Exercise 1. Control in Matlab

This exercise is intended to give a basic introduction to Matlab. The main focus will be on the use of Control Systems Toolbox for control system analysis and design. This toolbox will be used extensively during the upcoming exercises and laboratories in the course. A short Matlab reference guide and a guide to the most commonly used commands from Control Systems Toolbox can be found at the end of this exercise.

Getting started

Matlab is started by issuing the command

```
> matlab
```

which will bring up a Java-based interface with three different frames showing the current directory, your defined variables, and the Matlab command window. This gives a good overview, but may be slow.

An alternative way to start Matlab is by the command

```
> matlab -nodesktop
```

which will only open up the command window. You can then use the commands `ls` and `whos` to examine the current directory and your defined Matlab variables.

All Matlab commands have a help text, which is displayed by typing

```
>> help <command>
```

Try for example

```
>> help help
```

Use the help command frequently in the following exercises.

Matrices and system representations

Matrices in Matlab are created with the following syntax

```
>> A = [1 2;3 4]

A =

     1     2
     3     4
```

with semi-colons being used to separate the lines of the matrix. The transpose of a matrix is written as

```
>> A'

ans =

     1     3
     2     4
```

1.1 Consider the following state-space model describing the dynamics of an inverted pendulum

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)\end{aligned} \tag{1.1}$$

Enter the system matrices A, B, and C in Matlab and use the command `eig` to determine the poles of the system.

In Control System Toolbox, the basic data structure is the linear time-invariant (LTI) model. There is a number of ways to create, manipulate and analyze models (type, e.g., `help ltimodels`). Some operations are best done on the LTI system, and others directly on the matrices of the model.

1.2 Define an LTI model of the pendulum system with the command `ss`. Use the command `tf` to determine the transfer function of the system.

1.3 Zeros, poles and stationary gain of an LTI model is computed with the command `zpkdata`. Use this command on the inverted pendulum model. Compare with 1.1.

The command `tf` is used to create an LTI model from a transfer function. This is done by specifying the coefficients of the numerator and denominator polynomials, e.g.

```
>> %% Continuous transfer function P(s) = 1 / (2s+1)
>> P = tf(1, [2 1]);
```

Another convenient way to create transfer function models is by the following commands

```
>> s = tf('s');
>> P = 1 / (2*s + 1);
```

To specify a time delay, set the property `InputDelay` of the LTI model

```
>> P.InputDelay = 0.5;
```

To display all properties of an LTI model and their respective values, type

```
>> get(P)
```

1.4 Define an LTI model of the continuous-time transfer function

$$P(s) = \frac{1}{s^2 + 0.6s + 1} \cdot e^{-1.5s} \tag{1.2}$$

and use the commands `step`, `nyquist`, and `bode` to plot time- and frequency responses of the system. Is the system stable? Will the closed-loop system be stable if unit gain negative feedback is applied?

State feedback example

Now return to the model of the inverted pendulum (1.1). We want to design a state-feedback controller

$$u(t) = -Lx(t)$$

such that the closed loop system gets the characteristic equation

$$s^2 + 1.4s + 1 = 0$$

1.5 Is the system controllable? Use the command `ctrb` to compute the controllability matrix. Then use the command `place` to determine the state-feedback vector L.

Connecting systems

LTI systems can be interconnected in a number of ways. For example, you may add and multiply systems (or constants) to achieve parallel and series connections, respectively. Assume that the system (1.2) *without* time delay is controlled by a PD controller, $C(s) = K(1 + sT_d)$, with $K = 0.5$ and $T_d = 4$, according to the standard block diagram to the left in Figure 1.1:

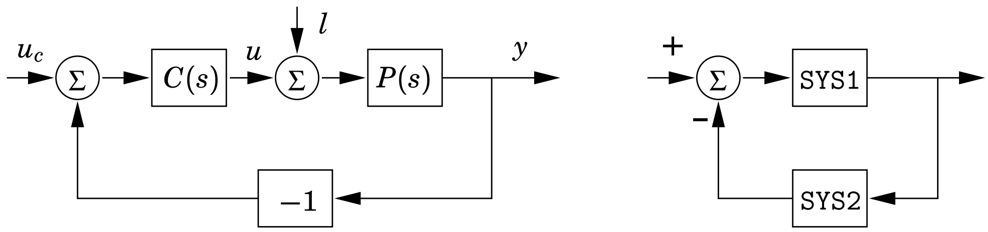

Figure 1.1 A closed loop control system.

