqrt{\omega^2+1}} = 0$  when  $\omega = 0$  the controller  $C(s)$  must contain an integrator. To avoid instability we must also lift the phase curve through adding a zero and decrease the gain in the open-loop,  $P(s)C(s)$ . A controller on the form

$$C(s) = K \cdot \frac{s/b + 1}{s}$$

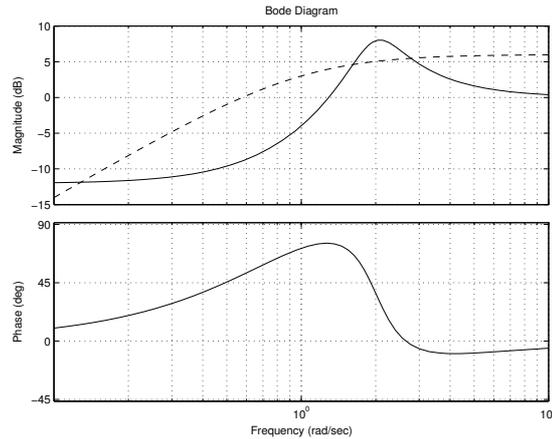
with e.g.  $K = 0.17$ ,  $b = 0.5$  will do the job.

Hint: To plot the specification on top of the Bode plot of  $S$  the following Matlab commands can be used:

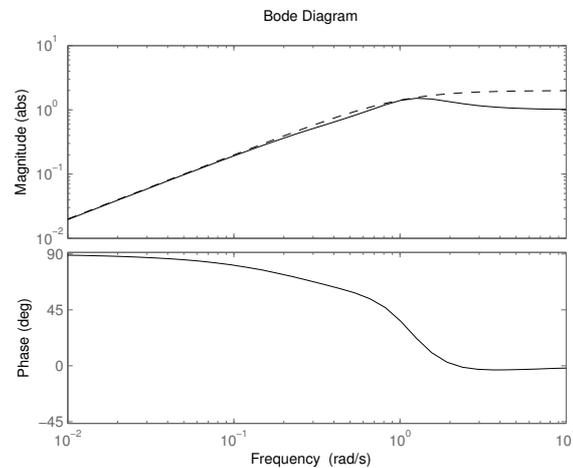
```
>>[mag, fas, w] = bode(S);
>>loglog(w, 2.*w./sqrt(w.^2+1), 'r --')
>>hold on
>>bode(S)
```



**Figure 6.1** Bode plot of  $P(s)$  in Problem 6.3.



**Figure 6.2** Sensitivity function when  $C(s) = 1$  and the specification (dashed) in Problem 6.3.



**Figure 6.3** Sensitivity function for the proposed controller and the specification (dashed) in Problem 6.3.

**6.4** The first case is impossible, because there is a time-delay of 2 seconds in the plant, so the control signal will affect the output with this delay. Thus, the controller would need to be non-causal to achieve the specification.

The second specification in the figure says that the gain should be below 2 (actually the requirement is closer to  $\approx 1.6$ ).

From theorem 7.3 in the lecture notes we see that if there is an unstable pole  $p$  and an unstable zero  $z$ , we have the following fundamental limitation:

$$\|S\|_{\infty} \geq \left| \frac{z + p}{z - p} \right|$$

We know that  $S + T = 1$ , so in this case, we have

$$\|T\|_{\infty} = \|S - 1\|_{\infty} \geq \|S\|_{\infty} - 1 \geq \left| \frac{20 + 10}{20 - 10} \right| - 1 = 2$$

where the first inequality follows from the reversed triangle inequality

$$|||x| - |y|| \leq \|x - y\|$$

It is also possible to get to the same conclusion without using the unstable zero, since the existence of a fast unstable pole is enough to make the specification impossible to achieve.

The third case is possible. If proportional control,  $C(s) = K$  is used the closed loop transfer function becomes  $G(s) = \frac{K}{s-3+K}$ . For stability it is required that  $K > 3$ . The static gain is given by  $\frac{K}{K-3}$ . Since it is a first order system there will be no overshoot in the step response, which means that a P-controller with  $K > 6$  will fulfill the specification to stay in the interval  $[0, 2]$ .

- 6.5 a.** Assume  $\sup_{\omega} |W_S(i\omega)S(i\omega)| \leq 1$  and  $\sup_{\omega} |W_T(i\omega)T(i\omega)| \leq 1$  are satisfied.

We know that  $1 = |S(s_0) + T(s_0)| \leq |S(s_0)| + |T(s_0)|$  (triangle inequality).

If  $|W_S(s_0)| > 2$  for some right half place  $s_0$ , then  $|S(s_0)| < 1/2$ , since

$$\sup_{\omega} |W_S(i\omega)S(i\omega)| = \sup_{\text{Re}(s) \geq 0} |W_S(s)S(s)| \leq 1 \text{ (Maximum Modulus Theorem).}$$

Analogously we get  $|T(s_0)| < 1/2$ . Then

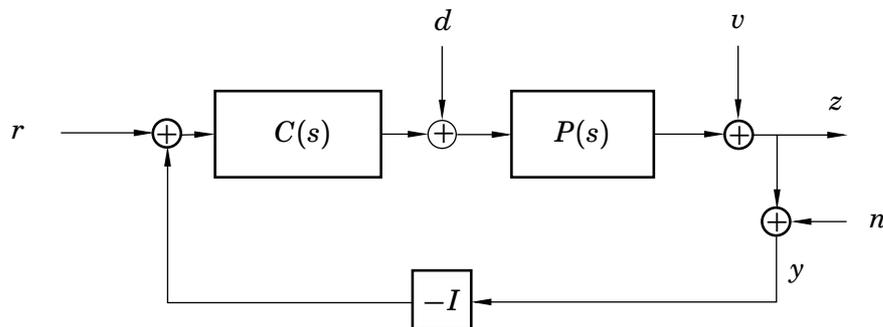
$$1 = |S(s_0) + T(s_0)| \leq |S(s_0)| + |T(s_0)| < 1$$

and we arrive to contradiction. Hence either  $|W_T(s)T(s)| > 1$  or  $|W_S(s)S(s)| > 1$  and the corresponding specification must fail.

- b.** We have

$$W_S(1) = \left(\frac{1+0.1}{1}\right)^n = \left(\frac{1+10}{10}\right)^n = W_T(1)$$

and the value is larger than 2 for  $n \geq 8$ . Hence, the statement in **a** shows that the specifications are incompatible.



**Figure 6.4** System in Problem 6.6

**6.6**

- a. The requirements on  $|S(i\omega)| = \bar{\sigma}(S(i\omega))$  and  $|T(i\omega)| = \bar{\sigma}(T(i\omega))$  may be formulated as

$$\begin{aligned} |S(i\omega)| &\leq \frac{1}{10}, \quad \omega \leq 0.1, & |T(i\omega)| &\leq \frac{1}{10}, \quad \omega \geq 2 \\ |S(0)| &\leq \frac{1}{100} \end{aligned}$$

- b. The specifications in a can be formulated with weighting functions  $W_S$  and  $W_T$  as

$$\begin{aligned} |S(i\omega)| &\leq |W_S^{-1}(i\omega)|, \quad \forall \omega \\ |T(i\omega)| &\leq |W_T^{-1}(i\omega)|, \quad \forall \omega \end{aligned}$$

If e.g.  $W_S^{-1}$  and  $W_T^{-1}$  are chosen according to

$$W_S^{-1}(s) = a_1 \left(1 + \frac{s}{b_1}\right), \quad W_T^{-1}(s) = \frac{a_2}{s} \left(1 + \frac{s}{b_2}\right)$$

we get

$$W_S^{-1}(s) = \frac{1}{100}(1 + 100s), \quad W_T^{-1}(s) = \frac{0.14}{s} \left(1 + \frac{s}{2}\right)$$

## 6.7

- a. The constraint

$$|S(i\omega)| \leq \frac{\omega\sqrt{\omega^2 + 36}}{\omega^2 + a^2} \quad \forall \omega$$

is equal to

$$\sup_{\omega} \left| \frac{\omega^2 + a^2}{\omega\sqrt{\omega^2 + 36}} S(i\omega) \right| \leq 1.$$

This can be rewritten as

$$\sup_{\omega} |W_S(i\omega)S(i\omega)| \leq 1$$

where

$$|W_S(i\omega)| = \frac{\omega^2 + a^2}{\omega\sqrt{\omega^2 + 36}} \Rightarrow W_S(s) = \frac{(s + a)^2}{s(s + 6)}.$$

From theorem 7.4 in [Glad, Ljung] we know that this specification can only be satisfied if

$$|W_S(z)| \leq 1.$$

Since

$$|W_S(2)| = \frac{(2 + a)^2}{16}$$

the specification is impossible to satisfy for  $a > 2$ .

**b.** With

$$C(s) = \frac{2(s^2 + 5s + 6)}{s(s + 6)}$$

the sensitivity function becomes

$$S(s) = \frac{1}{1 + PC} = \frac{s(s + 6)}{(s + 2)^2}.$$

This gives

$$|S(i\omega)| = \frac{\omega\sqrt{\omega^2 + 36}}{\omega^2 + 4}$$

which satisfies the specification for  $\alpha = 2$ .

## Solutions to Exercise 7. Controller Structures and Preparations for Laboratory Exercise 2

**Note:** Exercises 7.1-7.3 serve as preparation for Laboratory Exercise 2.

**7.1 a.** The relative gain array for a complex valued matrix is given by

$$\text{RGA}(G) = G .* (G^\dagger)^T$$

where  $\dagger$  denotes the pseudo-inverse of  $G$ , and  $.*$  denotes element wise multiplication. For a process  $G(s)$  the RGA is usually computed for the DC-gain ( $\omega = 0$ ) and the cut-off frequency ( $\omega = \omega_c$ ). By inspecting the elements in the RGA-matrix, we can often decide what output should be controlled with what input. We should choose a pairing that gives the diagonal elements close to 1 and avoid pairings that give negative diagonal elements.

**b.**

$$\text{RGA}(G(0)) = G(0) .* G^{-T}(0) = \begin{pmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{pmatrix}$$

**c.** Since we should avoid negative diagonal elements and keep the diagonal elements close to 1, we should choose the pairing  $y_1 \leftrightarrow u_2$  and  $y_2 \leftrightarrow u_1$ .

**7.2** We have

$$P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.01 & 0.1 \\ 0.1 & 1 & 0 \end{pmatrix}$$

and

$$P(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -0.1 & 0 & 1 \\ 0.01 & 10 & -0.1 \end{pmatrix}$$

$$\text{RGA}(P(0)) = P(0) .* (P(0)^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The RGA suggests that we should control output 1 with input 1, output 2 with input 3, and output 3 with input 2.

**7.3 a.** We see from the flow equation

$$A_i \frac{d\Delta h_i}{dt} = -a_i \sqrt{\frac{g}{2h_i^0}} \Delta h_i + \Delta q_{in} \quad (7.1)$$

that the outflow from tank  $i$  is

$$q_{out} = a_i \sqrt{\frac{g}{2h_i^0}} \Delta h_i.$$

The inflows into the tanks are found as the sum of the outflow from the tank above and the flow from the pumps into the respective tanks. Writing down equation (7.1) for each of the four tanks now gives the dynamics.

Substituting the time constants  $T_i$  into the dynamics, and arranging them into matrix form then gives the state space form.

**b.** The transfer matrix is given by

$$\begin{aligned}
 P(s) &= C(sI - A)^{-1}B = \\
 &= \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} \begin{pmatrix} s + \frac{1}{T_1} & 0 & -\frac{A_3}{A_1 T_3} & 0 \\ 0 & s + \frac{1}{T_2} & 0 & -\frac{A_4}{A_2 T_4} \\ 0 & 0 & s + \frac{1}{T_3} & 0 \\ 0 & 0 & 0 & s + \frac{1}{T_4} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1 - \gamma_2) k_2}{A_3} \\ \frac{(1 - \gamma_1) k_1}{A_4} & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{\gamma_1 c_1}{1 + s T_1} & \frac{k_2}{k_1} \cdot \frac{(1 - \gamma_2) c_1}{(1 + s T_1)(1 + s T_3)} \\ \frac{k_1}{k_2} \cdot \frac{(1 - \gamma_1) c_2}{(1 + s T_2)(1 + s T_4)} & \frac{\gamma_2 c_2}{1 + s T_2} \end{pmatrix}
 \end{aligned}$$

**c.** The zeros are given by the equation

$$T_3 T_4 s^2 + (T_3 + T_4) s + 1 - \frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2} = 0$$

The two first coefficients are always positive, since  $T_3, T_4 > 0$ . The last coefficient is positive (and both zeros are thus stable) iff

$$\frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2} < 1 \quad \Leftrightarrow \quad \gamma_1 + \gamma_2 > 1$$

In the case  $\gamma_1 = \gamma_2 = 0.7$  we get a minimum-phase system which should be easier to control than the non-minimum-phase system we get in the case  $\gamma_1 = \gamma_2 = 0.3$ .

