Lecture 3

Disturbance modelling

This section reviews the main aspects in disturbance modelling and the corresponding relations of descriptions in the time and frequency domain, respectively.

We will also consider the two related questions illustrated in Fig.3.1;

(i) Given a known input spectra and known transfer function, what is the spectral density of the output

(ii) Given a known spectral density for a signal, find a stable linear system with *white noise* input which gives the same spectral density on its output.

The latter problem is called the *spectral factorization problem* and will be used to rewrite systems with coloured disturbances to an equivalent system with white noise input, which will be used as a standard form for different estimation and prediction problems later on in course.

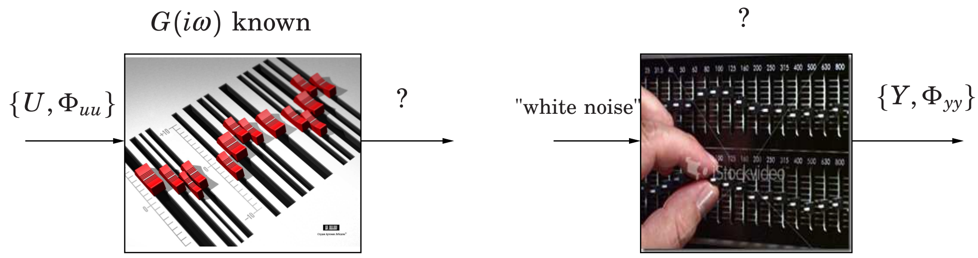


Figure 3.1 Illustration of two main questions of disturbance modelling in this chapter; *(left)* What is the output spectral density Φ_{yy} given that we know the input spectral density Φ_{uu} and and the linear filter $G(i\omega)$? *(right)* Knowing the output spectral density, find a stable linear filter which gives the same output spectral density if fed by white noise.

3.1 Disturbances

In the basic control diagram of Fig. 3.2 we consider load disturbances d and measurement noise n

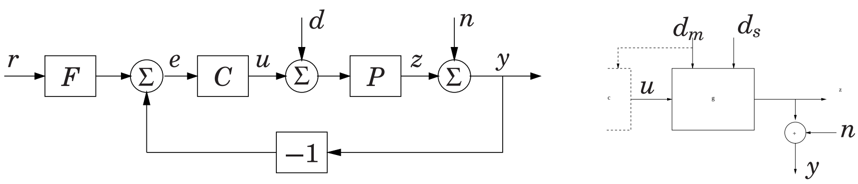


Figure 3.2 *(Left)* The basic control loop with load disturbances d and measurement noise n. *(Right)* Load disturbances which can be measured, d_m, e.g., changes in outer temperature, can be (partially) compensated for by feedforward to the control signal.

The load disturbance d drives the system from its desired state whereas the measurement noise n corrupts the feedback information about z. Load disturbances can be divided into measurable load disturbances, d_m, which partially can be compensated by feedforward, and load disturbances which can't be measured. Even if we can't measure d_s in Fig. 3.2, statistical information like covariance or spectral density will help us to design controllers which reduces/supresses the effect of the disturbances with respect to e.g., average and variance of the control objective z.

Example 1

In paper production there are a lot of disturbances which affect the paper quality and the paper thickness. One objective is to keep down the variation in the paper thickness, see Fig. 3.3.

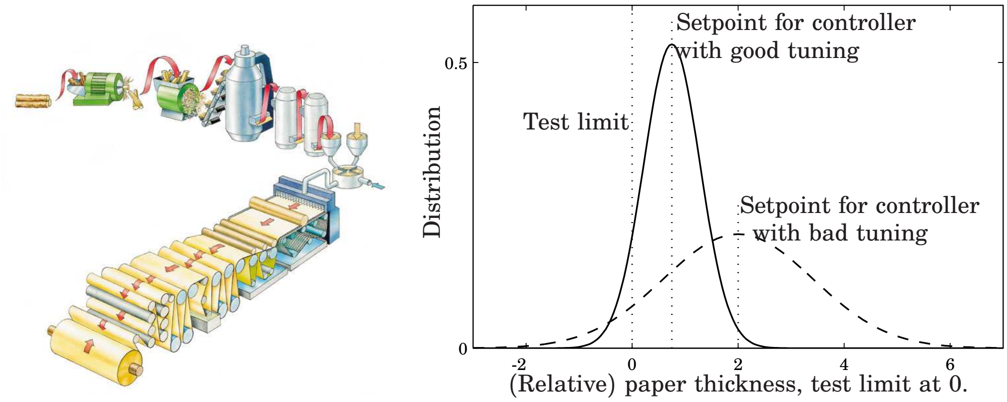


Figure 3.3 To be of acceptable quality, products must exceed a certain threshold. By minimizing the variance of the thickness we see that the average of the paper thickness can be reduced significantly (we come closer to the test limit) for the same yield. This may save a lot in production costs regarding both energy and raw material.

