Previous Lecture: Linear Quadratic Optimal Control

Problem:

Minimize $\int_0^\infty \Big(x(t)^T Q_1 x(t) + 2x(t)^T Q_{12} u(t) + u(t)^T Q_2 u(t)\Big) dt$

subject to $\dot{x} = Ax(t) + Bu(t), \qquad x(0) = x_0$

Solution: Assume (A, B) controllable. Then there is a unique $S > 0$ solving the Riccati equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with $L = Q_2^{-1}(SB + Q_{12})^T$.
The minimal value is $x_0^T S x_0$.

Question after lecture: What is it good for?

Why Linear Quadratic Optimal Control?

- Structured approach to MIMO systems
- Always stabilizing
- Guaranteed robustness in state feeback case
- Well developed theory

Introduction - Telescopes

Telescopes:

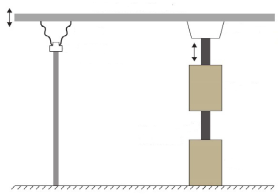
- Collect light to form pictures of stars and planets.
- Problem: Atmospheric turbulence gives optic distortion.

Adaptive optics:

- Counteract the distortion.
- Traditionally by small mirrors located late in the optical chain.
- New approach: large deformable primary or secondary mirror.

Introduction - Mirror Properties

- Large deformable **secondary** mirror
- Mirror in one solid piece
- Material: Borosilicate
- Outer diameter: 1 m
- Inner rim diameter 5 mm (where the mirror is attached to the telescope)
- Thickness: 2 mm
- Actuators = voice-coils
- sensors = microphones



Introduction - Mathematical Model

- Mirror modeled by partial differential equations.
- Finite element analysis gives

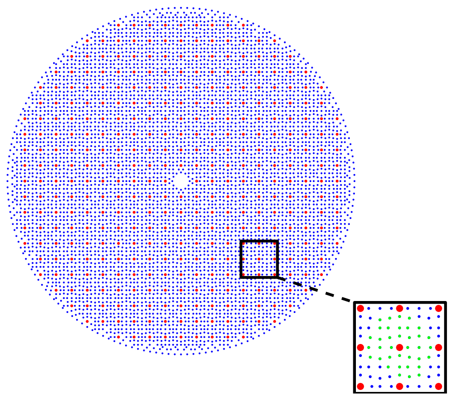
$$\mathcal{M}\ddot{\boldsymbol{\xi}} + \mathcal{C}\dot{\boldsymbol{\xi}} + \mathcal{K}\boldsymbol{\xi} = \boldsymbol{F}$$

  $\boldsymbol{\xi}$ translational and angular displacements
  $\boldsymbol{F}$ external forces.
- Equivalently

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

- E and A of dimension 36768×36768, but only 0.06% non-zero elements.
- B of dimension 36768×372 and 372 non-zero elements.

Schematic View of the Mirror



Introduction - Objectives

- Determine stabilizing controller
- Distributed structure of controller
- Good control theoretic performance
- Good performance in terms of astronomical measures
- Reduce effects of atmospheric distortion

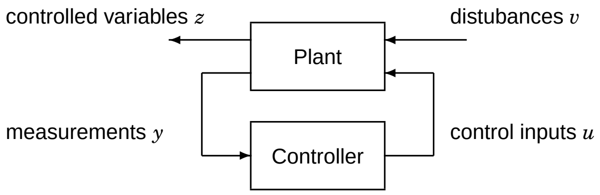
Design by Linear Quadratic Optimal Control!

Lecture 10: Optimal Kalman Filtering

- Observer Based Feedback
- The Optimal Kalman filter
- LQG by Separation
- Stochastic interpretations

Textbook sections 9.1-9.4 and 5.7

Linear Quadratic Gaussian Control (LQG)

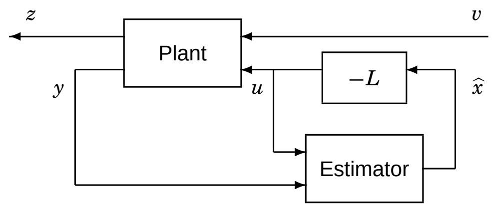


For a linear plant, minimize a quadratic function of the map from disturbance v to controlled variable z

Minimize trace $\int_{-\infty}^{\infty} QG_{zv}(i\omega)G_{zv}(i\omega)^* d\omega$

Last week: State feedback solution.