**d.** We have

$$P(0) = \begin{pmatrix} \gamma_1 c_1 & \frac{k_2}{k_1} (1 - \gamma_2) c_1 \\ \frac{k_1}{k_2} (1 - \gamma_1) c_2 & \gamma_2 c_2 \end{pmatrix}$$

and

$$P(0)^{-1} = \frac{1}{c_1 c_2 (\gamma_1 + \gamma_2 - 1)} \begin{pmatrix} \gamma_2 c_2 & -\frac{k_2}{k_1} (1 - \gamma_2) c_1 \\ -\frac{k_1}{k_2} (1 - \gamma_1) c_2 & \gamma_1 c_1 \end{pmatrix}$$

$$\begin{aligned} \text{RGA}(P(0)) &= P(0) .* (P(0)^{-1})^T = \\ &= \frac{1}{c_1 c_2 (\gamma_1 + \gamma_2 - 1)} \begin{pmatrix} \gamma_1 c_1 \gamma_2 c_2 & -(1 - \gamma_2) c_1 (1 - \gamma_1) c_2 \\ -(1 - \gamma_2) c_1 (1 - \gamma_1) c_2 & \gamma_2 c_2 \gamma_1 c_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - 1} & 1 - \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - 1} \\ 1 - \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - 1} & \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - 1} \end{pmatrix} = \begin{pmatrix} \lambda & 1 - \lambda \\ 1 - \lambda & \lambda \end{pmatrix} \end{aligned}$$

In the case  $\gamma_1 = \gamma_2 = 0.7$  we get

$$\text{RGA}(P(0)) = \begin{pmatrix} 1.225 & -0.225 \\ -0.225 & 1.225 \end{pmatrix}$$

The RGA suggests we should control output 1 with input 1 and output 2 with input 2.

In the case  $\gamma_1 = \gamma_2 = 0.3$  we get

$$\text{RGA}(P(0)) = \begin{pmatrix} -0.225 & 1.225 \\ 1.225 & -0.225 \end{pmatrix}$$

The RGA suggests that in this case we should control output 1 with input 2 and output 2 with input 1.

**7.4 a.** We compute the RGA for stationarity, i.e.  $s = 0$ .

$$\text{RGA}(G(s)) = \begin{pmatrix} \frac{s-1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+1} & \frac{s-1}{s+1} \end{pmatrix}$$

gives

$$\text{RGA}(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

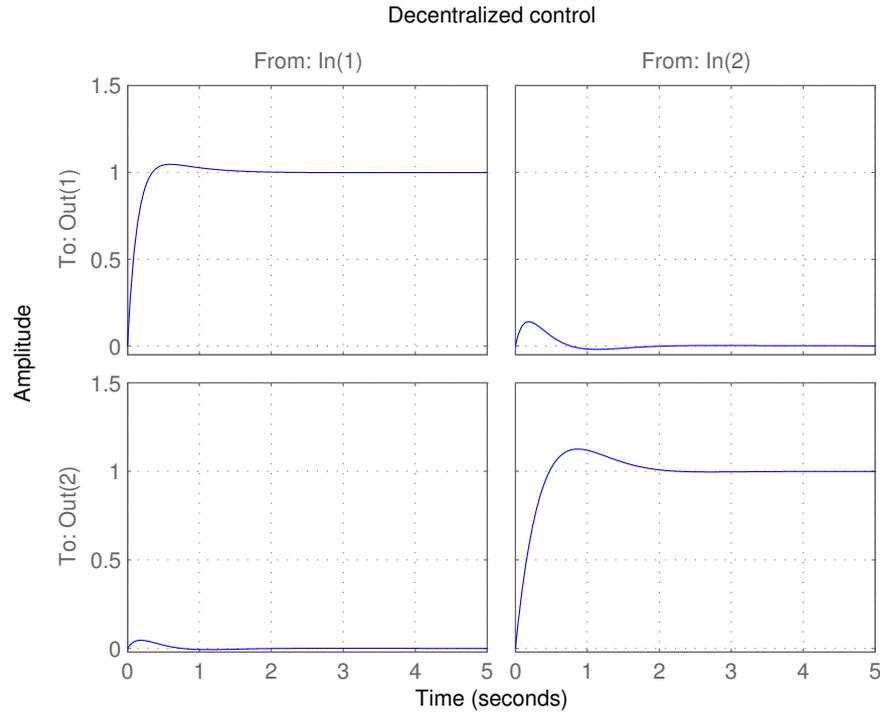
Since you should avoid pairing that gives negative diagonal elements we choose  $y_1 \leftrightarrow u_2$  and  $y_2 \leftrightarrow u_1$ .

**b.** We have that

$$G(0) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

Using a decoupled controller structure with  $W_1 = G^{-1}(0)$  and  $W_2 = I$  we get a decoupled system in stationarity. (See Glad&Ljung ch. 8.3.) The controller is

$$F(s) = W_1 F^{\text{diag}}(s) W_2 = \begin{pmatrix} -F_{11}(s) & 2F_{22}(s) \\ -F_{11}(s) & F_{22}(s) \end{pmatrix}.$$



**Figure 7.1** Decentralized control

- 7.5** 1. Decentralized control. First we calculate the RGA of the process,

$$\text{RGA}(G(0)) = G(0) \cdot * G^{-T}(0) = \begin{pmatrix} 1.2308 & -0.2308 \\ -0.2308 & 1.2308 \end{pmatrix}.$$

We see that we should choose  $y_1 \leftrightarrow u_1$  and  $y_2 \leftrightarrow u_2$ . A reasonable tuning, either by pole placement or hand tuning, gives PI-controllers with parameters close to

$$F(s) = \begin{pmatrix} 2(1 + \frac{1}{0.5s}) & 0 \\ 0 & 2(1 + \frac{1}{0.5s}) \end{pmatrix}.$$

See figure 7.1 for step responses.

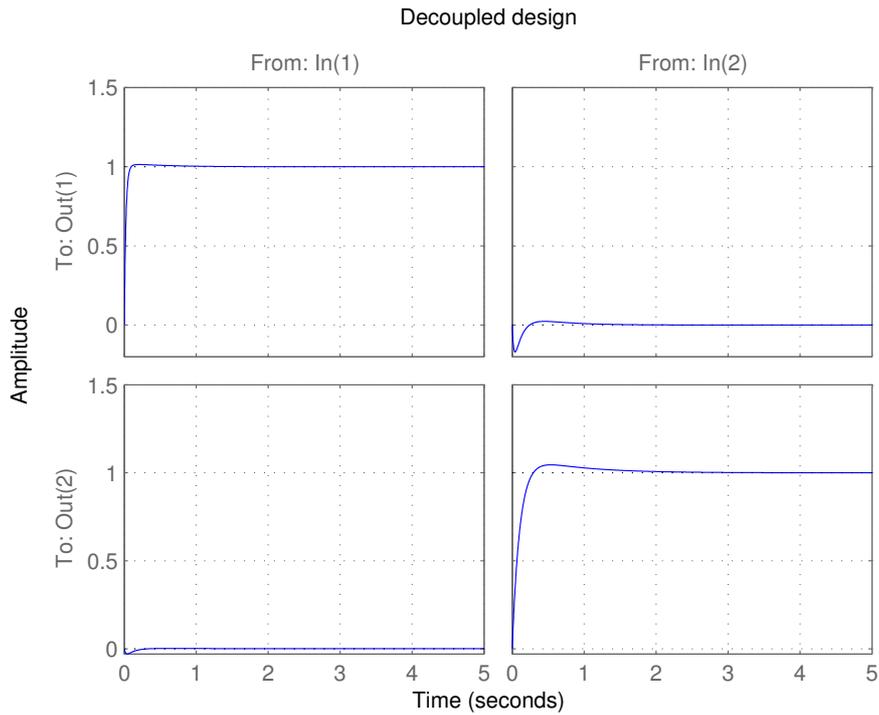
2. Decoupled control. The inverse of the static gain matrix is given by

$$G^{-1}(0) = \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}^{-1}$$

Thus, for decoupling, we use  $W_1 = G^{-1}(0)$  and  $W_2 = I$ . Hand-tuning of the PI-controllers gives

$$F(s) = \begin{pmatrix} 40(1 + \frac{1}{0.5s}) & 0 \\ 0 & 20(1 + \frac{1}{0.8s}) \end{pmatrix}.$$

See figure 7.2 for step responses.



**Figure 7.2** Decoupled control

```

close all
clear all
s = tf('s');

G = [4/(s+1) 3/(3*s+1); 1/(3*s+1) 2/(s+0.5)];

%Decentralized control
RGA = dcgain(G).*(inv(dcgain(G)))'
F = [2*(1+1/(0.5*s)) 0;0 2*(1+1/(0.5*s))];
figure(1)
step(feedback(G*F, eye(2)),5)
title('Decentralized control');grid

% Decoupled design
Go= dcgain(G)
F = [40*(1+1/(0.5*s)) 0;0 20*(1+1/(0.8*s))];
figure(2);
step(feedback(G*inv(Go)*F, eye(2)),5);
title('Decoupled design');grid

```

## Solutions to Exercise 8. Linear Quadratic Optimal Control

### 8.1

- a. The Riccati equation becomes ( $A = a$ ,  $B = 1$ ,  $M = 1$ ,  $Q_1 = 1$ ,  $Q_2 = R$ )

$$2Sa + 1 - SR^{-1}S = 0$$

This gives

$$S = aR + \sqrt{(aR)^2 + R}$$

( $S = aR - \sqrt{(aR)^2 + R}$  is not a solution since  $S$  has to be positive definite.)

Thus the optimal control is given by

$$L = \frac{S}{R} = a + \sqrt{a^2 + \frac{1}{R}}.$$

The closed loop system is hence, using  $u(t) = -Lx(t) + L_r r(t)$

$$\begin{aligned}\dot{x}(t) &= -\sqrt{a^2 + \frac{1}{R}}x(t) + L_r r(t) \\ y(t) &= x(t)\end{aligned}$$

$L_r$  has to be chosen so that we get a stationary gain of 1 from the reference to the output, i.e.  $G_{r \rightarrow y}(0) = C(-A + BL)^{-1}BL_r + D = 1$ .

We get  $L_r = (L - a) = \sqrt{a^2 + \frac{1}{R}}$ .

- b. See Matlab code below and Figure 8.1. Conclusion: Less weight on  $u$  gives a faster system since we are allowed to move the control signal more, and vice versa.

```
clear all
close all

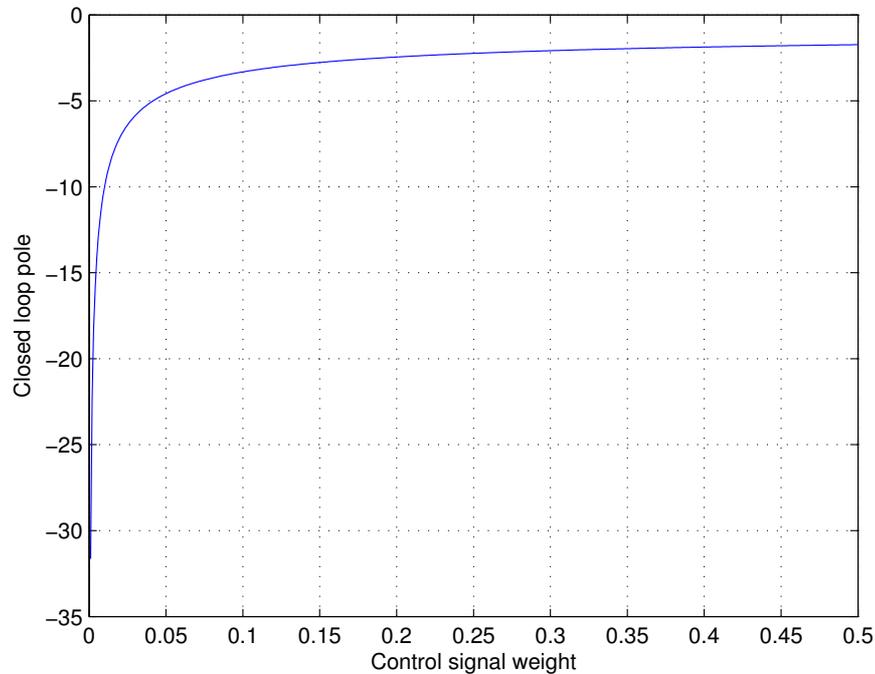
A= 1;
B = 1;
C = 1;

P = ss(A, B, C, 0);

Q = 1;
Rvec=0.001:0.001:0.5;

for i=1:length(Rvec)
    R = Rvec(i);
    [L, S, E] = lqr(P,Q, R);
    Evec(i) = E;
end
```

```
plot(Rvec, Evec)
xlabel('Control signal weight')
ylabel('Closed loop pole')
grid
```