All paper production below the test limit is wasted. Good control allows for lower setpoint with the same yield. By having a lower variance of the production, the average paper thickness can thus also be lower, which saves significant costs in both energy and raw material. Keeping down the variance of the output will be an important control objective for us in this course. □

A first glimpse at linear stochastic control

In the previous example we saw that one objective could be to minimize the variance of the output or of a state. In a more general setting one can choose a trade-off with respect to how much control action one will use by introducing a cost for this as well.

Consider a system with state-disturbances w and measurement-disturbances v. The LQ-problem (Linear system, Quadratic cost function) is then described as follows:

$$
\begin{aligned}
&\text{Minimize} && \int \left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right) dt \\
&\text{subject to} && \dot{x} = Ax + Bu + w \\
& && y = Cx + Du + v
\end{aligned}
$$

where v and w is white noise with intensity R_1 and R_2 respectively.

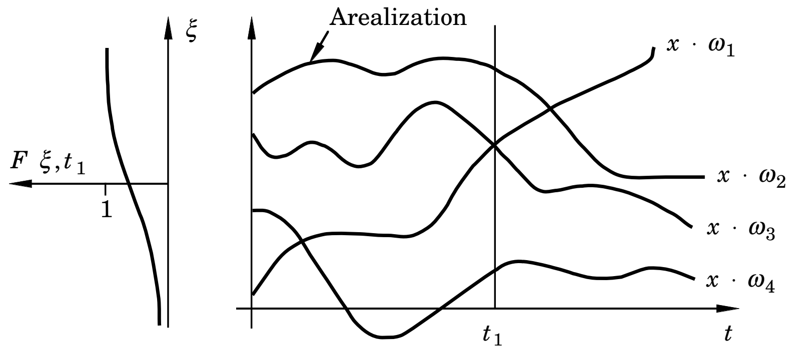


Figure 3.4 A stochastic process: For a fixed ω we call it a realization, for fixed time t_1 it will correspond to a random variable with a distribution $F(\xi, t_1) = Prob\{x(t_1) \leq \xi\}$.

As we will see later on in this course, one can solve this as two independent sub-problems thanks to the *separation principle* by considering

- Controller design for full state information, $u = -Lx$
- Optimal estimation of states (Kalman filter),

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y})$$

combination $\Longrightarrow$ Output feedback using observer $u = -L\hat{x}$...

Before we do this we will have a closer look on how to describe disturbances using statistical properties.

Stochastic processes

A **stochastic process** (random process, random function) is a family of stochastic variables $\{x(t), t \in T\}$ where t represents time. The stocastic process can be viewed as a function of two variables $x(t, \omega)$. For a fixed $\omega = \omega_0$ it gives a time function $x(\cdot, \omega_0)$, often called a realization, whereas if we fix the time $t = t_1$ it gives a random variable $x(t_1, \cdot)$ with a certain distribution, see Fig. 3.4.

For a zero-mean *stationary* stochastic processes the distribution is independent of t. We refer to the basic course in statistics for more details on the following concepts:

Mean-value function

$$\mathbf{E}x(t) \equiv 0$$

Covariance function. A zero mean Gaussian process x is completely determined by its covariance function:

$$R_x(\tau) = \mathbf{E}x(t+\tau)x(t)^T$$

Cross-covariance function

$$R_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T$$

Spectral density (defined for (weakly) stationary processes). The spectral density is the Fourier transform of the covariance function

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(t)e^{-it\omega}\,dt$$

and

$$R_{xy}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \Phi_{xy}(\omega)\, d\omega$$

In particular, we get the following expressions for the *covariance matrix*:

$$\mathbf{E}xx^T = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega$$

When x is scalar, this is simply the variance of x. (Notation: We will use Φ_y as short for Φ_{yy}.)

For relations between covariance function, spectral density and a typical realization, see Fig. 3.5, where one may notice that the realizations seem to be "more random" the flatter the spectra is (over a larger frequency range) while peaks in the spectral density corresponds to periodic covariance functions.

