

# Multivariable Control

## Laboratory Exercise 2

### The Quadruple Tank<sup>1</sup>

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## 1. Introduction

The aim of this laboratory exercise is to study some different aspects of multivariable control systems. The ideas are illustrated on a quadruple-tank process with two inputs and two outputs. The process can be modified to have different dynamics: In one configuration the process is minimum phase, and in another configuration the process is nonminimum phase. In the lab we will study the pairing of inputs and outputs, interaction between the control loops, decoupling, and fundamental limitations introduced by a nonminimum phase zero.

### Preparations

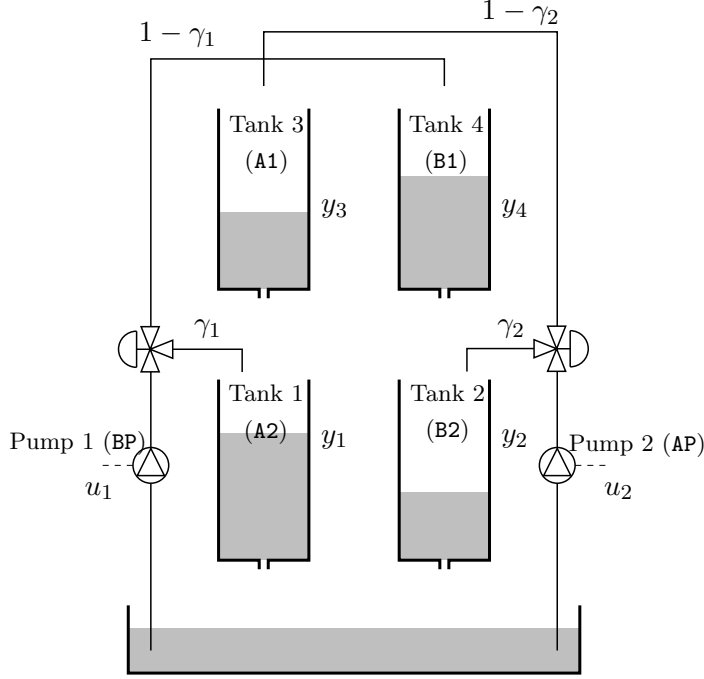
Solve problems 1-3 in exercise session 7. Read this lab manual carefully as well as the lecture notes on fundamental limitations. At the beginning of the lab, you should also be able to discuss the points below:

#### Discussion points

1. Give real-world examples of systems where the step response initially goes in the wrong direction.
2. In which of the two possible process configurations is it easier to control the level of the lower tanks? Can you give an intuitive explanation?
3. Consider the configuration where most of the water enters the upper tanks. Simultaneously increase the flow through pump 1 and decrease the flow through pump 2 by the same amount. What short-term effects do you expect? What long-term effects do you expect?
4. Imagine a configuration where the flow from each pump is divided 50–50, i.e.  $\gamma = 0.5$ . Can you achieve different levels in the lower left and right tanks?
5. We start with non-linear equations but end up with a linear model. What happened in between? Where will the linear model be a good approximation of the non-linear one?

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**Figure 1** The quadruple-tank process.

## 2. The Process

The quadruple-tank process is shown in Figure 1. The goal is to control the water level in the lower tanks using the two pumps.

The control inputs  $u_1$ ,  $u_2$  are the voltages (0–10 V) applied to the two pumps. The control outputs  $y_1$ ,  $y_2$  are voltages (0–10 V) representing the levels in the lower tanks.

### Nonlinear Model

In this section, a nonlinear model of the quadruple-tank process is derived. For each tank  $i = 1 \dots 4$ , mass balance and Bernoulli's law give that

$$A_i \frac{dh_i}{dt} = -a_i \sqrt{2gh_i} + q_{in_i}$$

where  $A_i$  [cm<sup>2</sup>] is the cross-section of the tank,  $h_i$  [cm] is the water level,  $a_i$  [cm<sup>2</sup>] is the cross-section of the outlet hole,  $g$  [cms<sup>-2</sup>] is the acceleration of gravity, and  $q_{in_i}$  [cm<sup>3</sup>s<sup>-1</sup>] is the inflow to the tank.