**Figure 8.1** Control signal weight versus closed loop pole

**8.2** See Figure 8.2 and Matlab code below

```
clear all
close all

A = [1 0;1 0];
B = [1 0]';
C = [1 1];

%using lqry
sys = ss(A, B, C, 0);
[L2, S, E]=lqry(sys, R,R)
eig(A-B*L2)

%simulate the system with initial conditions
sys = ss(A-B*L2, B, C, 0);
x0 = [1 1];
initial(sys,x0);grid

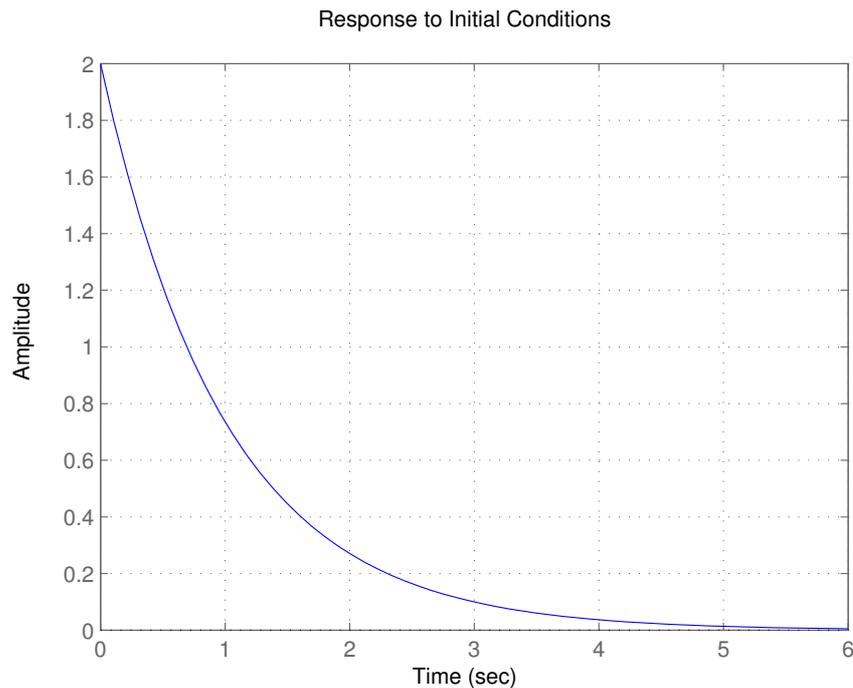
%Solving the Riccati equation
```

```

Q = C'*C;
R = 1;
S = zeros(2,1);
E = eye(2);

[X, K, G]=care(A, B, Q, R, S, E);
L1 = R\B'*X
eig(A-B*L1)

```



**Figure 8.2** Response to initial conditions

**8.3** The loop gain is

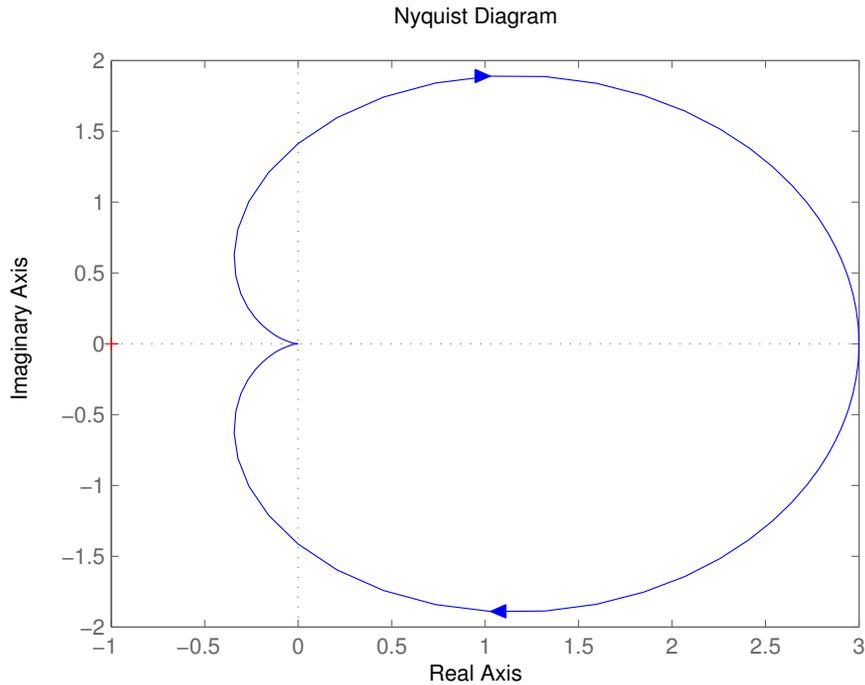
$$L(sI - A)^{-1}B = \frac{6}{(s+1)(s+2)}$$

Thus, as seen in figure 8.3, the Nyquist curve will approach the origin with a phase of  $-180^\circ$ . LQ-optimal loop gain always has an asymptotic phase of  $-90^\circ$ . Therefore, it can not be an LQ-optimal state feed back vector.

**8.4** The system has two unstable poles in 2 and 3. If the cost function should be less than  $\infty$  then the system must be stabilizable, i.e. all unstable poles must be controllable (due to  $Q_1 > 0$ ). The controllability matrix is given by

$$W_c = (B \quad AB) = \begin{pmatrix} -4 & -12 \\ 8 & 24 \end{pmatrix}$$

which is a rank 1 matrix. Thus, only one of the modes is controllable meaning that there is an uncontrollable, unstable mode, and hence, we can not make the cost function less than  $\infty$ .



**Figure 8.3** Nyquist plot.

**8.5**

- a. The cost function is  $J = \int_0^{\infty} x^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t) + u^T(t) R u(t) dt$ ,  $R = 0.01, 10, 1000$ .
- b. See Figure 8.4 for step responses, and Matlab code below.

```
clear all
close all
```

```
A=[1 3; 4 8]; B=[1; 0.1]; M=[0 1];
P=ss(A,B,M,0);
```

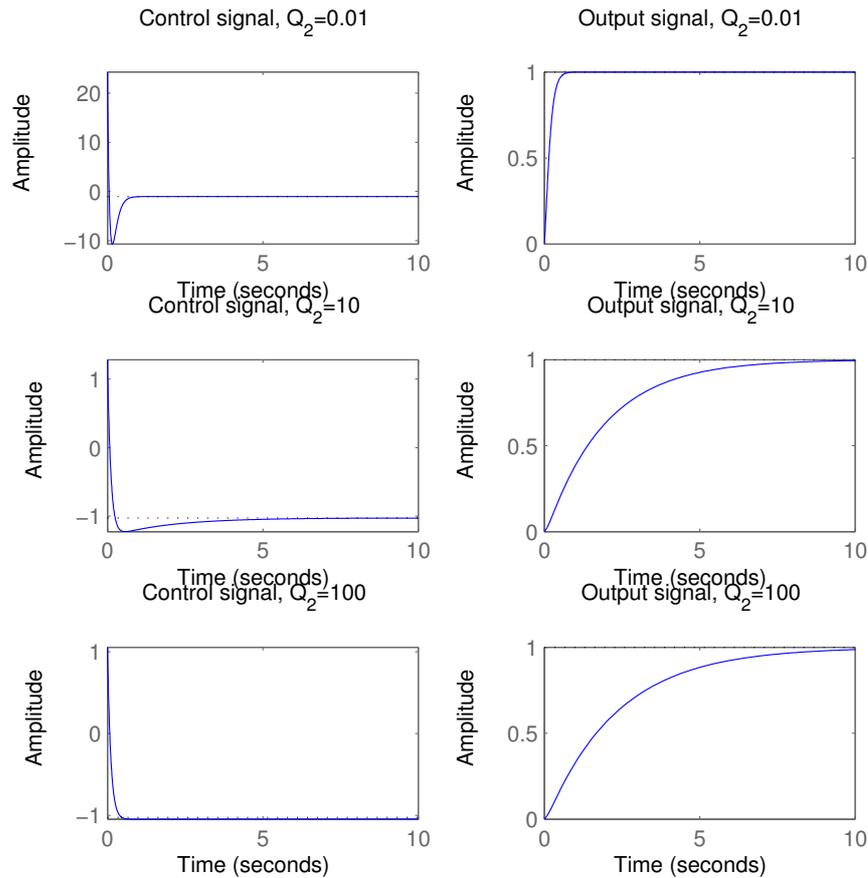
```
Q1=[1 0; 0 2]; Q2_vector=[0.01 10 100];
```

```
for i=1:length(Q2_vector)
    [L,S,E]=lqr(P,Q1,Q2_vector(i));
```

```
    % Calculating Lr (static gain to output must be 1)
    Lr=1/(M/(B*L-A)*B);
```

```
    % Calculating the control signal:
    to_control_signal=Lr-L*ss(A-B*L,B*Lr,eye(2),0);
```

```
    % Calculating the output signal:
    to_output_signal=ss(A-B*L,B*Lr,M,0);
```



**Figure 8.4** Step responses for different weight on control signal.

```

% Plotting step responses
figure(11)
subplot(3,2,i*2-1)
step(to_control_signal)
axis([0 10 -Inf Inf])
title(['Control signal, Q_2=' num2str(Q2_vector(i))])
subplot(3,2,i*2)
step(to_output_signal)
axis([0 10 -Inf Inf])
title(['Output signal, Q_2=' num2str(Q2_vector(i))])
poles{i}=E;

end
poles{:}

```

## 8.6

a. Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

and solve the Riccati equation

$$Q_1 + A^T S + SA - SBQ_2^{-1}B^T S = 0.$$

This gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{0.1} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0,$$

with the solution

$$\begin{aligned} s_1 &= \sqrt{2} \cdot 10^{-1/4}, \\ s_2 &= 10^{-1/2}, \\ s_3 &= \sqrt{2} \cdot 10^{-3/4}. \end{aligned}$$

The optimal controller is given by

$$L = Q_2^{-1}B^T S = (\sqrt{10} \quad \sqrt{2} \cdot 10^{1/4}).$$

To get  $y = r$  in stationarity:

$$1 = G(0) = M(-A + BL)^{-1}BL_r \Rightarrow L_r = \sqrt{10}.$$

**b.** Both  $x_1$  and  $x_2$  must be measured, e.g.

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**c.** 3. is the only case with a cost on the velocity  $x_2$ . This makes the controller try to avoid rapid variations in  $x_1$ , so we get 3. –  $D$ ), the only step response without overshoot. The weight,  $Q_2$ , on the control signal determines the speed of the system. A low weight on the control signal gives a faster system since we are allowed to use more control signal. This gives 1. –  $C$ ), 2. –  $A$ ), 4. –  $B$ ).

## 8.7

**a.** Weighting matrices  $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  och  $Q_2 = \eta$ . The Riccati equation to be solved with respect to  $S$  is

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

which gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

We see, by insertion, that

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

**b.** The optimal state feedback is

$$\begin{aligned} L &= Q_2^{-1} B^T S = \frac{1}{\eta} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot (\eta^{1/2} \quad \sqrt{2}\eta^{3/4}) = (\eta^{-1/2} \quad \sqrt{2} \cdot \eta^{-1/4}) \end{aligned}$$

The poles are the eigenvalues to  $A - BL$ . Put  $\mu = \eta^{-1/4} \Rightarrow L = (\mu^2 \quad \sqrt{2} \cdot \mu)$ . This gives

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

that is

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If  $\eta$  is reduced, the distance between the poles and the origin will increase. This means that  $u(t)$  will increase. Check the criterion!

## Solutions to Exercise 9. Kalman Filtering/LQG

- 9.1 a.** We have that  $A = B = C = N = M = 1$ . The Riccati equation thus reduces to

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

which has the positive semi-definite solution  $P = R_2 + R_2\sqrt{1 + \frac{R_1}{R_2}}$ . Thus, the Kalman filter gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1 + \beta}.$$

- b.** The Kalman filter dynamics are given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where  $y(t) = Cx(t) + v_2(t)$ . Using the values  $A = B = C = N = M = 1$  we have the error dynamics

$$\dot{e}(t) = (A - KC)e(t) - Kv_2(t) + v_1(t) = -\sqrt{1 + \beta}e(t) - (1 + \sqrt{1 + \beta})v_2(t) + v_1(t)$$

- c.** The position of the Kalman filter pole is  $-\sqrt{1 + \beta}$ . We can see that if  $\beta \rightarrow \infty$ , the pole of the Kalman filter  $\rightarrow -\infty$ . Hence, the estimation error dynamics are fast, we believe very much in our measurements. On the other hand, if  $\beta \rightarrow 0$ , the Kalman filter pole tends to -1, that is, as fast as the process pole. Now, we trust the model more than the measurements.