1.6 Define an LTI model of the controller, $C(s)$. Compute the amplitude margin and phase margin for the loop transfer function. Use the Matlab command `margin`.

There is a function `feedback` for constructing feedback systems. The block diagram to the right in Figure 1.1 is obtained by the call `feedback(SYS1,SYS2)`. Note the sign conventions. To find the transfer function from the set point u_c to the output y, you identify that `SYS1` is $P(s)C(s)$ and `SYS2` is 1.

1.7 Compute the transfer function for the closed-loop system, both using the `feedback` command and by direct computation (use `minreal` to simplify) according to the formula

$$G_{cl}(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$

1.8 Plot the step response of the closed-loop system. What is the stationary gain from u_c to y?

1.9 The systems you have seen so far have been SISO (Single-Input, Single-Output) systems. In this course we will also work with MIMO (Multiple-Input, Multiple-Output) systems. This problem gives you an example of a MIMO system.

A rough model for the pitch dynamics of a JAS 39 Gripen is given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 1 & 0 & -1/2 & 0 \\ 4 & -1 & 0 & -25 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 3/2 & 1/2 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} x \end{aligned} \tag{1.3}$$

Enter this system into Matlab using the `ss` command and determine if the system is stable/controllable/observable? What is your conclusion from this analysis?

Can you find the transfer function from the second input (the elevator rudder command) to the first output (the pitch rate)?

1.10 (*) In Matlab, a transfer function can be represented either on `tf` format, corresponding to $G_1(s)$ and $G_3(s)$ below, or on `zpk` format, corresponding to $G_2(s)$ and $G_4(s)$.

Calculate the poles of the following systems

$$G_1(s) = \frac{1}{s^3 + 3s^2 + 3s + 1}$$

$$G_2(s) = \frac{1}{(s+1)^3}$$

During calculations, numerical round-off errors can arise, which can lead to changed dynamics of the system. Calculate the poles of the following systems where one coefficient has been modified.

$$G_3(s) = \frac{1}{s^3 + 2.99s^2 + 3s + 1}$$

$$G_4(s) = \frac{1}{(s+0.99)^3}$$

In view of your results on the poles of the four systems $G_1(s)$ to $G_4(s)$, discuss which format that is numerically the best.

1.11 (*) Consider the following state-space model of a system

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 3 & 4 \end{pmatrix} x(t) \end{aligned} \tag{1.4}$$

Is the system observable? Is the system controllable? Motivate your answer by calculations, but also with an insight how the system behaves.

1.12 **(*)** Given a mass-spring system in state-space form without a damper with $m = 0.5$ kg and $k = 10$ N/m, compute the transfer function of the system using the commands `ss` and `zpk`. Design a PID controller such that the closed loop system gets the characteristic equation

$$s^3 + s^2(2\zeta\omega + \omega) + s(\omega^2 + 2\zeta\omega^2) + \omega^3 = 0$$

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{aligned} \tag{1.5}$$

As design parameters use $\omega = 1$ and $\zeta = 0.4$ as start values. Then try different values of ω and ζ, such that during a step response, the settling time is less than 1.0 seconds and the maximum overshot is less than 15%. The settling time is defined as the minimum time T such that $1-p < y(t) < 1+p$ for all $t > T$ where p can be 5%.

1.13 **(*)** Given a transfer function for the following system

$$P(s) = \frac{3-s}{(s+1)(s+2)}$$

Compute a state-space realization using the command `ssdata`. Is the system controllable? Design a state feedback controller, such that the closed loop system gets the characteristic equation

$$s^2 + 5.6s + 16 = 0$$

Simulate a step response of the closed-loop system. What is the static gain? What is the system called when the step response starts “in the wrong direction”? How can we directly see this property in the transfer function of the process?