Each pump  $j = 1, 2$  gives a flow proportional to the control signal,

$$q_{pump_j} = k_j u_j$$

where  $k_j$  [cm<sup>3</sup>V<sup>-1</sup>s<sup>-1</sup>] is a pump constant. The flows from the pumps are divided according to the two parameters  $\gamma_1, \gamma_2 \in [0, 1]$ . The flow to Tank 1 is  $\gamma_1 k_1 u_1$  and the flow to Tank 4 is  $(1 - \gamma_1) k_1 u_1$ . Symmetrically, the flow to Tank 2 is  $\gamma_2 k_2 u_2$  and the flow to Tank 3 is  $(1 - \gamma_2) k_2 u_2$ .

The measured level signals are  $y_1 = k_c h_1$  and  $y_2 = k_c h_2$ , where  $k_c$  [Vcm<sup>-1</sup>] is a measurement constant.

Considering the flow in and out of all tanks simultaneously, the dynamics of the quadruple-tank process are given by

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1 \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2 \\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2 \\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1\end{aligned}$$

**Stationary Points** For a stationary operating point  $(h_1^0, h_2^0, h_3^0, h_4^0, u_1^0, u_2^0)$ , the equations above give that

$$\begin{aligned}\frac{a_3}{A_3}\sqrt{2gh_3^0} &= \frac{(1-\gamma_2)k_2}{A_3}u_2^0 \\ \frac{a_4}{A_4}\sqrt{2gh_4^0} &= \frac{(1-\gamma_1)k_1}{A_4}u_1^0\end{aligned}\tag{1}$$

and thus

$$\begin{aligned}\frac{a_1}{A_1}\sqrt{2gh_1^0} &= \frac{(1-\gamma_2)k_2}{A_3}u_2^0 + \frac{\gamma_1 k_1}{A_1}u_1^0 \\ \frac{a_2}{A_2}\sqrt{2gh_2^0} &= \frac{(1-\gamma_1)k_1}{A_4}u_1^0 + \frac{\gamma_2 k_2}{A_2}u_2^0\end{aligned}\tag{2}$$

If we chose stationary levels in the lower tanks,  $h_1^0, h_2^0$ , we can obtain the stationary control signals  $u_1^0, u_2^0$  by solving the linear system of equations (2). The stationary levels in the upper tanks,  $h_3^0, h_4^0$ , are then obtained from (1).

### Linear Model

Let  $\Delta u_i = u_i - u_i^0$ ,  $\Delta h_i = h_i - h_i^0$ , and  $\Delta y_i = y_i - y_i^0$ . Introducing

$$u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \quad x = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix}, \quad y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

and linearizing around a stationary point gives the linear system

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{pmatrix} x + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} x,\end{aligned}$$

where

$$T_i = \frac{A_i}{a_i} \sqrt{\frac{2h_i^0}{g}}$$

The transfer matrix from  $u$  to  $y$  is given by

$$G(s) = \begin{pmatrix} \frac{\gamma_1 c_1}{1 + sT_1} & \frac{(1 - \gamma_2)c_1}{(1 + sT_1)(1 + sT_3)} \\ \frac{(1 - \gamma_1)c_2}{(1 + sT_2)(1 + sT_4)} & \frac{\gamma_2 c_2}{1 + sT_2} \end{pmatrix}$$

where  $c_1 = T_1 k_1 k_c / A_1$  and  $c_2 = T_2 k_2 k_c / A_2$ .

### Multivariable Zeros

As shown in the preparatory exercises, the zeros of  $G(s)$  are given by the equation

$$(1 + sT_3)(1 + sT_4) - \frac{(1 - \gamma_1)(1 - \gamma_2)}{\gamma_1 \gamma_2} = 0$$

The system is found to be minimum phase (i.e., all zeros are in the left half plane) if  $1 < \gamma_1 + \gamma_2 \leq 2$  and nonminimum phase (i.e., at least one zero is in the right half plane) if  $0 \leq \gamma_1 + \gamma_2 < 1$ .

### Relative Gain Array

The relative gain array (RGA) is a useful tool for determining what outputs should be controlled by what inputs for decentralized controllers.