- 9.2** See Matlab code below.

```
>> clear all
>> close all

>> A = [0 1; 1 0];
>> C = [1 0];
>> N = [1 1]';

a. >> %Using care
>> Q = N*N';
>> R = 1;
>> S = zeros(2,1);
>> E = eye(2);

>> [X, K, G]=care(A', C', Q, R, S, E);

>> K1 = X*C'

K1 =
```

2.4142

2.4142

The poles of the Kalman filter should be in the LHP.

```
>> eig(A-K1*C)
```

ans =

-1.4142

-1.0000

**b.** >> %Using lqe

```
>> [K2, P, E]=lqe(A,N, C, 1,1,0)
```

K2 =

2.4142

2.4142

P =

2.4142 2.4142

2.4142 2.4142

E =

-1.4142

-1.0000

The poles of the Kalman filter should be in the LHP.

```
>> eig(A-K2*C)
```

ans =

-1.4142

-1.0000

**9.3 a.** The noise model has the following state space realization,

$$\dot{x}_w(t) = -\delta x_w(t) + n(t)$$

$$w(t) = x_w(t)$$

Extending the state space model of the process with the noise model gives,

$$\dot{x}_e(t) = \begin{pmatrix} -1 & 0 \\ 0 & -\delta \end{pmatrix} x_e(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_{1e}(t)$$

$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x_e(t) + v_2(t)$$

$$z(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x_e(t)$$

Note that  $z(t)$  contains the noise state  $x_w(t)$ , so that, if we design a LQG controller we will also try to minimize the disturbance state effect.

**b.**

$$\begin{aligned}\hat{x}(t) &= A_e \hat{x}(t) + B_e u(t) + K(t)(y(t) - C \hat{x}(t)) \\ K(t) &= P(t) C_e^T R_2^{-1} \\ \dot{P}(t) &= A_e P(t) + P A_e^T - K(t) R_2 K^T(t) + N R_1 N \\ \phi_{v_i} &= 1 \\ u(t) &= -L \hat{x}(t) \\ L &= Q_2^{-1} B_e^T S \\ 0 &= A_e^T S + S A + M^T Q_1 M - S B_e Q_2^{-1} B_e^T S\end{aligned}$$

We are looking for the stationary Kalman filter and therefore solve for  $\dot{P}(t) = 0$  as before.  $R_i$  are noise intensities and  $Q_i$  are the weighting matrices for the LQ-problem.

See Matlab code in **(d)**. Why do we need a small weight on  $u(t)$ ? Since integral action requires the magnitude of the control signal to be large at low frequencies, we have to let the control signal to be large. If not, the low frequency gain will be limited independent of noise model.

- c.** If we change the cut-off frequency  $\delta$  of the noise filter  $H(s)$ , we change the cut-off frequency of the low frequency gain of the controller, this is shown in Figure 9.1.

If we on the other hand change the noise intensity, we indirectly change the gain of the noise filter. Hence, we will increase the controller gain for all frequencies, see Figure 9.2.

- d.** Response to constant load disturbances will *always* have a static error since we do not have infinite gain at low frequencies. That is, we do not have pure integral action, only approximative.

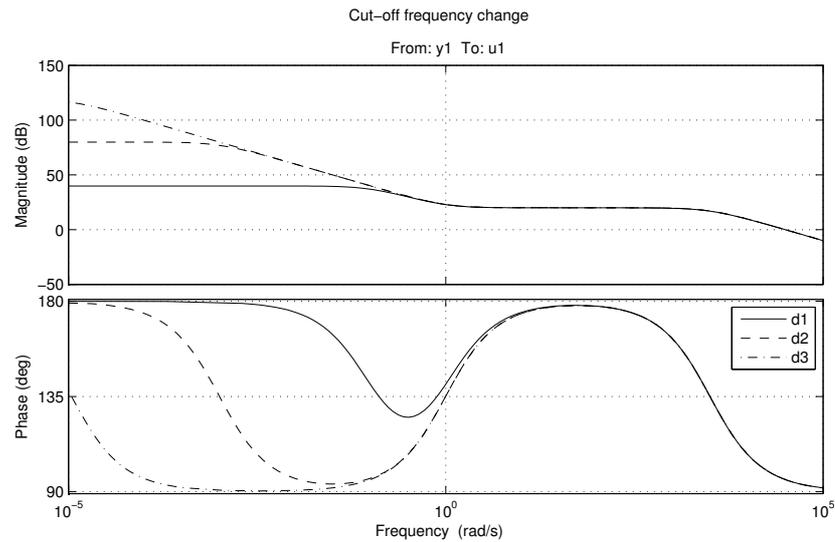
See below for Matlab code,

```
clear all
close all

B = [1; 0];
C = [1 1];
D = 0;
N = eye(2);
H = [0 0];

d1 = 1e-1;
d2 = 1e-3;
d3 = 1e-5;

R1 = [1 0; 0 1];
R2 = [1 0; 0 10];
R3 = [1 0; 0 100];
```



**Figure 9.1** Change of cut-off frequency

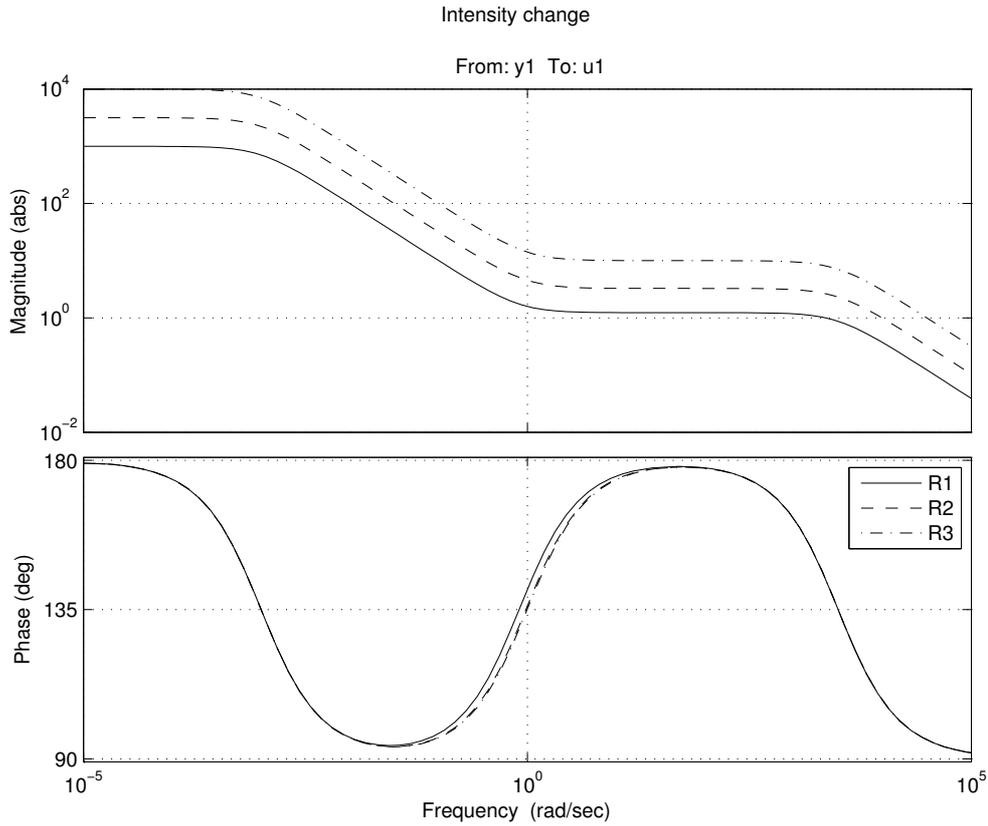
%Different values of cut-off frequency of noise filter

%Note that the values in the diagonal of the disturbance  
%filter input intensity matrix [1 0; 0 100] are arbitrary;  
%their relation will later be varied

```
A = [-1 0;0 -d1];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R3, 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
reg1 = lqgreg(Kest,K);
```

```
A = [-1 0;0 -d2];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R3, 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
reg2 = lqgreg(Kest,K);
```

```
A = [-1 0;0 -d3];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R3, 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
reg3 = lqgreg(Kest,K);
figure(1)
bode(reg1, reg2,'--', reg3,'-.'');grid
legend('d1', 'd2', 'd3')
```



**Figure 9.2** Change of noise intensity

```
title('Cut-off frequency change')
```

```
%Different values of the disturbance filter input intensity
```

```
A = [-1 0;0 -d2];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R1, 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
reg1 = lqgreg(Kest,K);
```

```
A = [-1 0;0 -0.001];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R2, 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
reg2 = lqgreg(Kest,K);
```

```
A = [-1 0;0 -0.001];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, R3, 1);
P = ss(A, B, C, D);
```

```
[K,S,E] = lqry(P,1,0.0000001,0);
reg3 = lqgreg(Kest,K);

figure(2)
bode(reg1, reg2,'--', reg3,'-.');grid
legend('R1', 'R2', 'R3')
title('Intensity change')
```

- 9.4 a.** To get a small complementary sensitivity at the oscillation frequency, we need the LQG controller to have a low gain at this frequency; effectively ignoring corresponding oscillations in the output  $y$ . This can be achieved by modelling the influence of the oscillatory system as a disturbance  $w$  on  $y$  according to

$$\begin{aligned}\dot{x} &= Ax + Bu + Nv_1 \\ y &= Cx + w + v_2\end{aligned}$$

To model the oscillatory characteristics of  $w$ , we can consider  $w$  to be generated by passing white noise  $n$  through a second-order filter with a resonance peak at  $\omega_0 = 0.5$  rad/s and a zero at  $s = 0$ , with transfer function

$$H(s) = \frac{K_v s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

The zero at  $s = 0$  is placed there to avoid an increased gain at low frequencies, which would otherwise follow. It is not necessary unless it is important to avoid this phenomenon and the exercise can be solved without it, which will then yield a slightly different solution to the one below.

The parameter  $\zeta$  determines the magnitude of the resonance peak, and we can choose e.g.  $\zeta = 0.02$ .

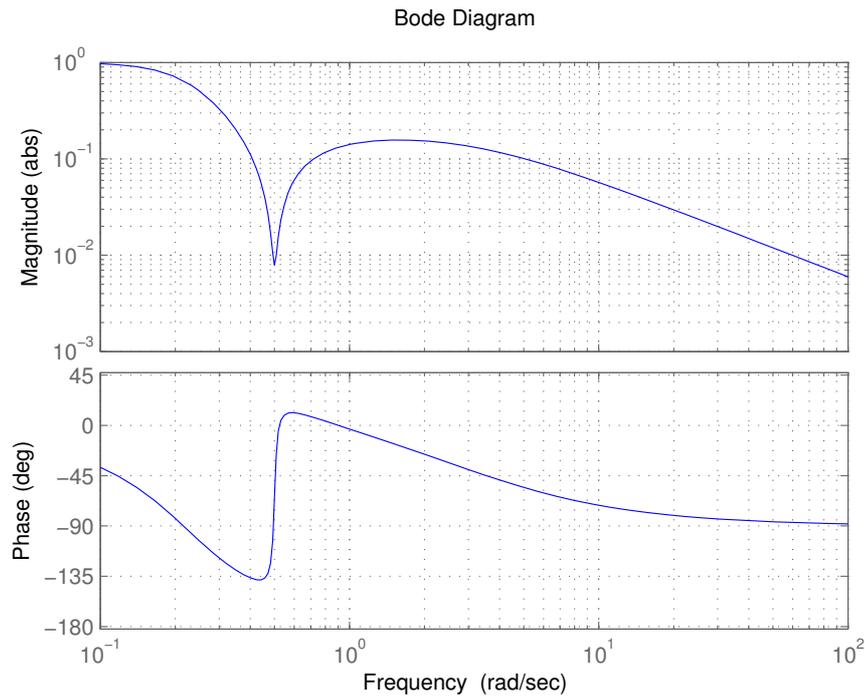
On state-space form, the filter is given by

$$\begin{aligned}\dot{x}_v(t) &= \begin{pmatrix} -0.02 & -0.25 \\ 1 & 0 \end{pmatrix} x_v(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} n(t) \\ w(t) &= (K_v \ 0) x_v(t)\end{aligned}$$

Extend the original state space form with the noise model

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.02 & -0.25 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix} \\ y(t) &= (1 \ 0 \ K_v \ 0) x(t) + v_2(t) \\ z(t) &= (1 \ 0 \ 0 \ 0) x(t)\end{aligned}$$

If this model is used to compute  $K$  in the Kalman filter, for an appropriate value of  $K_v$ , we get suppression of the resonance frequency. The intensity of the added noise input can e.g. be set to 1 since we can control the amplitude



**Figure 9.3** Attenuation of oscillative disturbance

of the disturbance by changing  $K_v$ . Thus, we have the intensity matrices  $R_1 = \text{diag}(1, 1)$ ,  $R_2 = 0.1$ .

Note that  $z(t)$  do not depend on the  $x_v$ -states, i.e., if we are about to design a LQG controller, we have no weight on the added noise. The added noise is only used for specifying at what frequencies our measurements are uncertain.