Quick Matlab Reference

Some Basic Commands

Note: the command syntax is case-sensitive!

help <command>	display the Matlab help for <command>.
who	lists all of the variables in your matlab workspace.
whos	list the variables and describes their matrix size.
clear	deletes all matrices from active workspace.
clear x	deletes the matrix x from active workspace.
save	saves all the matrices defined in the current session into the file matlab.mat.
load	loads contents of matlab.mat into current workspace.
save filename	saves the contents of workspace into filename.mat
save filename x y z	saves the matrices x, y and z into the file titled filename.mat.
load filename	loads the contents of filename into current workspace; the file can be a binary (.mat) file or an ASCII file.
!	the ! preceding any unix command causes the unix command to be executed from matlab.

Matrix commands

[1 2; 3 4]	create the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
zeros(n)	creates an nxn matrix whose elements are zero.
zeros(m,n)	creates a m-row, n-column matrix of zeros.
ones(n)	creates a n x n square matrix whose elements are 1's
ones(m,n)	creates a mxn matrix whose elements are 1's.
ones(A)	creates an m x n matrix of 1's, where m and n are based on the size of an existing matrix, A.
zeros(A)	creates an mxn matrix of 0's, where m and n are based on the size of the existing matrix, A.
eye(n)	creates the nxn identity matrix with 1's on the diagonal.
A'	(complex conjugate) transpose of A
diag(V)	creates a matrix with the elements of V on the diagonal.
blkdiag(A,B,C)	creates block matrix with the matrices A, B, and C on the diagonal

Plotting commands

plot(x,y)	creates an Cartesian plot of the vectors x & y.
stairs(x,y)	creates a stairstep of the vectors x & y.
semilogx(x,y)	plots log(x) vs y.
semilogy(x,y)	plots x vs log(y)
loglog(x,y)	plots log(x) vs log(y).
grid	creates a grid on the graphics plot.
title('text')	places a title at top of graphics plot.
xlabel('text')	writes 'text' beneath the x-axis of a plot.
ylabel('text')	writes 'text' beside the y-axis of a plot.
gtext('text')	writes text according to placement of mouse
hold on	maintains the current plot in the graphics window while executing subsequent plotting commands.
hold off	turns off the 'hold on' option.
print filename -dps	writes the contents of current graphics to 'filename' in postscript format.

Misc. commands

length(x)	returns the number elements in a vector.
size(x)	returns the size m(rows) and n(columns) of matrix x.
rand	returns a random number between 0 and 1.
randn	returns a random number selected from a normal distribution with a mean of 0 and variance of 1.
rand(A)	returns a matrix of size A of random numbers.
fliplr(x)	reverses the order of a vector. If x is a matrix, this reverse the order of the columns in the matrix.
flipud(x)	reverses the order of a matrix in the sense of exchanging or reversing the order of the matrix rows. This will not reverse a row vector!
reshape(A,m,n)	reshapes the matrix A into an mxn matrix from element (1,1) working column-wise.
squeeze(A)	remove empty dimensions from A
A.x	access element x in the struct A