As shown in the preparatory exercises, the RGA of  $G(0)$  is given by

$$\begin{pmatrix} \lambda & 1 - \lambda \\ 1 - \lambda & \lambda \end{pmatrix}$$

where  $\lambda = \gamma_1 \gamma_2 / (\gamma_1 + \gamma_2 - 1)$ .

### Process Data

The quadruple tank has approximately the following physical constants:

$$\begin{aligned} A_i &= 4.9 \text{ cm}^2 \\ a_i &= 0.03 \text{ cm}^2 \\ k_i &= 1.6 \text{ cm}^3/\text{V} \\ k_c &= 0.5 \text{ V/cm} \\ g &= 981 \text{ cm/s}^2 \end{aligned}$$

The flow from each pump is divided so that roughly 70 % goes into one tank and 30 % into the other. In the minimum phase case, the 70 % goes into the lower tanks and we have  $\gamma_1 = \gamma_2 = 0.7$ . In the nonminimum phase case, the 30 % goes into the lower tanks and we have  $\gamma_1 = \gamma_2 = 0.3$ .

The operating point in the lower tanks are chosen as  $h_1^0 = h_2^0 = 10 \text{ cm}$ . In the minimum phase case the stationary levels in the upper tanks are calculated to be  $h_3^0 = h_4^0 = 0.9 \text{ cm}$  and in the nonminimum phase the stationary levels are  $h_3^0 = h_4^0 = 4.9 \text{ cm}$ . In both cases, the stationary control signals are  $u_1^0 = u_2^0 = 2.6 \text{ V}$ .

### 3. The Lab Interface

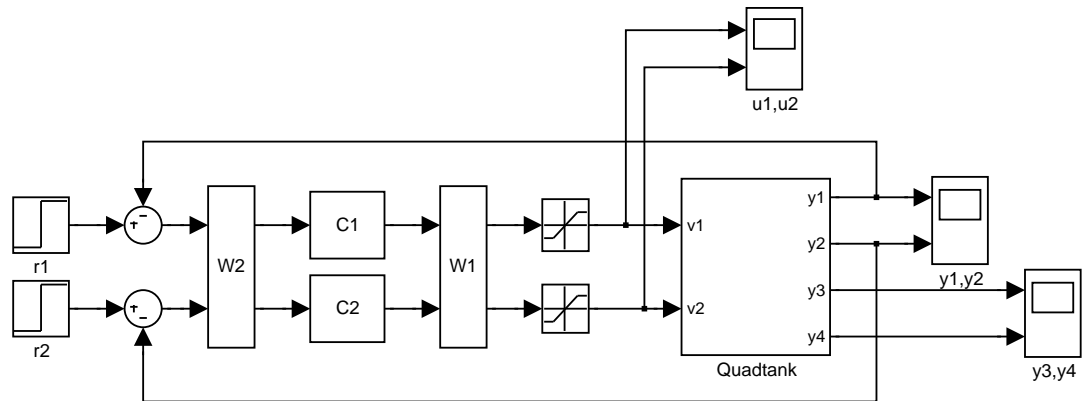
The controllers are designed and tested in MATLAB/Simulink. There is one Simulink model for simulation `quadtank_simulated` and another one for experiments on the real process `quadtank_real`. There is also a model for manually driving the physical process `quadtank_real_manual`.

The Simulink model for the simulated process is shown in Figure 2. The *Quad-tank* block contains the non-linear model of the quadruple tank process. Clicking on the block allows you to specify  $\gamma_1$  and  $\gamma_2$  for the process.

Before a simulation, you must define the two controllers **C1** and **C2** in the workspace. In the lab, the controllers will be of either PI or PID type. The controllers are conveniently designed using the commands `designpi` and `designpid` (use `help` on them).

Run the script `define_process` to define the two systems **Gmin** and **Gnonmin** representing the linear process in the minimum phase and nonminimum phase case respectively.

It is also possible to specify two decoupling matrices **W1** and **W2** (see Section 5.3). The default choice is  $W_1 = W_2 = I$  (the identity matrix), i.e. no decoupling.