- b.** See figure 9.3 for the Bode plot of the transfer function from measurement  $y(t)$  to estimated output  $\hat{y}(t)$  using  $K_v = 1$ . We see a large attenuation of frequencies at  $\omega = 0.5$  rad/s.

Matlab code

```
clear all
close all

A = [0 1 0 0; 0 -1 0 0; 0 0 -0.02 -0.25; 0 0 1 0];
B = [0 1 0 0]';
C = [1 0 1 0];
N = [0.1 0; 0.1 0; 0 1; 0 0];

[K, P, E] = lqe(A,N,C,blkdiag(1,1),0.1);

Cnom = [1 0 0 0];
tf(ss(A-K*C, K, Cnom, 0))
bode(ss(A-K*C, K, Cnom, 0),{0.1, 100})
grid
```

**9.5 a.** We have the state-space representation

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1(t) \\ y(t) &= (1 \ 0) x(t) + v_2(t)\end{aligned}$$

(If a different state-space representation is chosen, the solution will look different although the steps will be similar.)

The Riccati-equation

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$

is solved by letting  $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$ . The equations become,

$$\begin{aligned}2p_2 - p_1^2 &= 0 \\ p_3 - p_1p_2 &= 0 \\ 1 - p_2^2 &= 0\end{aligned}$$

The solution is thus

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

with the optimal gain

$$K = PC^T = (\sqrt{2} \ 1)^T$$

**b.** The poles of the Kalman filter are the eigenvalues of  $A - KC$ ,

$$A - KC = \begin{pmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues  $\lambda_j = \frac{1}{\sqrt{2}}(-1 \pm i)$ .

# Solutions to Exercise 10. LQG and Preparations for Laboratory Exercise 3

## 10.1

- a.**
- ```

clear all
close all
clc
A = [0 1;0 0];
B = [1 6;0 4];
C = [1 1];
D = zeros(1,2);

%State feedback design
process = ss(A, B, C, D);
Q1 = 1;
Q2 = eye(2);
[L, S, E] = lqry(process,Q1,Q2);

%Kalman filter design
G = eye(2);
H =zeros(1,2);
syskalman = ss(A, [B G], C, [D H]);
R1 = eye(2);
R2 = 1;
[Kest, K, E] = kalman(syskalman,R1,R2);

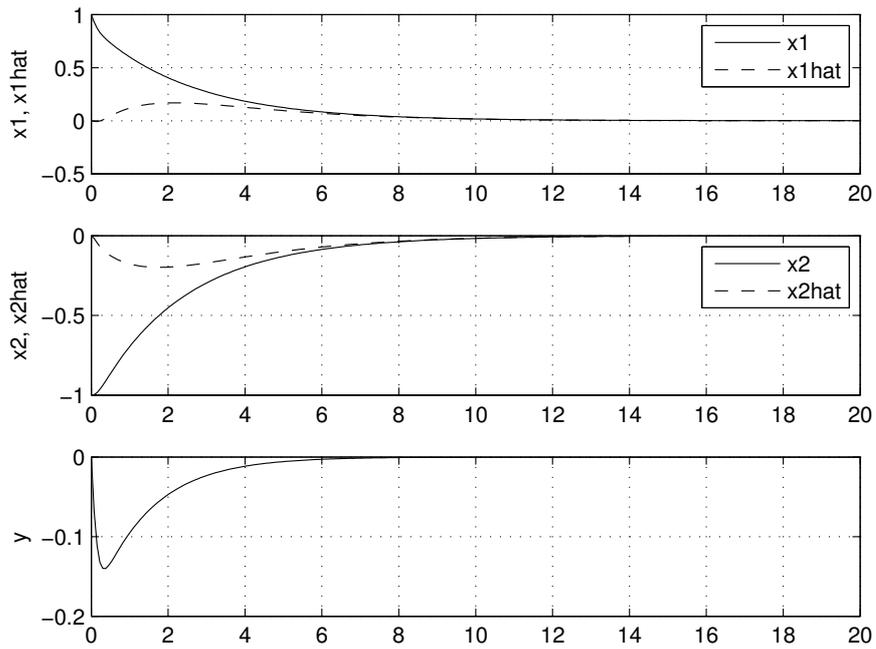
%Construct controller
reg = lqgreg(Kest, L);

```
- b.** Using the state vector  $x_e = (x^T \hat{x}^T)^T$  and the obvious notation  $A, B, C$ , we get the system

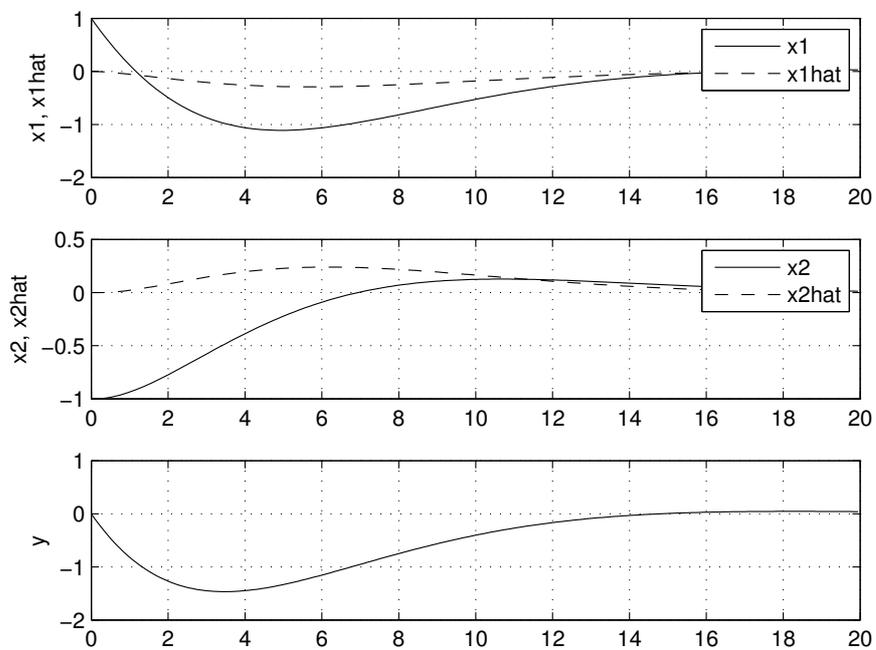
$$\begin{aligned} \dot{x}_e &= \begin{pmatrix} A & -BL \\ KC & A - BL - KC \end{pmatrix} x_e + \begin{pmatrix} I \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ K \end{pmatrix} v_2 \\ z &= (C \ 0) x_e \end{aligned}$$

- c.** With less measurement noise the estimated states converge faster to the actual states, and the output  $z$  converge faster to zero. See Figures 10.1-10.2 and Matlab code.

As shown in exercise 9.1, only the relation between process noise and measurement noise matters. More process noise will therefore have the same effect as less measurement noise.



**Figure 10.1** Initial response if little measurement noise ( $\frac{R_2}{R_1} = 0.1$ ).



**Figure 10.2** Initial response if much measurement noise ( $\frac{R_2}{R_1} = 100$ ).

```

%Construct closed loop
closed_loop = feedback(process, -reg);

%Plot response
[Y, T, X] = initial(closed_loop, [1 -1 0 0], 0:0.01:20);
figure(1)
subplot(311)
plot(T, X(:,1));hold on;plot(T, X(:,3),'--');grid
legend('x1', 'x1hat');ylabel('x1, x1hat')
subplot(312)
plot(T, X(:,2));hold on;plot(T, X(:,4),'--');grid
legend('x2', 'x2hat');ylabel('x2, x2hat')
subplot(313)
plot(T,Y); grid; ylabel('y');

```

**10.2** First of all, we see that we can not measure the states we want to control, so we need a Kalman filter. We start by setting up the problem in the standard form

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1 \\ y &= (1 \ 0) x + v_2 \\ z &= x \end{aligned}$$

where  $v_2$  is white noise with intensity 1. The cost function is

$$J = \int_0^{\infty} (z^T Q_1 z + u Q_2 u) dt$$

with  $Q_1 = I_2$  and  $Q_2 = 1$ .

For the state feedback gain, we have to solve the Riccati equation

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

This gives the following equations,

$$\begin{aligned} 1 - s_2^2 &= 0 \\ s_1 - s_2 s_3 &= 0 \\ 2s_2 + 1 - s_3^2 &= 0 \end{aligned}$$

with the solution  $s_1 = s_3 = \sqrt{3}$ ,  $s_2 = 1$ . This gives the state feedback vector  $L = B^T S = (1 \ \sqrt{3})$ .

For the Kalman filter we must solve the Riccati equation

$$AP + PA^T + R_1 - PC^T CP^T = 0$$

with  $R_1 = I_2$ , which gives

$$\begin{aligned} 2p_2 + 1 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

Using the solution for  $S$  we have that  $p_1 = p_3 = \sqrt{3}$  and  $p_2 = 1$  and  $K = PC^T = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

The controller is given by

$$\begin{aligned}\hat{x} &= (A - BL - KC)\hat{x} + Ky \\ u &= -L\hat{x}\end{aligned}$$

and we have that

$$A - BL - KC = \begin{pmatrix} -\sqrt{3} & 1 \\ -2 & -\sqrt{3} \end{pmatrix}$$

**10.3** No solution provided.

# Solutions to Exercise 11. Youla Parametrization and Dead Time Compensation

11.1 a. We can divide  $P$  even further into smaller parts such that

$$P_{zw} = \begin{pmatrix} P_{xd} & P_{xn} \\ P_{vd} & P_{vn} \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_{xu} \\ P_{vu} \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_{yd} & P_{yn} \end{pmatrix}$$

Looking at the block diagram of the closed-loop system, we see that

$$\begin{aligned} P_{xd} &= P_0, & P_{xn} &= 0, & P_{vd} &= 1, & P_{vn} &= 0 \\ P_{xu} &= P_0, & P_{vu} &= 1 \\ P_{yd} &= P_0, & P_{yn} &= 1 \\ P_{yu} &= P_0 \end{aligned}$$

Note that you have to determine the open-loop transfer functions, as if  $C = 0$ . The results gives us the following transfer matrix  $P$ :

$$P = \begin{pmatrix} P_0 & 0 & P_0 \\ 1 & 0 & 1 \\ P_0 & 1 & P_0 \end{pmatrix},$$

where

$$P_{zw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_0 \\ 1 \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_0 & 1 \end{pmatrix}, \quad P_{yu} = P_0$$

b.

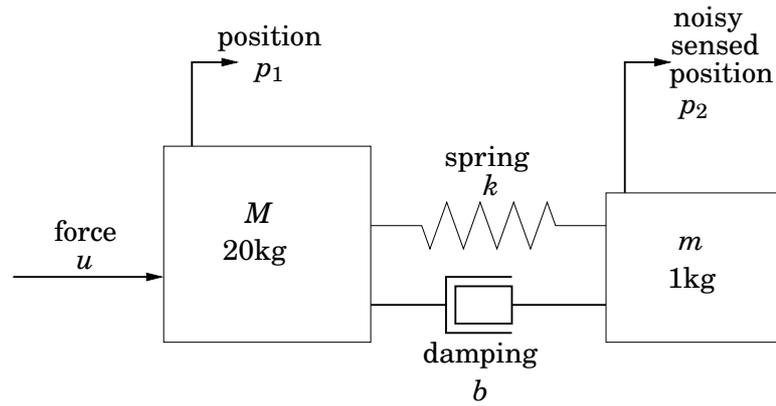
$$u = Cy$$

$$y = P_{yu}u + P_{yw}w = P_{yu}Cy + P_{yw}w \Rightarrow y = \frac{1}{1 - P_{yu}C} P_{yw}w$$

$$z = P_{zw}w + P_{zu}u = P_{zw}w + P_{zu}Cy = (P_{zw} + P_{zu}C \frac{1}{1 - P_{yu}C} P_{yw})w$$

c. Using the formula, we get

$$\begin{aligned} H &= P_{zw} + P_{zu}C(1 - P_{yu}C)^{-1}P_{yw} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} P_0 \\ 1 \end{pmatrix} C(1 - P_0C)^{-1} \begin{pmatrix} P_0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{C}{1 - P_0C} \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} \\ &= \frac{1}{1 - P_0C} \begin{pmatrix} (P_0 - P_0^2C) + P_0^2C & P_0C \\ (1 - P_0C) + P_0C & C \end{pmatrix} = \begin{pmatrix} P_0S & -T \\ S & CS \end{pmatrix} \end{aligned}$$



**Figure 11.1** Mass spring system in Exercise 11.2.

This means that the closed loop transfer function  $H$  consists of the gang of four. Note that

$$T = 1 - S = -\frac{P_0 C}{1 - P_0 C}$$

in this case where we have no explicit minus sign in the feedback loop.

