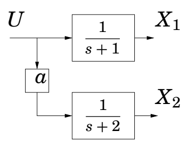
Limitations: Controllability [from lec 6]



System $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ a \end{bmatrix} u$

State x_2 is *uncontrollable* for $a = 0$ and "hard to control" for small values of a.

Controllability matrix

$W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ a & -2a \end{bmatrix}$

Controllability gramian S

$AS + SA^T + BB^T = 0 \Longrightarrow$

$S = ... = \begin{bmatrix} \frac{1}{2} & \frac{1}{3}a \\ \frac{1}{3}a & \frac{1}{4}a^2 \end{bmatrix}$

Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot S^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$ corresponds to what states we can reach by $\int_0^{t_1} |u(t)|^2 dt = 1$.

Lecture 7: Fundamental Limitations

- Limitations from unstable poles and zeros: Intuition
- A back-wheel steered bicyle?
- Limitations from unstable poles and zeros: Hard proofs
- Bode's integral formula
- Bode's relation: Coupling magnitude and phase

See lecture notes and [G&L Ch. 7]

Unstable poles — "intuitive reasoning"

An unstable pole p makes the output signal for a bounded input grow exponentially as $\sim e^{pt}$. To stabilize this system, one has to act fast, on a time scale proportional to $\sim 1/p$.

Intuitive conclusion: *Unstable poles give a lower bound on the speed of the closed loop.*

Systems with time-delay

Assume that the plant contains a time-delay T. This means e.g. that a load disturbance is not visible in the output signal until after at least T time units. Of course, this puts a hard constraint on how quickly a feedback controller can reject the disturbance!

Intuitive conclusion: *Time delays give an upper bound on the speed of the closed loop.*

Unstable zeros — "intuitive reasoning"

The step response of a system with a process *zero in the right half plane* (i.e, with positive real part) goes initially in the "wrong direction".

Intuitive conclusion: *Unstable zeros give an upper bound on the speed of the closed loop.*

Why the wrong direction? The Laplace transform of the system output signal $G(s)U(s)$ will be 0 if we evaluate it at $s = z$ where z is a process zero. If we in particular look at the step response, call it $y(t)$, and its Laplace transform we get

$$0 = Y(z) = \int_0^\infty y(t) \underbrace{e^{-zt}}_{>0} dt$$

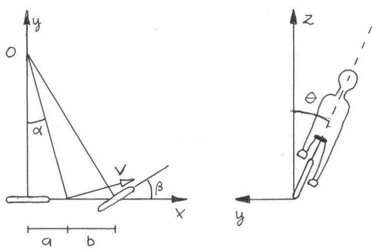
Hence, $y(t)$ must takey both positive and negative values!

Mini-problems

1. Give examples of systems that initially respond in the "wrong" direction.
2. Which of the intuitive arguments can be applied to
   - an inverted pendulum?
   - a rear wheel steered bicycle?

Bike example

A (linearized) torque balance for a bicycle can be approximated as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Bike example, cont'd

$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

where the physical parameters have typical values as follows:

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed:	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g = 9.81$ ms^{-2}

The transfer function from β to θ is

$$P(s) = \frac{mV_0\ell}{b}\frac{as + V_0}{Js^2 - mg\ell}$$

Bike example, cont'd

The system has an unstable pole p with time-constant

$$p^{-1} = \sqrt{\frac{J}{mg\ell}} \approx 0.4 \text{ s}$$

The closed loop system must be at least as fast as this. Moreover, the transfer function has a zero z with

$$z^{-1} = -\frac{a}{V_0} \approx -\frac{0.3\text{m}}{V_0}$$

For the back-wheel steered bike we have the same poles but different sign of V_0 and the zero will thus the be unstable!

An unstable pole-zero cancellation occurs for $V_0 \approx 0.75$m/s.

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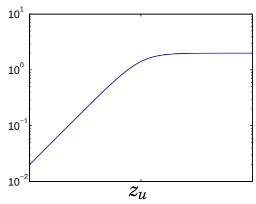
Hard limitations from unstable zeros

If the plant has an unstable zero z_u, then the specification

$$\left|\frac{1}{1+P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{1+z_u^2/\omega^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.

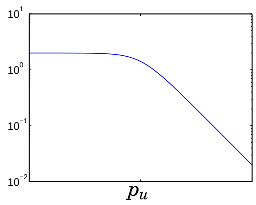


Hard limitations from unstable poles

If the plant has an unstable pole p_u, then the specification

$$\left|\frac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{1+\omega^2/p_u^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.