**Figure 2** Simulink model *quadtank\_simulated*.

### 4. Control in the Minimum Phase Case

We will start by controlling the process in the minimum phase case, where most water goes directly into the lower tanks. For simulations, make sure to enter the correct  $\gamma$ -values into the mask of the Quadtank block. For experiments on the physical process, make sure that it is configured so that 70 % of the flow from Pump 1 goes into Tank 1, and that 70 % of the flow from Pump 2 goes into Tank 2. This is achieved with the following valve configuration: **AV1**, **AV2** and **BV1** should be pulled out while **BV2** should be pushed in (from the users perspective). Also make sure valves **AV3**, **AV4**, **BV3**, **BV4** and **V5** are all pushed down.

## 4.1 Choice of Control Structure and Design

**Exercise 1** Have a look at the transfer matrix  $G_{min}$  in MATLAB and also compute the zeros of the system:

```
>> Gmin
>> zero(minreal(ss(Gmin)))
```

Also compute the RGA of  $G_{min}(0)$ :

```
>> G0 = evalfr(Gmin,0*i)
>> G0.*transpose(inv(G0))
```

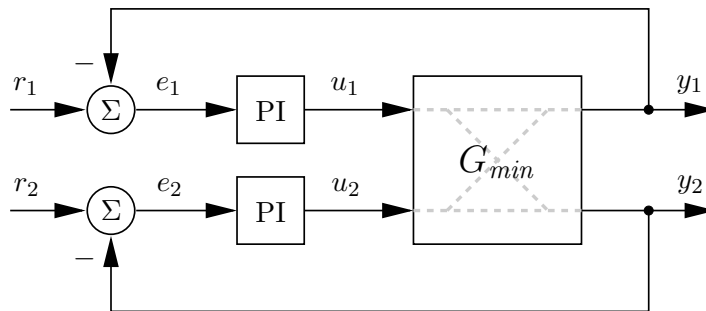
Do the results agree with the preparatory exercises?  $\diamond$

Now compute the RGA matrix for higher frequencies, say  $\omega = 0.1, 0.5, 1, 10, 100$  rad/s, that is, we control the closed loop system faster and faster. How does the RGA matrix change with increasing frequency?

The RGA analysis suggests that we should control Tank 1 with Pump 1 and Tank 2 with Pump 2. If we ignore the cross-terms in the transfer matrix, we see that the problem is reduced to controlling the two single-tank processes

$$G_{min11}(s) = \frac{\gamma_1 c_1}{1 + sT_1}, \quad G_{min22}(s) = \frac{\gamma_2 c_2}{1 + sT_2}$$

From the basic course we know that we can control a single-tank process without stationary error using a PI controller. The resulting control structure is shown in Figure 3.



**Figure 3** The control structure in the minimum phase case.

We will use pole-placement to design the PI controllers in the two loops. The desired characteristic polynomial of each closed loop is specified as

$$s^2 + 2\zeta\omega s + \omega^2$$

## 4.2 Experiments

**Exercise 2** Make sure that the Simulink models agree with the control structure described above. Double-click on the process block in Simulink to set the value of  $\gamma_1$  and  $\gamma_2$ . Use the command `designpi` to design the two PI

controllers **C1** and **C2** for the subsystems  $G_{min11}$  and  $G_{min22}$ . Use  $\zeta = 0.7$  and experiment with different  $\omega$ . Try the controller in simulations, and, when it works well, also on the real process.

Set the step time of step block **R1** to 50 and **R2** to 100 and the simulation stop time (found in the Simulation menu under Configuration Parameters) to 150. If you have difficulties to find a controller which settles within these times, you may increase them. However, have in mind that this implies longer experiment durations.

If you try large values of  $\omega$ , say  $\omega > 0.5$ , you may see large overshoots in the water levels and in some cases even combinations of over- and undershooting. Explain this behaviour! How can you eliminate this undesired phenomena? *Hint:* Study the plot of the control signals.

Can you notice any cross-coupling in the system response? How fast can you make the system? What limits the achievable performance?  $\diamond$

### 4.3 Estimating $\gamma$

The  $\gamma$ -values given in this manual are from nominal construction data and values of individual processes do differ. If you experience differences between simulations and real process experiments, this might be a possible cause. The following steps suggest an experiment for identifying  $\gamma_1$  and  $\gamma_2$ .