- d.** Go back to the formula  $H = P_{zw} + P_{zu} C(1 - P_{yu} C)^{-1} P_{yw}$ , but replace  $C(1 - P_{yu} C)^{-1}$  with  $Q$ . This gives

$$H = P_{zw} + P_{zu} Q P_{yw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + Q \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} = \begin{pmatrix} P_0 + P_0^2 Q & P_0 Q \\ 1 + P_0 Q & Q \end{pmatrix}$$

where each element of  $H$  is linear in  $Q$ .

- e.** Rearranging the  $Q$  formula a bit, we get

$$Q = C(1 - P_0 C)^{-1} \Rightarrow Q - Q P_0 C = C \Rightarrow (1 + Q P_0) C = Q, \Rightarrow C = (1 + Q P_0)^{-1} Q$$

where the last equality gives us the same controller as (8.18) in Glad&Ljung with the only difference that the feedback sign is a part of the controller here.  $P_0$  is assumed to be the model we have of the system.

**11.2 a.** From the equation for the plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ p_1 &= C_1 x \\ p_2 &= C_2 x \end{aligned}$$

and the block diagram of the closed-loop system, we can see that

$$\begin{aligned}
 \dot{x} &= Ax + B(u + d) = Ax + \begin{pmatrix} 0 & 0 & B \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + Bu \\
 &= Ax + B_w w + Bu \\
 z &= \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix} = \begin{pmatrix} C_1 x \\ u + d \\ r - p_1 \end{pmatrix} = \begin{pmatrix} C_1 x \\ u + d \\ r - C_1 x \end{pmatrix} \\
 &= \begin{pmatrix} C_1 \\ 0 \\ -C_1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\
 &= C_z x + D_{zw} w + D_{zu} u \\
 y &= \begin{pmatrix} r \\ p_2 + n \end{pmatrix} = \begin{pmatrix} r \\ C_2 x + n \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ C_2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \\
 &= C_y x + D_{yw} w + D_{yu} u
 \end{aligned}$$

- b.** The constraint on the maximum control signal,  $\|u(t)\| \leq u_{max}$ , will correspond to the closed loop transfer matrix  $H_{u_o r}$ , with index (2, 1). In problem 11.1 we saw that the transfer function  $H_{u_o d}$  will correspond to the sensitivity function  $S$ . The  $M_s$  constraint will therefore correspond to the index (2, 3). The objective function will be related to two indices, namely those associated with  $H_{er}$  and  $H_{u_o r}$ , (3, 1) and (2, 1).
- c.** We have the formula  $H = P_{zw} + P_{zu} Q P_{yw}$ . Since  $P_{zu}$  is a  $3 \times 1$  system and  $P_{yw}$  is a  $2 \times 3$  system,  $Q$  must be  $1 \times 2$ . Therefore we have that  $Q = [Q_1 \quad Q_2]$ .

**11.3** The system is non-minimum phase. There are many ways to choose the  $Q$  filter for IMC, but we have to respect some fundamental limitations. Here we will use a simple choice of  $Q$ . We try to cancel the process dynamics with  $Q(s)$ , but use the stable counterpart  $6 + 3s$  of the zero instead. We also need to add a pole to  $Q(s)$  to make it proper, which we place in  $s = \lambda^{-1}$ . We get

$$Q(s) = \frac{s^2 + 5s + 6}{(6 + 3s)(\lambda s + 1)} = \frac{s + 3}{3(\lambda s + 1)}$$

The controller becomes

$$C(s) = \frac{s^2 + 5s + 6}{s(3\lambda s + 6(\lambda + 1))}$$

which can be rewritten as

$$C(s) = \frac{5}{6(1 + \lambda)} \left( 1 + \frac{6}{5s} + \frac{s}{5} \right) \frac{1}{\frac{3\lambda}{6(\lambda+1)}s + 1}$$

This corresponds to a PID controller in series with a lowpass filter.

**11.4** As before, there are many ways to apply IMC. Here we try the two approaches to deal with time delays described in Glad&Ljung section 8.3.

1. Choose to ignore the time delay when the  $Q(s)$  transfer function is calculated, but not when  $F_y(s)$  is calculated.

Thus, we get

$$Q(s) = \frac{(P(s)e^{4s})^{-1}}{\lambda s + 1}$$

Hence, the controller is given by

$$F_y(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s + 1}{\lambda s + 1 - e^{-4s}}$$

2. Approximate the time delay with a first order Padé approximation,

$$G(s) \approx \frac{1}{s + 1} \frac{1 - 2s}{1 + 2s}.$$

When we calculate the  $Q(s)$ -transfer function, we exclude  $1 - 2s$ . Thus, we now have

$$Q(s) = \frac{(s + 1)(2s + 1)}{(\lambda s + 1)^2}.$$

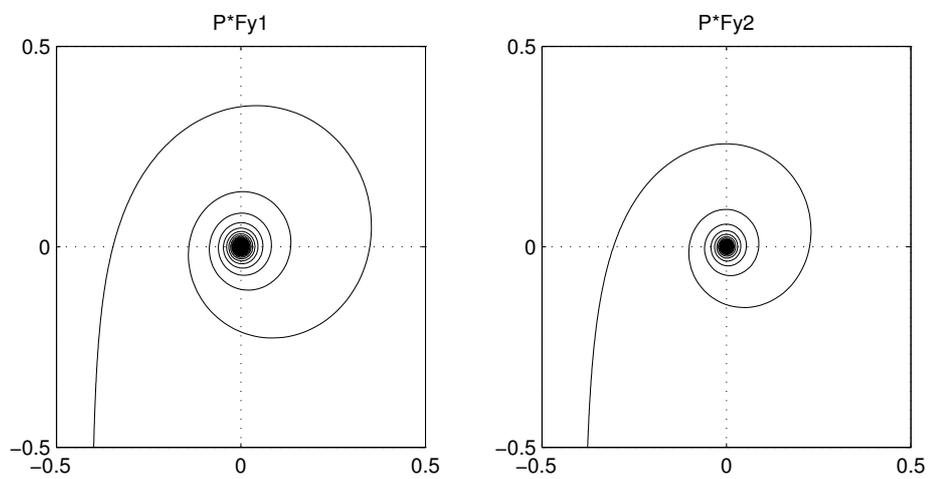
Hence we have the controller

$$F_y(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{(s + 1)(2s + 1)}{(\lambda s + 1)^2 - (1 + 2s)e^{-4s}}$$

The Nyquist plots can be generated in Matlab, using the following lines of code. In this case, lambda is chosen to 3.

```
>> lambda = 3;
>> w = logspace(-2,2,1000);
>> P = 1./(1+i*w).*exp(-4*i*w);
>> Fy1 = (i*w+1)./(lambda*i*w+1-exp(-4*i*w));
>> Fy2 = (i*w+1).*(1+2i*w)./((lambda*i*w+1).*(lambda*i*w+1)-(1+2*i*w).*exp(-4*i*w));
>> figure
>> plot(P.*Fy1)
>> grid
>> figure
>> plot(P.*Fy2)
>> grid
```

From the plots (Figure 11.2) we see that neither encircles  $-1$  and the closed loop systems are stable in both cases.



**Figure 11.2** Nyquist plots of the loop transfer functions in Problem 11.4. The left plot shows the first alternative and the right plot shows the second.

# Solutions to Exercise 12. Synthesis by Convex Optimization

## 12.1

- a. The inputs to the controller are  $r$  and  $y_0$ , i.e.

$$y = \begin{pmatrix} r \\ y_0 \end{pmatrix}.$$

The input to  $P$  is  $\begin{pmatrix} w \\ u \end{pmatrix}$ , which contains 4 signals, and the output is  $\begin{pmatrix} z \\ y \end{pmatrix}$ , which contains 4 signals as well. Thus  $P$  must be  $4 \times 4$ .

- b. We know that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} e \\ u \\ r \\ y_0 \end{pmatrix}, \quad \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} d \\ n \\ r \\ u \end{pmatrix}.$$

The block diagram gives that

$$\begin{aligned} e &= r - x = r - P_0(d + u), \\ u &= u, \\ r &= r, \\ y_0 &= n + P_0(d + u). \end{aligned}$$

Arranging this into matrix form gives the answer:

$$P = \begin{pmatrix} -P_0 & 0 & 1 & -P_0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ P_0 & 1 & 0 & P_0 \end{pmatrix}.$$

- c. The control objective a) is convex in  $H$ , and  $H$  is a linear function of  $Q$ , so the control objective a) is convex in  $Q$ . Since it is satisfied for  $Q_1$  and  $Q_2$ , it is thus satisfied for any convex combination

$$Q = wQ_1 + (1 - w)Q_2, \quad w \in [0, 1].$$

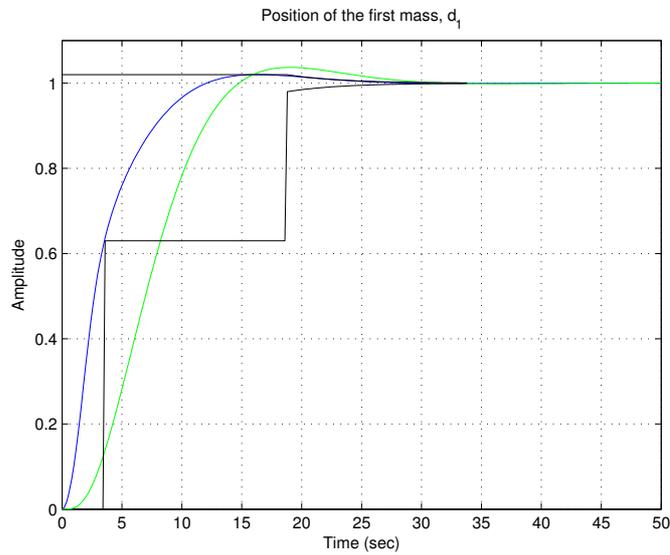
We see from the impulse responses that neither  $Q_1$  nor  $Q_2$  satisfies b) or c). However, a convex combination of  $Q_1$  and  $Q_2$  will give the same convex combination of the disturbance responses. Taking e.g.  $w = 0.7$ ,

- The control signal satisfies  $|u(t)| \leq 0.7 \cdot 0.4 + 0.3 \cdot 2 = 0.88$ , since  $|u(t)| \leq 0.4$  with  $C_1$  and  $|u(t)| \leq 2$  with  $C_2$ .
- When  $t \geq 2$ , the control error satisfies  $|e(t)| \leq 0.7 \cdot 1 + 0.3 \cdot 0.1 = 0.73$ , since  $|e(t)| \leq 1$  with  $C_1$  and  $|e(t)| \leq 0.1$  with  $C_2$ .

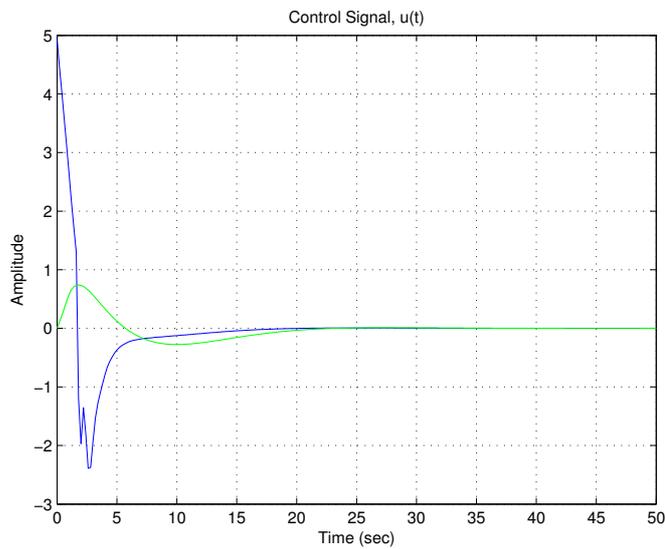
Thus we can use  $Q = 0.7Q_1 + 0.3Q_2$ .

12.2

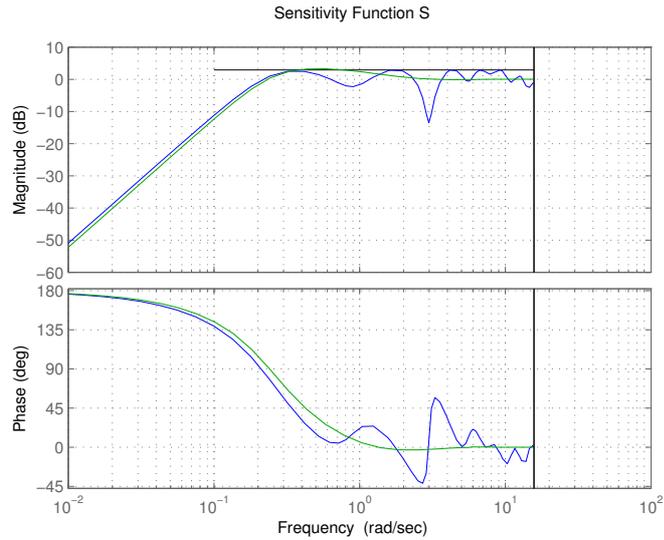
- a. The green curves in the plots correspond to the nominal controller. See the plots in Figures 12.1-12.3 and read the figure texts. From the plots we can also see that the only inactive constraint is the one posed on the control signal. Also see figure 12.4 for a plot of the open loop Nyquist curve together with the  $M_s$ -circle.