Some useful functions from Control Systems Toolbox

Do `help` *<function>* to find possible input and output arguments.

```
Creation and conversion of continuous or discrete time LTI models.
  ss            - Create/convert to a state-space model.
  tf            - Create/convert to a transfer function model.
  zpk           - Create/convert to a zero/pole/gain model.
  ltiprops      - Detailed help for available LTI properties.
  ssdata etc.   - Extract data from a LTI model.
  set           - Set/modify properties of LTI models.
  get           - Access values of LTI model properties.
  minreal       - Minimal realization and pole/zero cancellation.

Sampling of systems.
  c2d           - Continuous to discrete conversion.
  d2c           - Discrete to continuous conversion.

Model dynamics.
  pole, eig     - System poles.
  zero          - System zeros.
  pzmap         - Pole-zero map.
  covar         - Covariance of response to white noise.

State-space models.
  ss2ss         - State coordinate transformation.
  canon         - State-space canonical forms.
  ctrb, obsv    - Controllability and observability matrices.

Time response.
  step          - Step response.
  impulse       - Impulse response.
  initial       - Response of state-space system with given initial state.
  lsim          - Response to arbitrary inputs.
  ltiview       - Response analysis GUI.

Frequency response.
  bode          - Bode plot of the frequency response.
  margin        - Bode plot with phase and gain margins.
  sigma         - Singular value plot.
  nyquist       - Nyquist plot.
  nichols       - Nichols plot.
  ltiview       - Response analysis GUI.
  ctrlpref      - Open GUI for setting Control System Toolbox Preferences.

System interconnections.
  + and -       - Add and subtract systems (parallel connection).
  *             - Multiplication of systems (series connection).
  / and \       - Division of systems (right and left, respectively).
  inv           - Inverse of a system.
  [ ]           - Horizontal/vertical concatenation of systems.
  feedback      - Feedback connection of two systems.

Classical design tools.
  rlocus        - Root locus.
  place, acker  - Pole placement (state feedback or estimator).
  estim         - Form estimator given estimator gain.
  reg           - Form regulator given state-feedback and estimator gains.

LQG design tools.
  lqr,dlqr      - Linear-quadratic (LQ) state-feedback regulator.
  lqry          - LQ regulator with output weighting.
  lqrd          - Discrete LQ regulator for continuous plant.
  kalman, kalmd - Kalman estimator.
  lqgreg        - Form LQG regulator given LQ gain and Kalman estimator.

Matrix equation solvers.
  dlyap         - Solve discrete Lyapunov equations.
  dare          - Solve discrete algebraic Riccati equations.
```

Exercise 2. System Representations and Stability

2.1 A system is given by

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + u_1 \\ \dot{x}_2 &= -3x_2 + u_1 + 2u_2 \\ y_1 &= x_1 + x_2 \\ y_2 &= 2x_1 + u_1 \\ y_3 &= 2x_2 + u_2\end{aligned}$$

Express the system on state-space form by determining the matrices (A, B, C, D).

2.2 A system with two inputs and one output is modeled by a differential equation:

$$\ddot{y} + a_1\dot{y} + a_2 y = b_{11}\dot{u}_1 + b_{12}u_1 + b_{21}\dot{u}_2 + b_{22}u_2.$$

Find the transfer matrix.

2.3 A system has the following input-output relation:

$$y(t) = \int_0^t (t-\tau)e^{-2(t-\tau)}u(\tau)d\tau$$

a. Determine $g(t)$ (the open-loop impulse response) such that

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Also, if $u(t) = r(t) - y(t)$, find the closed-loop transfer function $G_c(s)$ such that

$$Y(s) = G_c(s)R(s)$$

b. Is the closed loop system input-output stable?

c. Estimate the L_2-gain of the closed-loop system.

2.4 In Figure 2.1 a feedback system is illustrated.

a. Determine the transfer function from disturbances v to the output y. This important function, called the *sensitivity function*, is denoted by $S(s)$. (Note: v is sometimes called *process noise* but in some literature also *output load disturbance*).

Also determine the transfer function from reference r to output y. This function, equally important, is called the *complementary sensitivity function*, and is denoted by $T(s)$.

What is the transfer function from measurement noise n to output y, expressed in S and T?

