Course outline

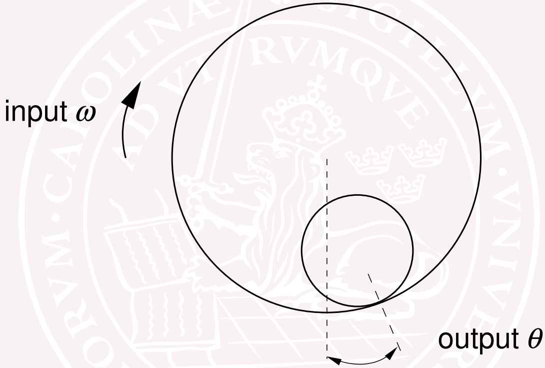
L1-L5	Specifications, models and loop-shaping by hand
L6-L8	Limitations on achievable performance
L9-L11	Controller optimization: Analytic approach
L12-L14	Controller optimization: Numerical approach

Lecture 6

- Controllability and observability
- Multivariable zeros
- Realizations on diagonal form

Examples: Ball in a hoop
Multiple tanks

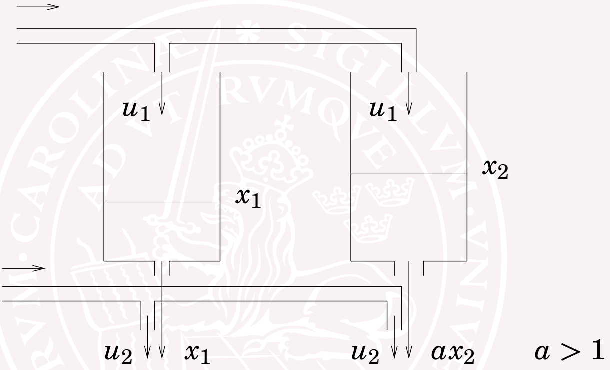
Example: Ball in the Hoop



$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$

Can you reach $\theta = \pi/4$, $\dot{\theta} = 0$? Can you stay there?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -ax_2 + u_1 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Controllability

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable , if for every $x_1 \in \mathbf{R}^n$ there exists $u(t), t \in [0, t_1]$, such that $x(t_1) = x_1$ is reached from $x(0) = 0$.

The collection of vectors x_1 that can be reached in this way is called the controllable subspace.

Controllability criteria

The following statements regarding a system $\dot{x}(t) = Ax(t) + Bu(t)$ of order n are equivalent:

(i) The system is controllable

(ii) rank $[A - \lambda I \quad B] = n$ for all $\lambda \in \mathbf{C}$

(iii) rank $[B \ AB \ldots A^{n-1}B] = n$

If A is exponentially stable, define the controllability Gramian

$$S = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$$

For such systems there is a fourth equivalent statement:

(iv) The controllability Gramian is non-singular

Interpretation of the controllability Gramian

The controllability Gramian measures how difficult it is in a stable system to reach a certain state.

In fact, let $S_1 = \int_0^{t_1} e^{At} B B^T e^{A^T t} dt$. Then, for the system $\dot{x}(t) = Ax(t) + Bu(t)$ to reach $x(t_1) = x_1$ from $x(0) = 0$ it is necessary that

$$\int_0^{t_1} |u(t)|^2 dt \geq x_1^T S_1^{-1} x_1$$

Equality is attained with

$$u(t) = B^T e^{A^T (t_1 - t)} S_1^{-1} x_1$$

Proof

$$0 \le \int_0^{t_1} [x_1^T S_1^{-1} e^{A(t_1-t)} B - u(t)^T][B^T e^{A^T(t_1-t)} S_1^{-1} x_1 - u(t)] dt$$

$$= x_1^T S_1^{-1} \int_0^{t_1} e^{At} B B^T e^{A^T t} dt \; S_1^{-1} x_1$$

$$- 2 x_1^T S_1^{-1} \int_0^{t_1} e^{A(t_1-t)} B u(t) dt + \int_0^{t_1} |u(t)|^2 dt$$

$$= -x_1^T S_1^{-1} x_1 + \int_0^{t_1} |u(t)|^2 dt$$

so $\int_0^{t_1} |u(t)|^2 dt \ge x_1^T S_1^{-1} x_1$ with equality attained for $u(t) = B^T e^{A^T(t_1-t)} S_1^{-1} x_1$. This completes the proof.