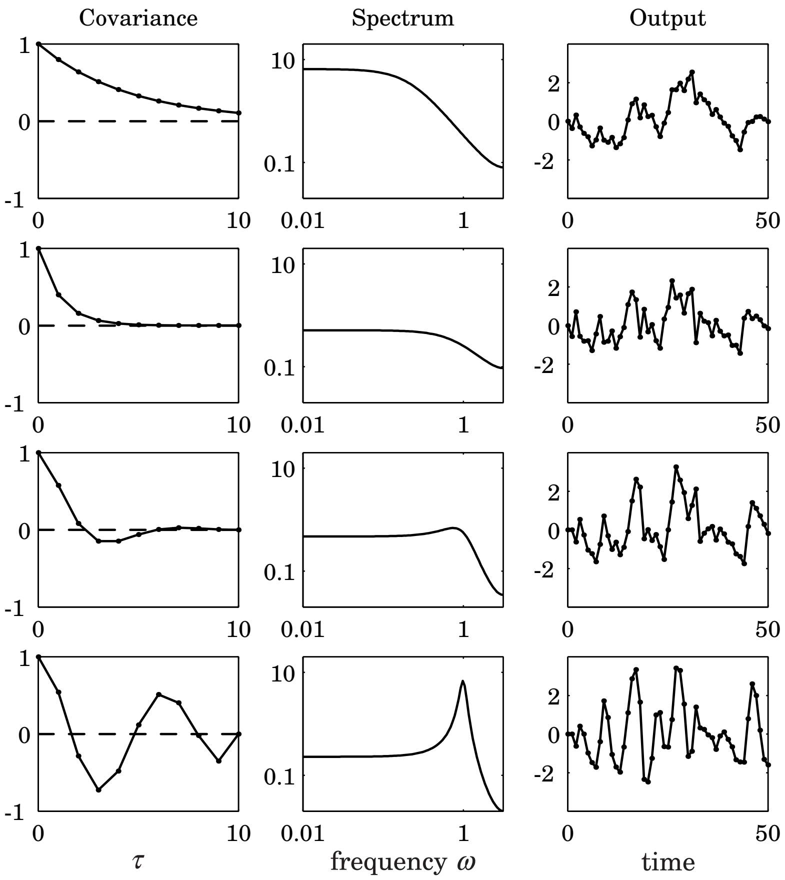


Figure 3.5 Relations between covariance function, spectral density and a typical realization. Correction: The spectra should be divided by 2π

White noise

A particular disturbance is so-called *white noise* e with *intensity* R. Here R is a constant matrix, which corresponds to a constant spectrum, totally flat and equal for all frequencies:

$$\Phi_e(\omega) = R$$

One effect of this definition is that the continuous-time version of white noise has infinite energy, and causes some issues to be handled mathematically rigorously, but we will not go into these details here.

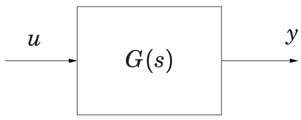
The most important property of white noise which we will use below, is that it can not be predicted; based on previous measurements there is no information about future values (infinite variance). From transform theory we also have that the Fourier transform of the Dirac pulse $\delta(t)$, is constant, which corresponds to an alternative interpretation: by applying a Dirac impulse as input to a linear system, the spectral density of the corresponding output (i.e., of the impulse response), will be like a finger-print of the system's frequency properties.

The two problems related to Fig. 3.1 can be formulated as

1. Determine the covariance function and spectral density of y when white noise u is filtered through the linear system

$$\begin{aligned} \dot{x} &= Ax + u(k) \\ y &= Cx \end{aligned}$$

2. Conversely, find filter parameters for a stable linear filter, A and C, to give the output y a desired spectral density.



What is the output spectral density for y if the input u has spectral density $\Phi_u(\omega)$?

We use the transfer function representation

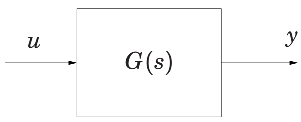
$$Y(i\omega) = G(i\omega)U(i\omega)$$

where $Y = \mathcal{F}\{y\}$, $U = \mathcal{F}\{u\}$ are the Fourier transforms. According to the definition, we get

$$\Phi_y(\omega) \hat{=} \Phi_{yy}(\omega) = Y(i\omega)Y(i\omega)^* = G(i\omega)U(i\omega)U(i\omega)^*G(i\omega)^*$$

where we can identify the spectral density of the output as

$$\Phi_{yy}(\omega) = G(i\omega)\Phi_{uu}(\omega)G(i\omega)^*$$



In similar way we find the cross-spectral density

$$\Phi_{yu}(\omega) = G(i\omega)\Phi_{uu}(\omega)$$

Spectral factorization

The next question is then how we go "backwards" according to Fig. 3.1 *right*, to find what linear filter which will do.

- Assume that the disturbance w has spectrum $\Phi_w(\omega)$
- *(Spectral factorization)* Assume that we can find a transfer function $G(s)$ such that $G(i\omega)RG(i\omega)^* = \Phi_w(\omega)$ for a constant R.

In that case we can consider w as an output from the linear system G with white noise as input, $\Phi_v(\omega) = R$ (equal energy for all frequencies/flat spectrum).