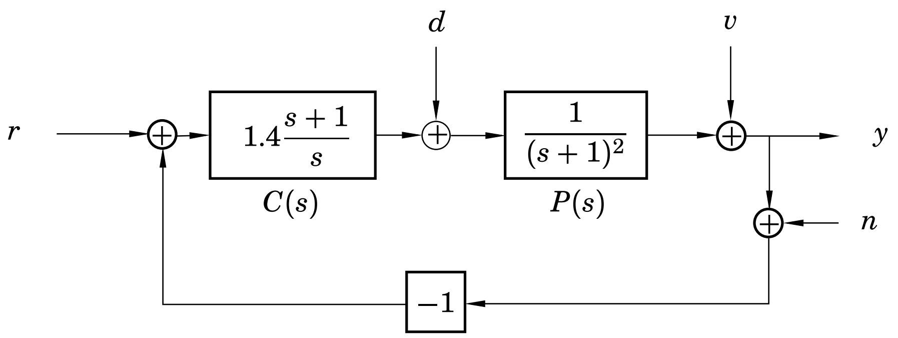

Figure 2.1 System in Problem 2.4

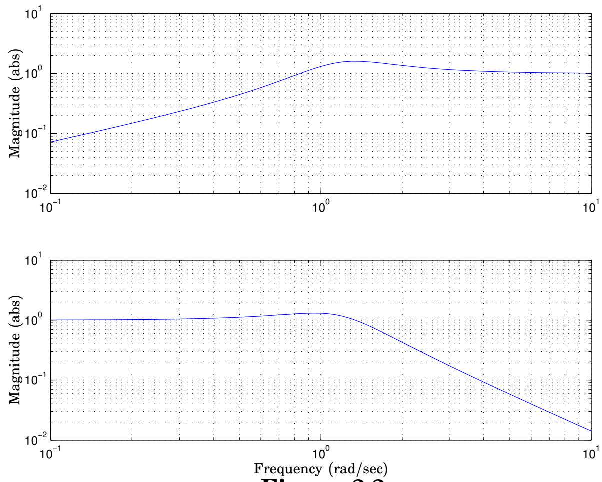

Figure 2.2

b. Figure 2.2 shows gain curves for the sensitivity function and the complementary sensitivity function. Which curve represents which function?

c. In which frequency region (roughly) is there good tracking of the reference value?

d. In which frequency region (roughly) is there good attenuation of the measurement noise, n?

2.5 Study the feedback control system in Figure 6.5, where the process, $P(s)$, is given by

$$P(s) = \frac{1}{(s+1)(s+2)}$$

The Bode diagram of $P(s)$ is shown in Figure 2.4.

Three different controllers were designed

$$C_1(s) = 10 \quad C_2(s) = 10\frac{s+1}{s} \qquad C_3(s) = 10\frac{s+1}{s}e^{-0.1s}$$

where the last one has a small delay.

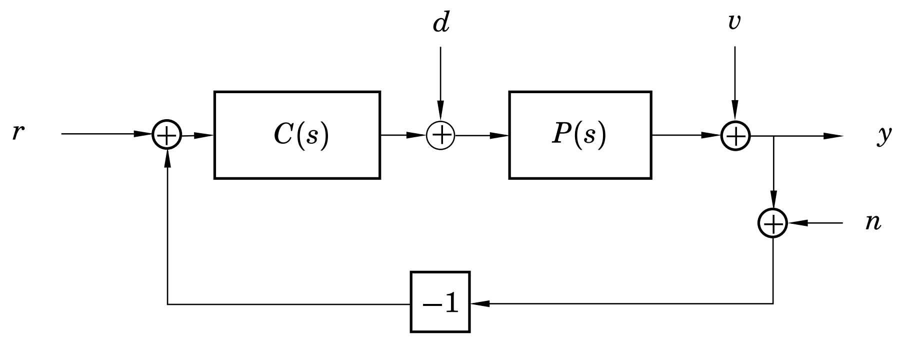

Figure 2.3 System in Problem 2.5

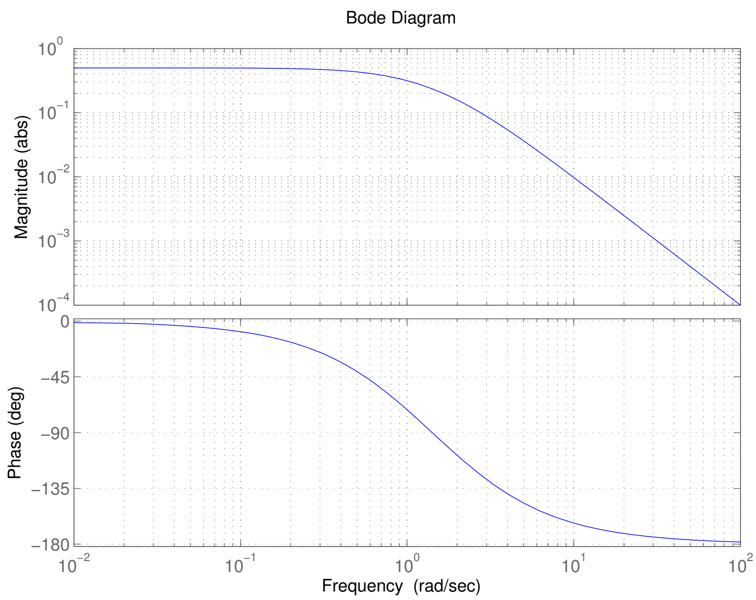

Figure 2.4 Bode diagram for $P(s)$ in Problem 2.5.