1. Push up **AV3**, **AV4**, **BV3**, **BV4** and down **V5** to close all tank outlets. Use either of the two following configurations to choose operating mode (minimum or nonminimum phase).
  - (a) **AV1**, **AV2** and **BV1** pulled out, **BV2**. pushed in
  - (b) **AV1**, **BV1** and **BV2** pulled out, **AV2** pushed in.
2. Open `quadtank_real_manual` and start a simulation
3. Use the slider gains to fill the tanks (how large control signal do you want to use?).
4. Stop the simulation when the tanks that fill fast have reached  $\approx 160$  mm.
5. Measure the water levels  $A1$ ,  $A2$ .  $B1$  and  $B2$ .
6. Use

$$\gamma_1 = \frac{A2}{A2 + B1}, \quad \gamma_2 = \frac{B2}{B2 + A1}$$

After this is done, do the following, in order to use your identified values.

- Edit `define_process` and enter your measured values for `gamma1` and `gamma2` under either `% Minimum phase case` or `% Nonminimum phase case`.
- Enter your values in the mask of the Quadtank block of `quadtank_simulated`
- Rerun `define_process` and the synthesis script for **C1** and **C2**.

## 5. Control in the Nonminimum Phase Case

We will now try to control the process in the nonminimum phase case, where most water goes to the upper tanks. You **must** reconfigure the physical process such that 70 % of the flow from Pump 1 goes into Tank 4, and that 70 % of the flow from Pump 2 goes into Tank 3. This is achieved with the following valve configuration: AV1, BV1 and BV2 should be pulled out while AV2 should be pushed in (from the user's perspective).

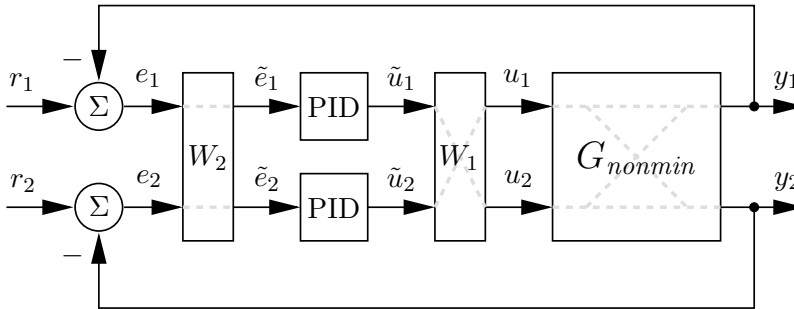
### 5.1 Choice of Control Structure and Design

**Exercise 3** Take a look at the transfer matrix  $G_{nonmin}$  in MATLAB. Compute the zeros of the system. Are they minimum phase? Also compute the RGA in stationarity, i.e.  $G_{nonmin}(0)$ , as well as for some higher frequencies, i.e.  $G_{nonmin}(\omega i)$  for say  $\omega = 0.02, 0.05, 0.1, 0.2, 1, 10$ . Do the results agree with the preparatory exercises? Does the RGA for higher frequencies agree with your intuition?  $\diamond$

The RGA analysis suggests that we should control Tank 1 with Pump 2 and Tank 2 with Pump 1. If we ignore the cross-terms in the transfer matrix, the problem is reduced to controlling the two double-tank processes

$$G_{nonmin12} = \frac{(1 - \gamma_2)c_1}{(1 + sT_1)(1 + sT_3)}, \quad G_{nonmin21} = \frac{(1 - \gamma_1)c_2}{(1 + sT_2)(1 + sT_4)}$$

From the basic course we know that we can control a double-tank process without stationary error using a PID controller. The resulting control structure is shown in Figure 4 with  $W_2 = I$  and  $W_1$  chosen according to your RGA reasoning.



**Figure 4** The control structure in the nonminimum phase case. Note the internal structure of the  $W_1$  and  $W_2$  blocks (dashed).