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$

Closed loop dynamics

Eliminate u and y:

$$\frac{d}{dt}x(t) = Ax(t) - BL\hat{x}(t) + v_1(t)$$
$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) - BL\hat{x}(t) + K[Cx(t) - C\hat{x}(t)] + Kv_2(t)$$

Introduce $\tilde{x} = x - \hat{x}$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}\begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_1(t) - Kv_2(t) \end{bmatrix}$$

Two kinds of closed loop poles

Process poles: $0 = \det(sI - A + BL)$

Observer poles: $0 = \det(sI - A + KC)$

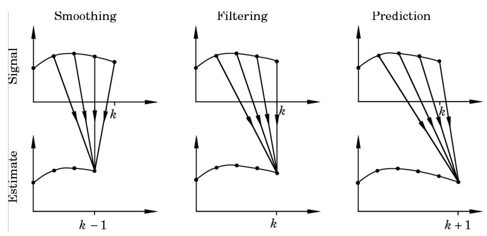
Rudolf Kalman, (born 1930)

Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering "for the devlopment and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products.”

Prediction and filtering

* Wiener (1949) Stationary I/O case
* Kalman and Bucy (1960) Time-varying state-space

Estimate $x(k+m)$ given $\{y(i), u(i) \,|\, i \le k\}$

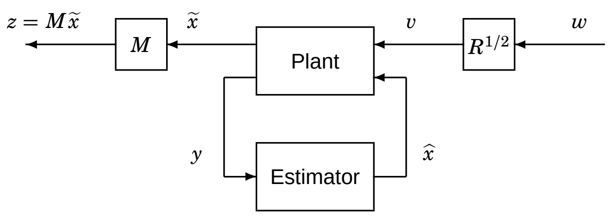


Examples

Smoothing	To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Thursday
Filtering	To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Wednesday (helps to reduce measurement error)
Prediction	To predict the Wednesday temperature based on temperature measurements from Sunday, Monday and Tuesday

Norbert Wiener, 1894–1964

The Kalman Filter Optimization Problem



Minimize error variance when v is white noise with intensity R:

$$\mathbf{E}|z|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} MG_{\tilde{x}v}(i\omega)\,R\,G_{\tilde{x}v}(i\omega)^* M^T d\omega$$

Equivalent reformulations

The time domain version of the optimization problem can be written

$$\text{Minimize } \int_0^\infty M g_{\tilde{x}v}(t)\, R\, g_{\tilde{x}v}(t)^T M^T dt$$

Given the error dynamics

$$\frac{d}{dt}\tilde{x}(t) = [A - KC]\tilde{x}(t) + v_1(t) - Kv_2(t)$$

the impulse response from v to $\tilde{x}$ is

$$g_{\tilde{x}v}(t) = e^{(A-KC)t}[I \quad -K]$$

so K should be chosen to

$$\text{Minimize} \quad \int_0^\infty M e^{(A-KC)t}[I \quad -K]R[I \quad -K]^T e^{(A-KC)^T t} M^T dt$$

Recall lecture 9: Linear Quadratic Optimal Control

For the system $\dot{x} = Ax(t) + Bu(t)$, $x(0) = x_0$ with control law $u = -Lx$ consider the cost

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt = \int_0^\infty x_0^T e^{(A-BL)^T t} \begin{bmatrix} I \\ -L \end{bmatrix}^T Q \begin{bmatrix} I \\ -L \end{bmatrix} e^{(A-BL)t} x_0 dt$$

The minimal cost is achieved by $L = Q_2^{-1}(SB + Q_{12})^T$, where $S > 0$ solves

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

The minimal value of the integral is $x_0^T S x_0$.

The solution can be reused to get the optimal Kalman filter!