a. Figure 2.5 shows sensitivity functions, corresponding to the three different control designs $C_1 - C_3$. Combine the controllers $C_1 - C_3$ with the sensitivity functions $A - C$. Motivate!

b. Figure 2.6 shows responses to a step load disturbance, d, corresponding to the three different control designs $C_1 - C_3$. Combine the controllers $C_1 - C_3$ with the load step responses $I - III$. Motivate!

2.6 Consider a water tank with a separating wall. The wall has a hole at the bottom, as can be seen in figure 2.7.

The input signals are the inflows of water to the left, u_1, and the right, u_2, halves of the tank, measured in cm^3/s. The water levels are denoted by h_1 cm and h_2 cm, respectively. The outflow y cm^3/s is considered proportional to the water level in the right half of the tank:

$$y(t) = \alpha h_2(t)$$

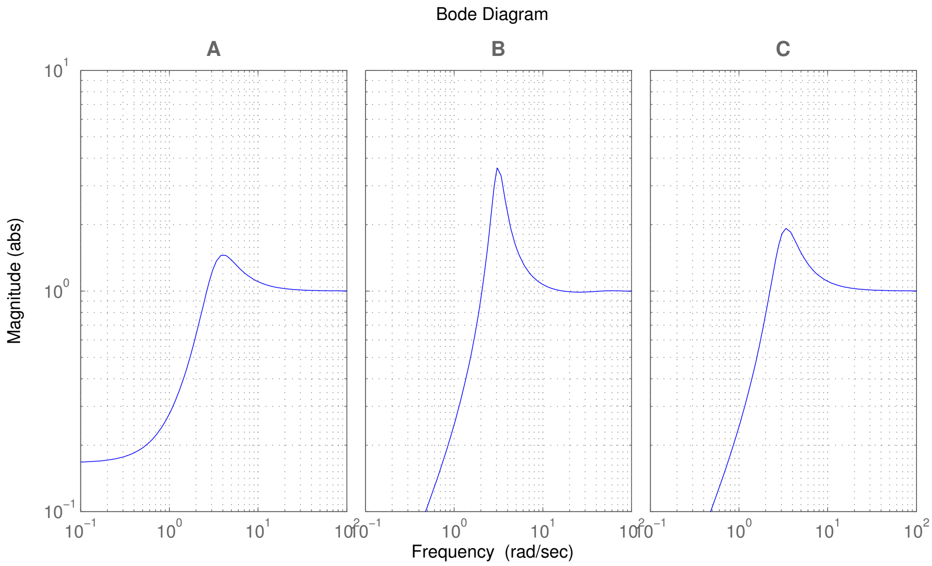

Figure 2.5 Sensitivity functions for Problem 2.5.

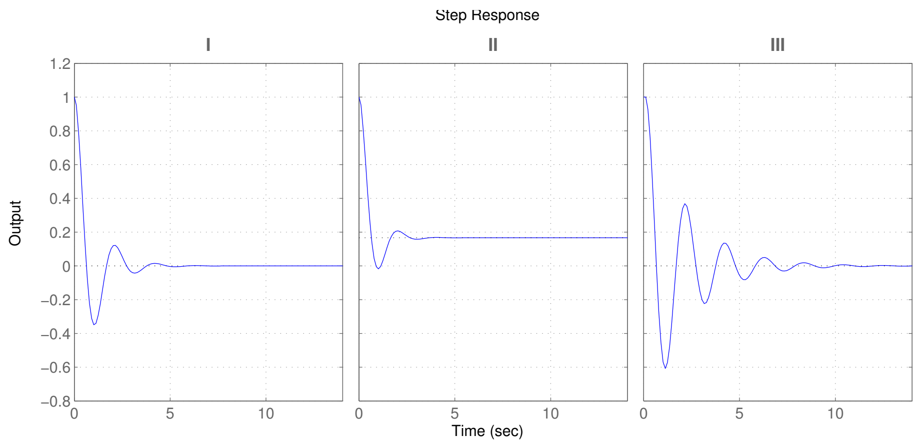

Figure 2.6 Step Load Disturbance Responses for Problem 2.5.