THEOREM 3.1
Spectral factorization [G&L 5.3] Assume that the real, scalar valued function $\Phi_w(\omega) \geq 0$ is a rational function of ω^2. Then there exists a rational function $G(s)$ with all poles strictly in the left half plane and all zeros in the left half plane or on the imaginary axis such that

$$\Phi_w(\omega) = |G(i\omega)|^2 = G(i\omega)G(-i\omega)$$

□

If v and w are *scalar valued* and $\Phi_w(\omega)$ is a rational function of ω^2 it is easy to follow the proof in [Glad & Ljung] and factorize to first or second order polynomials of ω^2 in both the numerator and the denominator. These can then be split in stable and unstable poles, respectively, and comes from the fact that if the characteristic polynomial for $G(i\omega)$ is $\Pi_{k=1}^{n}(i\omega - \lambda_k)$ then $G^* = G(-i\omega)$ will have its poles mirrored in the the imaginary axis. This is done in transfer function form, and in the next section we will see how this looks in state-space representation.

Assume we have state-space model with disturbances

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Nw_1(t) \\ z(t) &= Mx(t) + D_z u(t) \\ y(t) &= Cx(t) + D_y u(t) + w_2(t) \end{aligned}$$

where

- w_1 is called state- or system noise
- w_2 is called measurement- or output noise

The question is how to handle coloured noise?

If w_1 and w_2 is coloured noise with known or estimated spectral density then re-write w_1 and w_2 as output signals from linear systems with *white noise inputs* v_1 and v_2 .

$$w_1(t) = G_1(p)v_1(t), \qquad w_2(t) = G_2(p)v_2(t)$$

where $p = \frac{d}{dt}$ (corresponding to the Laplace s in frequency domain).

Make a state space realization of G_1 and G_2 and extend the system description with these states

$$\begin{aligned} \dot{\overline{x}}(t) &= \overline{A}\overline{x}(t) + \overline{B}\overline{u}(t) + \overline{N}v_1(t) \\ z(t) &= \overline{M}\overline{x}(t) + D_z u(t) \\ y(t) &= \overline{C}\overline{x}(t) + D_y u(t) + v_2(t) \end{aligned}$$

where the *extended state* $\overline{x}$ consists of the state x and the states from the state-space realizations of G_1 and G_2.

$\overline{A}$ is the corresponding system matrix for the extended system etc. We illustrate this procedure with an example.

Example 2
Consider the system

$$\begin{aligned} \dot{x}_1 &= -7x_1 + u + w_1 \\ y &= x_1 + w_2 \end{aligned}$$

where w_1 is coloured noise with spectral density

$$\Phi_{w_1} = \frac{9}{\omega^2 + 4} =_{\text{(spectral factorization)}}= \frac{3}{(i\omega + 2)} \frac{3}{(-i\omega + 2)}$$

We can then introduce a state-space form of this transfer function, representing the coloured noise w_1 as

$$\begin{aligned}\dot{x}_2 &= -2x_2 + 3v_1 \\ w_1 &= x_2\end{aligned}$$

where v_1 is white noise with intensity 1. The system can now be written as

$$\begin{aligned}\dot{x}_1 &= -7x_1 + u + x_2 \\ \dot{x}_2 &- 2x_2 + 3v_1 \\ y &= x_1 + w_2\end{aligned}$$

and we can proceed in the same way with the coloured noise w_2 □

Covariance and spectral density for a state vector

Consider the linear system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R_v$$

We can calculate the transfer function from noise to state as

$$G_{v\to x}(s) = (sI - A)^{-1}B$$

and the spectral density for x will thus be

$$\Phi_x(\omega) = (i\omega I - A)^{-1}BR_v\underbrace{B^*(-i\omega I - A)^{-T}}_{((i\omega I - A)^{-1}B)^*}$$

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R_v$$

One way to calculate the covariance matrix for state x is

$$\Pi_x = R_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_x(\omega)d\omega$$

However there is an alternative way of calculating Π_x

THEOREM 3.2—[GLAD&LJUNG 5.3]

If all eigenvalues of A are strictly in the left half plane (i.e. $Re\{\lambda_k\} < 0$) then there exists a unique matrix $\Pi_x = \Pi_x^T > 0$ which is the solution to the matrix equation

$$A\Pi_x + \Pi_x A^T + BR_vB^T = 0$$

□

We will see that a similar formula can be used to calculate the optimal gain K in the Kalman filter with respect to measurement and state noise covariances. An intuitive interpretation how large the gain K should be is that if we have much state noise but little output noise (i.e, reliable measurements), then the optimization chooses a large gain which "trusts" the measurements. With very large measurement noise, it will choose a low gain which means that the observer will almost run in open-loop; trusting the model and gaining very little information from the measurements.

Example 3
Consider the system

$$\dot{x} = Ax + Bv \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

where v is white noise with variance $R_v = 1$.

What is the covariance for x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!
Solve the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12} + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

In matlab: `lyap([-1 2; -1 0],[1 ; 0]*[1 0])`

□