Computing the controllability Gramian

The controllability Gramian $S = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ can be computed by solving the linear system of equations

$$AS + SA^T + BB^T = 0$$

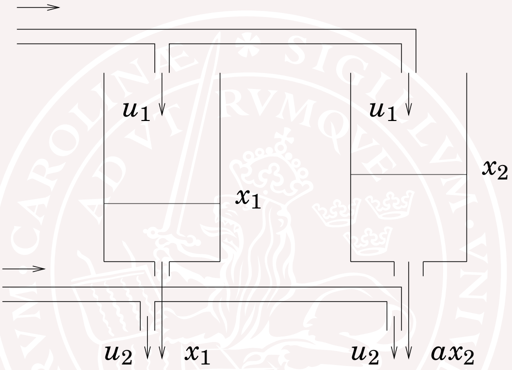
Proof. A change of variables gives

$$\int_h^\infty e^{At}BB^Te^{A^Tt}dt = \int_0^\infty e^{A(t-h)}BB^Te^{A^T(t-h)}dt$$

Differentiating both sides with respect to h and inserting $h = 0$ gives

$$-BB^T = AS + SA^T$$

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

The controllability Gramian $S = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

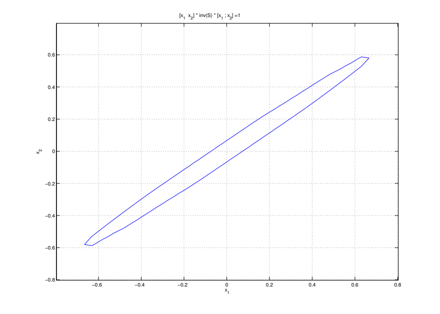
is close to singular when $a \approx 1$. Interpretation?

Example cont'd

In matlab you can solve the Lyapunov equation $AS + SA^T + BB^T = 0$ by `lyap`

```
>> a=1.25 ;   A=[-1 0 ; 0  -1*a ]; B=[1 ; 1] ;
```

```
>> Cs= [B  A*B] , rank(Cs)
Cs =
    1.0000   -1.0000
    1.0000   -1.2500
ans =
     2
>> S=lyap(A,B*B')
S =
   0.5000    0.4444
   0.4444    0.4000
>> invS=inv(S)
invS =
  162.0 -180.0
 -180.0  202.5
```



Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot S^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$

corresponds to the states we can reach by

$\int_0^\infty |u(t)|^2 dt = 1.$

Observability

The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

is observable, if the initial state $x(0) = x_0 \in \mathbf{R}^n$ is uniquely determined by the output $y(t), t \in [0, t_1]$.

The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the unobservable subspace.

Observability criteria

The following statements regarding a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) The system is observable

(ii) rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbf{C}$

(iii) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

If A is exponentially stable, define the observability Gramian

$$O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

For such systems there is a fourth equivalent statement:

(iv) The observability Gramian is non-singular

Interpretation of the observability Gramian

The observability Gramian measures how difficult it is in a stable system to distinguish two initial states from each other by observing the output.

In fact, let $O_1 = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$. Then, for $\dot{x}(t) = Ax(t)$, the influence from the initial state $x(0) = x_0$ on the output $y(t) = Cx(t)$ satisfies

$$\int_0^{t_1} |y(t)|^2 dt = x_0^T O_1 x_0$$

Computing the observability Gramian

The observability Gramian $O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be computed by solving the linear system of equations

$$A^T O + OA + C^T C = 0$$

Proof. The result follows directly from the corresponding formula for the controllability Gramian.

Poles and zeros

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$$

The points $p \in \mathbf{C}$ where $G(s) = \infty$ are called poles of G. They are eigenvalues of A and determine stability.

The poles of $G(s)^{-1}$ are called zeros of G.

Poles determine stability

All poles of $G(s) = C(sI - A)^{-1}B + D$ are eigenvalues of A.

The matrix A can always be written on the form

$$A = U \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^{-1}. \qquad \text{Hence } e^{At} = U \begin{bmatrix} e^{\lambda_1 t} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}$$

The diagonal elements are the eigenvalues of A.

e^{At} decays exponentially if and only if $Re\{\lambda_k\} < 0$ for all k.