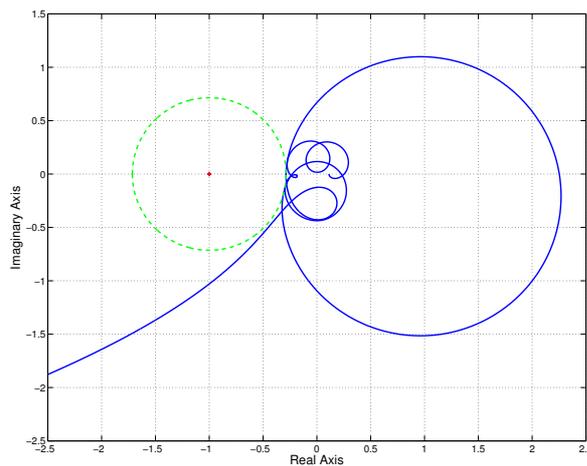
**Figure 12.1** Step responses from reference  $r$  to mass position  $d_1$ . The response violating the constraints we have posed is the one given with the nominal control and the other one comes from using our optimized controller.



**Figure 12.2** The nominal controller gives the control signal with the least energy of the two.



**Figure 12.3** The sensitivity function of the system. The plot that violates the constraint correspond to the nominal control.



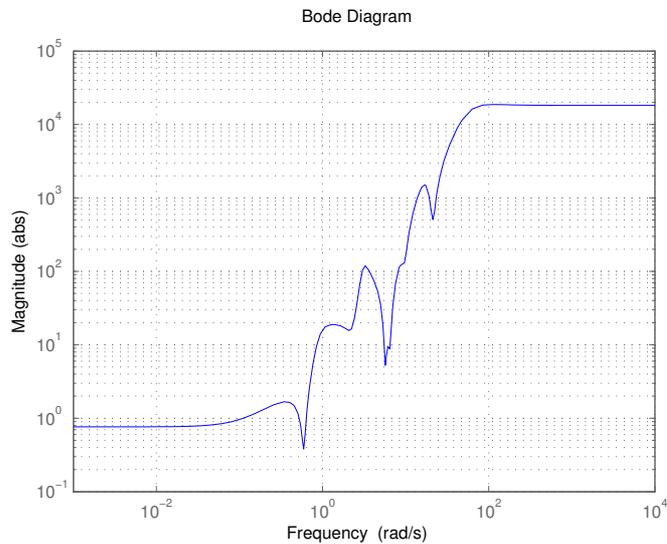
**Figure 12.4** The open loop Nyquist curve and the  $M_s$ -circle when the optimal controller is used.

**b.** `bodemag(K2c)`

The bode diagram is displayed in Figure 12.5. The shape of the magnitude plot is very similar to that of a PD controller with a first order filter. Since the D-part of a PID controller acts to damp out oscillations, it seems rather logic that we have this kind of similarity.

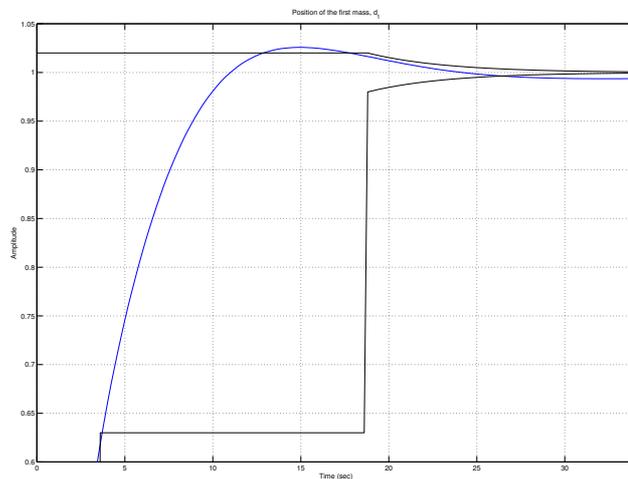
The system has its resonance frequency at 5.8 rad/s, which is almost exactly at the same frequency as the deepest dip in the controller magnitude plot. The logic behind the dip is that we do not want to augment signals

at this frequency. A PD controller does not have this kind of flexibility in its structure to damp out a certain frequency and is therefore not so well suited for highly oscillative systems like this one.



**Figure 12.5** Bode diagram of the controller.

- c. The least value on  $NQ$ , for which our problem is feasible, is 7. The constraints that fail are those posed on  $d_1$  during a step change in the reference. See Figure 12.6.

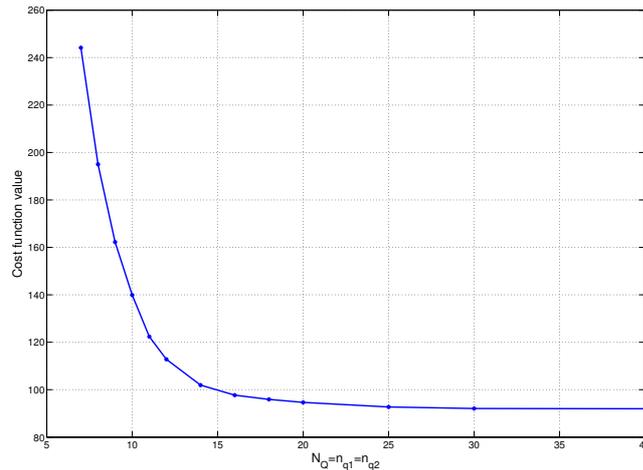


**Figure 12.6** The problem is infeasible with  $NQ=6$ .

- d. See Figure 12.7 for a plot of the cost function value versus the order of the  $Q$ -filters. When  $NQ$  reaches around 20, the control will gain very little from increased complexity of the  $Q$ -filter. We can then say that we have a good

estimate of the *Limit of Performance*, i.e. lowest cost that linear controller can achieve given the problem setup.

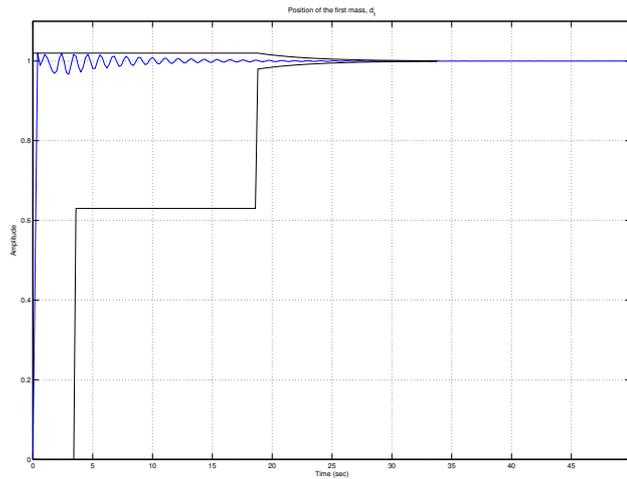
The maximum value of the control signal will decrease as the order of the  $Q$ -filters goes up. For  $N_Q = [7, 10, 15, 20]$ , we get  $u_{max} = [6.0, 4.9, 3.8, 3.7]$ . This means that the more complex the controller becomes, the more freedom it will have to choose its control signal. As it is good to have a control signal that is low on energy (due to the cost function), it is also likely that it goes down in magnitude if it has the possibility.



**Figure 12.7** Cost function value plotted against the complexity of the  $Q$ -filters.

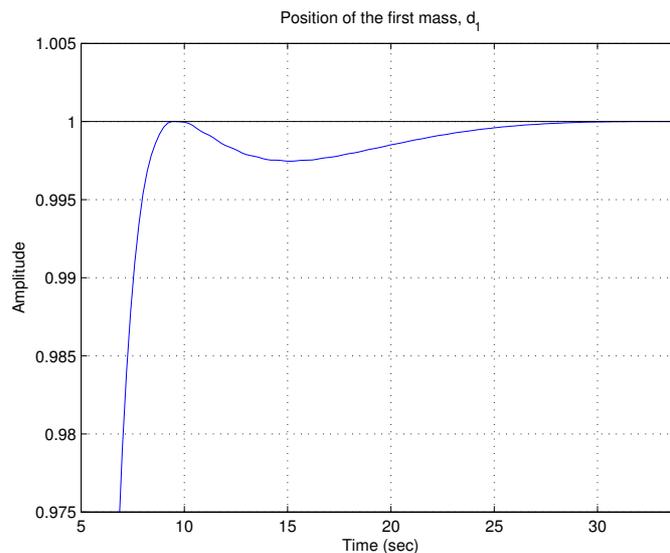
- e. If we start out by setting  $\rho = 0$  (i.e.  $\text{weight}_u = 0.0$ ), then we do not punish the control signal energy at all, which means that we may get very aggressive and poorly damped control. The constraint on  $u_{max}$  will to some extent prevent this, but if  $u_{max}$  is made arbitrarily large then we can get a step responses like the one in Figure 12.8. If instead  $\gamma = 0$  while  $\rho$  remains 1, then the solution will remain fairly unchanged. The reason is that both the constraints on rise time and the cost of having a large  $e$  will force  $u$  to be quite active still. If the step response constraints are made inactive, the solution will be fairly close to the one of the nominal LQG controller.
- f. The solution will still be feasible when the overshoot is constrained to 0.4% ( $\text{os}=1.004$ ). In order to make the least possible overshoot even smaller we can for instance set:
- $u_{max} = 7$ , which will give the control signal a bigger span to work with
  - $M_s = 1.6$ , gives the controller a higher gain (see Figure 12.4 and imagine how the open loop system can change if the  $M_s$ -circle is made smaller), making it be able take assume greater values, within the boundaries set by  $u_{max}$ .
  - $N_Q = 30$ , which will give the controller more freedom to shape the control signal optimally.

All these 3 changes will be able to handle an overshoot of 0.2% respectively.  $N_Q = 30$  will also be able to shape the step response such that there is



**Figure 12.8**  $d_1$  due to a unit step in the reference when  $\rho = 0$ . The control is very aggressive.

no overshoot (see Figure 12.9). Note that  $t_2$  had to be set to 34 (maximum time for which we have constraints) in order for the lower step constraints to not make the problem too difficult. Try with a lower value of  $t_2$  to see the effect.



**Figure 12.9**  $d_1$  due to a unit step in the reference when no overshoot is allowed.

**g.** In `plot_result.m` we find the following line of code

```
c1 = prob.T11 + prob.T12*[prob.Q{1} prob.Q{2}]*prob.T21;
```

which is used before the plots are created. This line corresponds to the formula stated in the previous exercise session, namely  $H = P_{zw} + P_{zu}QP_{yw}$ ,

where  $P$  includes the nominal controller. Copying this line of code into the Matlab prompt, followed by `bodemag(c1)` will give us the 9 desired plots. Looking at the middle one, corresponding to  $H_{u_0n}$ , we can see that it does not have high frequency roll-off. A high frequency measurement noise,  $n$ , may therefore lead to a very noisy control signal. This shows that it is very important to take all signals in a system into consideration and that a solution, even though optimal, may need to be reconsidered. If we were to modify the problem, a good idea may therefore be to put constraints on this closed loop transfer function as well.

- h.** No solution given.

## Solutions to Exercise 13. Controller simplification

### 13.1

- a. Inspection of the locations of the poles and zeros gives us the transfer function

$$G(s) = 1.04 \frac{s/1.3 + 1}{(s/1.2 + 1)(s^2 + 0.4s + 1.04)}$$

- b. The closeness of the pole-zero pair on the real axis suggests that a model reduction might be possible.
- c. A balanced realization and the Hankel singular values for the system can be calculated using the Matlab command

```
>>> s = tf('s');
>>> G = 1.04*(s/1.3+1)/((s/1.2+1)*(s^2+0.4*s+1.04));
>>> [balr,g] = balreal(G);
```

which gives the following Hankel singular values:

$$g = \begin{pmatrix} 1.5105 \\ 1.0196 \\ 0.0091 \end{pmatrix}$$

Elimination of the state in the balanced realization corresponding to the smallest Hankel singular value is done in Matlab by

```
>>> modsys = modred(balr,g<0.01)
>>> modsysG = tf(modsys)
```

This gives the following transfer function for the reduced order system:

$$G_{red}(s) = 0.0181 \frac{s^2 - 2.412s + 57.49}{s^2 + 0.4086s + 1.043}$$

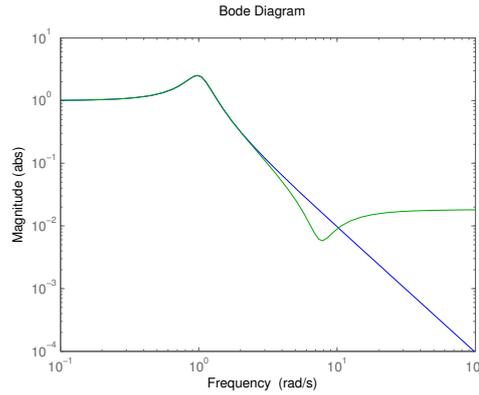
A Bode magnitude plot of the original system and the reduced system is shown in figure 13.1.