Again, we will use pole placement to design the PID controllers in the two loops. The desired characteristic polynomial of each closed loop is specified as

$$(s + \alpha\omega)(s^2 + 2\zeta\omega s + \omega^2)$$

### 5.2 Experiments

**Exercise 4** Modify the matrix  $W_1$  according to the RGA reasoning.



Set the step time of step block R1 to 100 and R2 to 200 and the simulation stop time to 300. If you have difficulties to find a controller which settles within these times, you may increase them. However, bear in mind that this implies longer experiment durations. Also, don't forget to change the values of  $\gamma_1$  and  $\gamma_2$  in the simulated process block as well.

Use the command `designpid` to design the two PID controllers C1 and C2. Should  $G_{nonmin12}$  or  $G_{nonmin21}$  be used for designing C1? Use  $\zeta = 0.7$ ,  $\alpha = 1$ , and experiment with different values of  $\omega$ . Try the controller in simulations, and, when it seems to work well, also on the real process.

Can you see the nonminimum phase behavior of the system in the step responses? How fast can you make the system? What limits the achievable performance? (*Warning:* The process is much slower and more difficult to control in this setting. A step response can take several minutes.)  $\diamond$

### 5.3 Decoupling

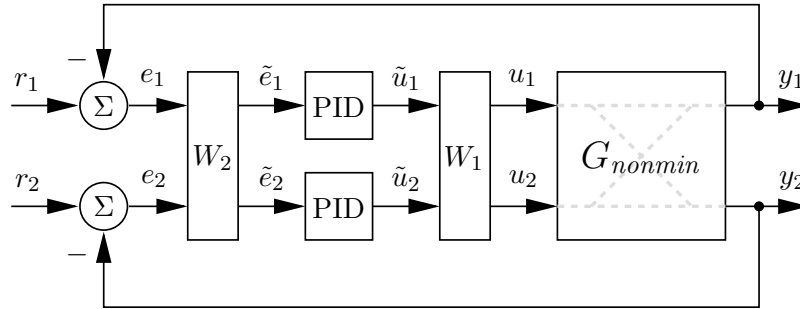
It is often possible to reduce the cross-coupling in a multivariable system by a suitable change of coordinates. The idea is to introduce new variables  $\tilde{y} = W_2 y$  and  $\tilde{u} = W_1^{-1} u$  such that the new system

$$\tilde{G}(s) = W_2 G(s) W_1$$

becomes as diagonal as possible. It is then possible to design a decentralized controller

$$\tilde{u} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \tilde{e}$$

for the decoupled system. The control structure is shown in Figure 5.



**Figure 5** Decoupling in the nonminimum phase case.

Since the process has  $G_{11} = G_{22}$  and  $G_{12} = G_{21}$  if  $\gamma_1 = \gamma_2$ , choosing

$$W_1 = W_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

will result in a complete diagonalization of the system. (Choosing  $W_1$  and  $W_2$  may be non-trivial in the general case). Notice that  $\tilde{y}_1$  corresponds to the sum of the lower tank levels, while  $\tilde{y}_2$  corresponds to the difference of the lower tank levels.

**Exercise 5** *Note: If you have estimated  $\gamma_1$  and  $\gamma_2$  and they are not equal you should make an average and have them equal in this part of the lab.*

Enter the matrices  $W_1$  and  $W_2$  in MATLAB and then compute the transformed system  $\tilde{G}$ :

```
>> Gtilde = minreal(W2*Gnonmin*W1)
```

Notice that the minimum phase dynamics are collected in  $\tilde{G}_{11}$ , while the non-minimum phase dynamics are collected in  $\tilde{G}_{22}$ . It should thus be harder to control  $\tilde{y}_2$  than  $\tilde{y}_1$ . Does this agree with your intuition?  $\diamond$

**Exercise 6** Design two PID controllers for the systems  $\tilde{G}_{11}$  and  $\tilde{G}_{22}$ . It is probably possible to achieve higher bandwidth in the first loop than the second one (why?).

Try the controller in simulations and also on the real process. Do you get better responses than you did in Exercise 4?

The system should be perfectly decoupled but you can probably still see some coupling in the graphs. Why? *Hint: What is decoupled?* Modify the simulink simulation model and verify that the system is indeed perfectly decoupled.  $\diamond$