The Maximum Modulus Theorem

The proofs will be based on the following theorem:

Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\max_{\text{Re } s \geq 0} |G(s)| = \max_{\omega \in \mathbf{R}} |G(i\omega)|$$

Sensitivity bounds from unstable zeros

It is easy to see that the sensitivity function must be equal to one at a righ-half-plane zero $s = z_u$ of the transfer function:

$$P(z_u) = 0 \qquad \Rightarrow \qquad S(z_u) := \frac{1}{1+\underbrace{P(z_u)}_{0}C(z_u)} = 1$$

Notice that the unstable zero in the plant can not be cancelled by an unstable pole in the controller, since this would give an unstable transfer function $C/(1+PC)$ from measurement noise to control input.

Sensitivity bounds from unstable poles

Similarly, the complimentary sensitivity must be one at an unstable pole p_u:

$$P(p_u) = \infty \qquad \Rightarrow \qquad T(p_u) := \frac{P(p_u)C(p_u)}{1+P(p_u)C(p_u)} = 1$$

In this case, cancellation by an unstable zero in the controller would give an unstable transfer function $P/(1+PC)$ from input disturbance to plant output.

Corollary of the Maximum Modulus Theorem

Suppose that the plant $P(s)$ has unstable zeros z_i and unstable poles p_j. Then the specifications

$$\sup_{\omega} |W_a(i\omega)S(i\omega)| \leq 1 \qquad \sup_{\omega} \left|W^b(i\omega)T(i\omega)\right| \leq 1$$

are impossible to meet with a stabilizing controller unless $\|W_a(z_i)\| \leq 1$ for every unstable zero z_i and $\|W^b(p_j)\| \leq 1$ for every unstable pole p_j.

In particular, if $W_a = (s+a)/(2s)$ and $W^b(s) = (s+b)/(2b)$, it is necessary that $a \leq \min_i z_i$ and $b \geq \max_j p_j$. This proves the statements on slide 12 & 13.

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Bode's Integral Formula ("The water bed effect")

For a system with loop gain $L = PC$ which has a relative degree ≥ 2 and unstable poles $p_1, \ldots, p_M$, the following *conservation law* for the sensitivity function $S = \dfrac{1}{1+L}$ holds.

$$\int_0^{+\infty} \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i)$$

See [G&L Theorem 7.3] for details/asumptions.

G. Stein: "Conservation of "dirt!""

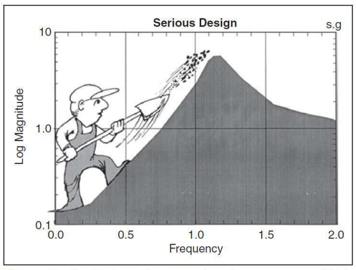
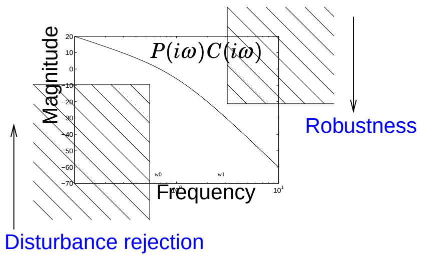


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Picture from Gunter Steins Bode Lecture (1985) "Respect the unstable". Reprint in [IEEE Control Systems Magazine (Aug 2003)]

Recall that the loop transfer matrix should have small norm $\|P(i\omega)C(i\omega)\|$ at high frequencies, while at low the frequencies instead $\|[P(i\omega)C(i\omega)]^{-1}\|$ should be small.



How quickly can we make the transition from high to low gain?

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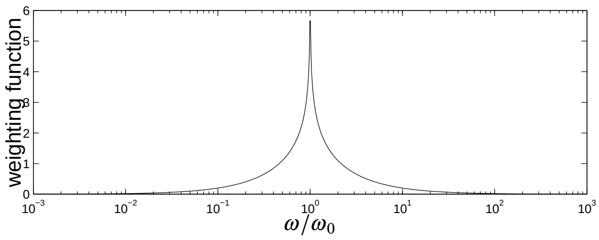
Bode's Relation — Exact version

If $G(s)$ is stable with no unstable zeros (*minimum-phase*), then

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi} \int_0^{\infty} \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{d \log |G(i\omega)|}{d \log \omega} \underbrace{\log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right|}_{\text{weighting function}} d \log \omega$$



Bode's Relation — Approximate version

If $G(s)$ is stable with no unstable zeros (*minimum-phase*), then

$$\arg G(i\omega_0) \approx \frac{\pi}{2} \left. \frac{d \log |G(i\omega)|}{d \log \omega} \right|_{\omega = \omega_0}$$

Otherwise the argument is even smaller.

As a consequence, the decay rate of the magnitude curve must be less than 2 at the cross-over frequency.

Summary: Fundamental Limitations

- Limitations from unstable poles and zeros: Intuition
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