- 13.2 a. With

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

we have

$$AS + SA^T + BB^T = \begin{pmatrix} -2 & 0 \\ -2 & -0.5 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



**Figure 13.1** Bode magnitude plot of the original and reduced system in problem 13.1

so  $S$  is the controllability gramian. Similarly, with

$$O = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$OA + A^T O + C^T C = \begin{pmatrix} -0.5 & 0 \\ -1 & -0.5 \end{pmatrix} + \begin{pmatrix} -0.5 & -1 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so  $O$  is the observability gramian.

**b.** The Hankel singular values are the eigenvalues of

$$SO = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so they are both 1.

**c.** The coordinate change  $\xi = Tx$  yields the new gramians  $S_\xi = TST^T$  and  $O_\xi = T^{-T}OT^{-1}$ . To find  $T$  we solve the equation  $S_\xi = O_\xi$ . Since both  $S$  and  $O$  are diagonal it seems reasonable that a diagonal  $T$  will work. With  $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$  we get the equations

$$TST^T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 2t_1^2 & 0 \\ 0 & t_2^2 \end{pmatrix}$$

and

$$T^{-T}OT^{-1} = \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} = \begin{pmatrix} 0.5/t_1^2 & 0 \\ 0 & 1/t_2^2 \end{pmatrix}$$

which gives

$$\begin{aligned} 2t_1^2 &= 0.5/t_1^2 &\Rightarrow t_1^4 &= 1/4 &\Rightarrow t_1 &= 1/\sqrt{2} \\ t_2^2 &= 1/t_2^2 &\Rightarrow t_2^4 &= 1 &\Rightarrow t_2 &= 1 \end{aligned}$$

(You could also use the direct formula for  $T$  in the proof on page 81 in [Glad & Ljung])

With this  $T$

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

the gramians become

$$S_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad O_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, a balanced realization is

$$\begin{aligned} \dot{\xi} &= \hat{A}\xi + \hat{B}u \\ y &= \hat{C}\xi + \hat{D}u \end{aligned}$$

where

$$\begin{aligned} \hat{A} &= TAT^{-1} = \begin{pmatrix} -1 & 0 \\ -\sqrt{2} & -0.5 \end{pmatrix} & \hat{B} &= TB = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\ \hat{C} &= CT^{-1} = (\sqrt{2} \quad 1) & \hat{D} &= D \end{aligned}$$

- d.** In this case, the Hankel singular values have the same size, therefore either could be removed. (However, this means that it is probably not a good idea to do any truncation at all!) If the second state is removed by letting  $\xi_2 = 0$ ,  $\xi_2$  can be expressed in terms of  $\xi_1$  through  $0 = \hat{A}_{21}\xi_1 + \hat{A}_{22}\xi_2 + \hat{B}_2u$ . The reduced realization then becomes

$$\begin{aligned} \dot{\xi}_1 &= (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2)u \\ y_r &= (\hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{D} - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2)u \end{aligned}$$

where for example  $\hat{A}_{21}$  is the element in the second row and first column in  $\hat{A}$ .

$$\begin{aligned} \dot{\xi}_1 &= -\xi_1 + \sqrt{2}u \\ y_r &= -\sqrt{2}\xi_1 + 12u \end{aligned}$$

The transfer function is obtained through the Laplace transform

$$G_1(s) = 12 - \frac{2}{s+1}$$

- 13.3 a.** The Matlab command `tf(ss(A,B,C,D))` gives

$$G(s) = \frac{10s^2 + 18s + 5}{s^2 + 1.5s + 0.5}$$

- b.** Plotting the Bode diagram for  $G(s) - G_1(s)$  through the command `bodemag(G-G1)` gives 2 as the maximal error, obtained at large frequencies. The error bound, twice the sum of the truncated singular values, also gives 2. In this case the error bound is tight.
- c.** Truncating both states gives

$$G_2 = \hat{D} - \hat{C}\hat{A}^{-1}\hat{B} = 10$$

- d. Plotting `bodemag(G-Gr)` gives 2 as the maximal error, near  $\omega = 1$ . The error bound  $2(1 + 1) = 4$  is conservative.

**13.4** Through partial fractions one can write

$$\frac{2s^2 + 2.99s + 1}{s(s + 1)^2} = \frac{1}{s} + \frac{s + 0.99}{(s + 1)^2}$$

The Matlab command

`[G3bal,sig]=balreal(tf([1 .99],[1 2 1]))` gives

$$sig = \begin{pmatrix} 0.4950 \\ 0.00001 \end{pmatrix}$$

so one state can be removed right away.

`G3red=modred(G3bal,(sig<0.1))` yields

$$\frac{-2.525 \cdot 10^{-5}s + 1}{s + 1.01} \approx \frac{1}{s + 1.01}$$

With the integrator we get the reduced system

$$\frac{1}{s} + \frac{1}{s + 1.01} = \frac{2s + 1.01}{s(s + 1.01)}$$

The commands `balreal` and `modred` can actually be used directly on systems with an integrator since they do the separation automatically.

## Solutions to Exercise 14. Old Exam

14.1 Partial fraction expansion gives

$$\begin{aligned} G(s) &= \left[ \frac{1}{(s+1)(s+2)} \quad \frac{s+3}{(s+1)(s^2+6s+8)} \right] \\ &= \frac{1}{s+1} \begin{bmatrix} 1 & \frac{2}{3} \end{bmatrix} + \frac{1}{s+2} \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} + \frac{1}{s+4} \begin{bmatrix} 0 & -\frac{1}{6} \end{bmatrix} \end{aligned}$$

so a realization on diagonal form can be written as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & \frac{2}{3} \\ -1 & -\frac{1}{2} \\ 0 & -\frac{1}{6} \end{bmatrix} u \\ y &= [1 \quad 1 \quad 1] x \end{aligned}$$

14.2 Denote the output by  $z$ . The spectral density of  $z$  is then

$$\begin{aligned} \Phi_z &= |G(i\omega)|^2 \Phi_n(\omega) = \frac{1}{i\omega + a} \frac{1}{-i\omega + a} = \\ &= \frac{1}{a^2 + \omega^2} \end{aligned}$$

14.3

- a. The bad damping in the disturbance response is a symptom of low phase margin, which is approximately  $20^\circ$  at  $\omega_c = 1$  (as seen in the bode diagram). The lead filter improves the phase margin, but the phase peak is located between the zero and pole at

$$\omega_p = \sqrt{1.79 \cdot 8.94} = 4 \text{ rad/s,}$$

which is far from  $\omega_c$ !

- b. One way to improve the control is to move the phase peak to  $\omega_c = 1$  by dividing the pole and zero by 4. The new controller is

$$C'(s) = K \left( 1 + \frac{1}{s} \right) \frac{s/0.45 + 1}{s/2.24 + 1}.$$

The gain  $K$  should be chosen so that the cutoff frequency is preserved, that is  $|C'(i\omega_c)| = |C(i\omega_c)|$ , which gives  $K = 0.45$ .

The new controller gives an increase in phase margin of  $19^\circ$ . The high frequency gain

$$\lim_{s \rightarrow \infty} C(s)$$

actually decreases from 4.39 to 2.24.

## 14.4

- a. Block scheme calculations gives

$$G_{y_2,m} = \frac{B_1(s)(s^2 - \omega_0^2)}{s[A_1(s)(s^2 - \omega_0^2) + B_1(s)\omega_0^2]}$$

Note that this transfer function can be considered as the process in the outer loop.

- b. A process zero at  $z = \omega_0$  imposes a constraint on the achievable bandwidth of the closed loop system - it is not possible to achieve a bandwidth significantly larger than  $\omega_0$ .
- c. Plot D shows too high bandwidth of the closed loop to be feasible. Plot B and C does not fulfill the  $|S + T| = 1$  constraint. Plot A shows a bandwidth of about 1 rad/s which is reasonable - hence plot A.

## 14.5

- a. The determinant of the system is

$$\det(G_2(s)) = \frac{1}{(s+1)(s+2)}$$

such that the least common denominator of all submatrices is  $p(s) = (s+1)(s+2)$ . Thus, the system have poles in  $-1$ ,  $-2$  and no zeros. Since the system has poles in the open left halplane and no non-minimum phase zeros, there are no fundamental limitations on the system bandwidth.

- b.

$$RGA = G_2(0) \cdot (G_2^T(0))^{-1} = \begin{pmatrix} -0.5 & 0 \\ -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the RGA is the identity matrix we can expect the system to be easily controlled with decentralized control at low frequencies. The identity matrix also gives us that it is suitable to pair input 1 with output 1 and input 2 with output 2.

- 14.6 Using the IMC method, we set  $Q(s)$  to

$$Q(s) = \frac{1}{(\lambda s + 1)^{n-m}} G^{-1}(s)$$

giving us the controller

$$\begin{aligned} C(s) &= (1 - Q(s)G(s))^{-1} Q(s) = \left(1 - \frac{1}{(\lambda s + 1)^{n-1}}\right)^{-1} \frac{(s+b)^n}{(s+a)(\lambda s + 1)^{n-1}} \\ &= \frac{(s+b)^n}{(s+a)((\lambda s + 1)^{n-1} - 1)} \end{aligned}$$

To match the structure of the PID controller

$$\frac{K}{sT_i} \frac{(T_i T_d s^2 + T_i s + 1)}{(s \frac{T_d}{N} + 1)}$$

we see that we will need to choose  $n = 2$ . This leaves us with

$$C(s) = \frac{b^2}{s\lambda a} \frac{(\frac{1}{b^2}s^2 + \frac{2}{b}s + 1)}{(\frac{s}{a} + 1)},$$

such that we can now determine the PID parameters one by one

$$T_i = \frac{2}{b}, \quad T_d = \frac{1}{T_i b^2} = \frac{1}{2b}, \quad K = \frac{b^2 T_i}{\lambda a} = \frac{2b}{\lambda a}, \quad N = T_d a = \frac{a}{2b}.$$

Since only  $K$  depends on  $\lambda$ , this is the only PID parameter that we have the possibility to tune ourselves.

**14.7**  $P$  can be said to consist of several submatrices

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix},$$

where

$$P_{zw} = \begin{pmatrix} P_{e_1 r_1} & P_{e_1 r_2} \\ P_{e_2 r_1} & P_{e_2 r_2} \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_{e_1 u_1} & P_{e_1 u_2} \\ P_{e_2 u_1} & P_{e_2 u_2} \end{pmatrix},$$

$$P_{yw} = \begin{pmatrix} P_{y_1 r_1} & P_{y_1 r_2} \\ P_{y_2 r_1} & P_{y_2 r_2} \\ P_{r_1 r_1} & P_{r_1 r_2} \\ P_{r_2 r_1} & P_{r_2 r_2} \end{pmatrix}, \quad P_{yu} = \begin{pmatrix} P_{y_1 u_1} & P_{y_1 u_2} \\ P_{y_2 u_1} & P_{y_2 u_2} \\ P_{r_1 u_1} & P_{r_1 u_2} \\ P_{r_2 u_1} & P_{r_2 u_2} \end{pmatrix}$$

We can now determine all transfer functions that make up  $P$ :

$$P_{zw} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad P_{zu} = \begin{pmatrix} -P_{11}^0 & -P_{12}^0 \\ -P_{21}^0 & -P_{22}^0 \end{pmatrix} = -P^0,$$

$$P_{yw} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad P_{yu} = \begin{pmatrix} P_{11}^0 & P_{12}^0 \\ P_{21}^0 & P_{22}^0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P^0 \\ 0 \end{pmatrix}$$

**14.8**

- a. Zeros of a multivariate process are defined as the points where the transfer matrix loses rank. For a quadratic matrix, losing rank is equivalent to the determinant being zero. The zeros are given by the equation:

$$\det P(s) = \frac{2s + 5}{(s + 4)(s + 2)(s + 1)} = 0$$

Thus  $s = -2.5$

**b.** Here  $RGA(0)$  is calculated.

$$G(0) = \begin{bmatrix} 0.5 & -0.5 \\ 1 & 0.25 \end{bmatrix}$$

Then  $RGA(0)$  becomes

$$RGA(0) = G(0) \cdot (G(0)^{-1})^T = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

and thus output 1 should pair with input 2 and output 2 with input 1.

**14.9** We see that A and C will give the same controller since we have just scaled the weights by 100, so A and C will correspond to Step response 1 and 2. Notice that the system is very oscillative. D has much larger weight on the control signal, thus we will not be able to get a fast system that dampens the oscillative system, hence D must correspond to Step response 3. B will correspond to Step response 4. We have very small weight on the control signal compared to output, which will give a fast system.