Duality between control and estimation

Optimal control	State estimation
A	A^T
B	C^T
Q_1	R_1
Q_2	R_2
Q_{12}	R_{12}
S	P
L	K^T

Optimal Kalman Filtering — The Solution

The Kalman filter $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$ gives the error covariance

$$\mathbf{E}|M\tilde{x}|^2 = \int_0^\infty M e^{(A-KC)t} \begin{bmatrix} I & -K \end{bmatrix} R \begin{bmatrix} I & -K \end{bmatrix}^T e^{(A-KC)^T t} M^T dt$$

The minimal error covariance is achieved by $K = (PC^T + R_{12})R_2^{-1}$

where $P > 0$ solves

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T$$

Remark: Notice that K is independent of M. Hence the same filter is optimal regardless of which state we want to estimate! The minimal error covariance is $\mathbf{E}\tilde{x}\tilde{x}^T = P$.

Example 1 – Kalman filter

$$\dot{x}(t) = v_1(t) \qquad v_1 \text{ is white noise with intensity } R_1$$
$$y(t) = x(t) + v_2(t) \qquad v_2 \text{ is white noise with intensity } R_2$$

$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

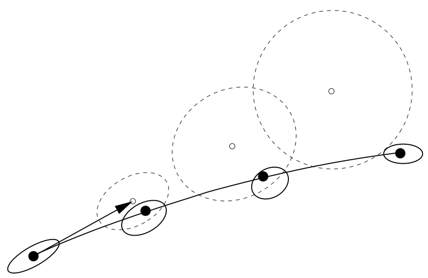
Riccati equation $\quad 0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1 R_2}$

Filter gain $\quad K = P/R_2 = \sqrt{R_1/R_2}$

Error dynamics $\quad \dfrac{d\tilde{x}}{dt} = -\sqrt{R_1/R_2}\,\tilde{x} + v_1 - \sqrt{R_1/R_2}\,v_2$

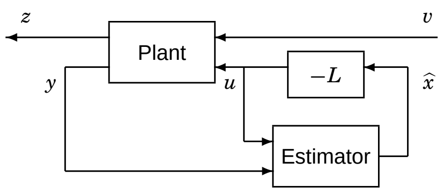
Error covariance $\quad \mathbf{E}\tilde{x}^2 = P = \sqrt{R_1 R_2}$

Example 2 – Tracking of a moving object



Dotted ellipses show estimates based on only a model with known initial state. Solid ellipses show Kalman filter estimates based on noisy measurements.

Output feedback using state estimates



Plant: $\quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\quad \begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$

Minimize $\quad \mathbf{E}|z|^2 = \mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$

when v is white noise of intensity R

The idea of separation

The state feedback control law is independent of R

The Kalman filter minimizes $\mathbf{E}|M\tilde{x}|^2$ independently of M

This makes it possible to optimize the control law $u(t) = -L\hat{x}(t)$ and the estimator separately.

Linear Quadratic Optimal Control (LQG)

Given the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \\ z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{cases} \qquad \begin{aligned} Q &= \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \\ R &= \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} \end{aligned}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The frequency integral

$$\text{trace} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} QG_{zv}(i\omega)RG_{zv}(i\omega)^* d\omega$$

is minimized when K and L satisfy

$$0 = Q_1 + A^TS + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T \qquad K = (PC^T + NR_{12})R_2^{-1}$$

The minimal value of the integral is

$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^TSB + Q_2)L]$$

Stochastic Interpretation of LQG Control

Given white noise (v_1, v_2) with intensity R and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The stationary variance

$$\mathbf{E}\left(x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u\right)$$

is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = Q_1 + A^TS + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$
$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^TSB + Q_2)L]$$

Example

Consider the problem to minimize $\mathbf{E}(Q_1x^2 + Q_2u^2)$ for

$$\begin{cases} \dot{x}(t) = u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

The observer based controller

$$\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$$

is optimal for K and L computed as follows:

$$0 = Q_1 - S^2/Q_2 \quad \Rightarrow \quad S = \sqrt{Q_1Q_2} \quad \Rightarrow \quad L = S/Q_2 = \sqrt{Q_1/Q_2}$$
$$0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1R_2} \quad \Rightarrow \quad K = P/R_2 = \sqrt{R_1/R_2}$$