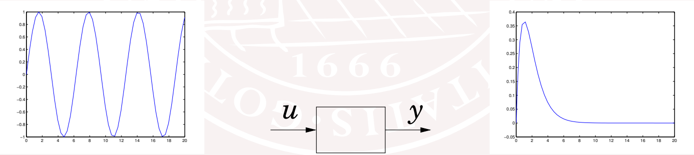
Interpretation of poles and zeros

Poles:

- A pole $s = a$ is associated with a time function $x(t) = x_0 e^{at}$
- A pole $s = a$ is an eigenvalue of A

Zeros:

- A zero $s = a$ means that an input $u(t) = u_0 e^{at}$ is blocked
- A zero describes how inputs and outputs couple to states



Pole polynomial and Zero polynomial

The following definitions can be used even when $G(s)$ is not a square matrix:

- The pole polynomial is the least common denominator of all minors (sub-determinants) to $G(s)$.
- The zero polynomial is the greatest common divisor of the maximal minors of $G(s)$.

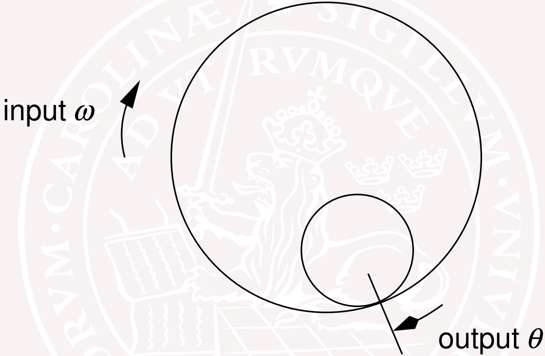
When $G(s)$ is square, the only maximal minor is $\det G(s)$, so the zeros are determined from the equation

$$\det G(s) = 0$$

Actually $s = z$ is a zero when the matrix $M(s)$ looses rank

$$M(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$

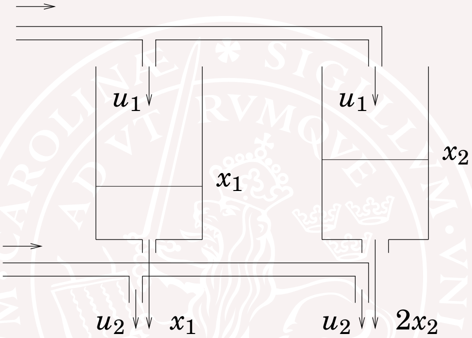
Example: Ball in the Hoop



$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$

The transfer function from ω to θ is $\frac{s}{s^2+cs+k}$. The zero in $s = 0$ makes it impossible to control the stationary position of the ball.

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -2x_2 + u_1 \qquad y_2 = 2x_2 + u_2$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

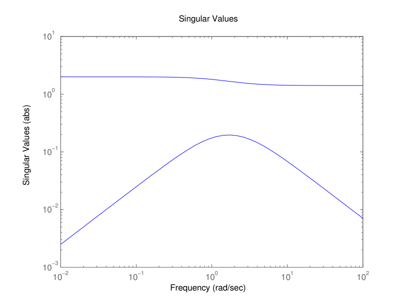
The system has a zero in the origin! At stationarity $y_1 = y_2$.

Plot Singular Values of $G(s)$ Versus Frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) ; plot singular values

% ALT. for a certain frequency:

» i=sqrt(-1)
» w=1;
» A=[1/(i*w+1) 1 ; 2/(i*w+2) 1]
» [U,S,V] = svd(A)
```



The largest singular value of $G(i\omega) = \begin{bmatrix} \frac{1}{i\omega+1} & 1 \\ \frac{2}{i\omega+2} & 1 \end{bmatrix}$ is fairly constant. This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference.

Singular values - continued

Revisit example from lecture notes 2:

The largest singularvalue of a matrix A, $\overline{\sigma}(A) = \sigma_{max}(A)$ is the square root of the largest eigenvalue of the matrix A^*A,
$\overline{\sigma}(A) = \sqrt{\lambda_{max}(A^*A)}$

Q: For frequency specifications (see prev lectures); When are we interested in the largest amplification and when are we interested in the smallest amplification?

Realization on diagonal form

Consider a transfer matrix with partial fraction expansion

$$G(s) = \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} x(t) + D u(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of columns in B_i and the multiplicity of the pole p_i.

Example: Realization of Multi-variable system

To find state space realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{s+2} - \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}}{s+3} - \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 0\end{bmatrix}}{s+4}$$

This gives the realization

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$