The flow between the tank halves is proportional to the difference in level:

$$f(t) = \beta(h_1(t) - h_2(t))$$

(flow from left to right)

The signals h_i, u_i and y are thought of as deviations from a linearization point, and may therefore be negative. Assume that the two tank halves each have area $A_1 = A_2 = 1$ cm^2.

a. Write the system on state-space form.

b. What is the transfer matrix from $(u_1 \quad u_2)^T$ to y?

c. What is the L_2-gain of the system when $\alpha = \beta = 1$? (Hint: Use Matlab.)

d. It turns out that the L_2-gain is larger than one. How is this possible? Can there be more water coming out from the tank than what is poured into it? Have we invented a water-producing device? Explain what's wrong here!

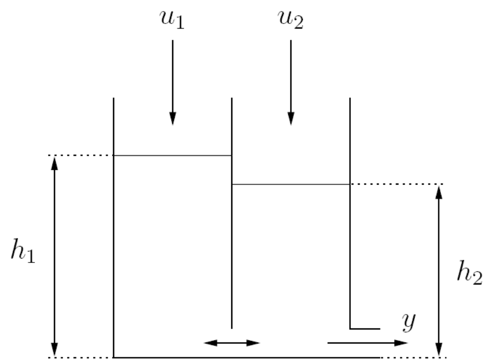

Figure 2.7

2.7 **(*)** A PI controller

$$U(s) = \left(1 + \frac{1}{s}\right)(R(s) - Y(s))$$

is used to control a servo process, which is modeled as

$$Y(s) = \frac{1}{ms^2 + 2s} U(s) \qquad m > 0$$

This problem is about investigating for what values of the mass m the closed loop system is stable, using the Nyquist criterion. Before we can do that however, we need to write the system in another form.

a. Find the characteristic polynomial of the closed-loop system. Show that the same characteristic polynomial can be generated by a feedback loop between some system $\frac{Q(s)}{P(s)}$ and d where $m = d + 1$. See figure 2.8. Determine $Q(s)$ and $P(s)$ so that they are not dependent on d.

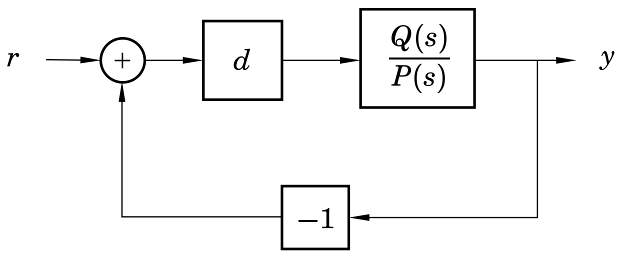

Figure 2.8 Closed-loop system for Problem 2.7.

b. Use the Nyquist criterion on $\frac{Q(s)}{P(s)}$ to determine the values of d (and hence m) for which the closed-loop system is stable. Use Matlab.

c. Let $m = 1$, but suppose there is a time delay τ in the measurement so that $U(s) = (1 + 1/s)e^{-s\tau}(R(s) - Y(s))$. How large time delays can be tolerated before the system becomes unstable?

d. Suppose that the servo instead has some unmodeled dynamics $\Delta(s)$

$$Y(s) = \left(\frac{1}{s^2 + 2s} (1 + s\Delta(s)) \right) U(s)$$

Rewrite the closed-loop system as a feedback loop between $\Delta(s)$ and a transfer function $H(s)$. Determine a number γ such that the small gain theorem guarantees stability for all $\Delta(s)$ with $\|\Delta\|_\infty < \gamma$.

Exercise 3. Disturbance Models and Robustness

3.1 **a.** Analyze the stability of the system to the left in Figure 3.1 using the small gain theorem.

b. Analyze how the stability of the system to the right in Figure 3.1 depends on the feedback gain K. If you get different results from **a**, explain why!

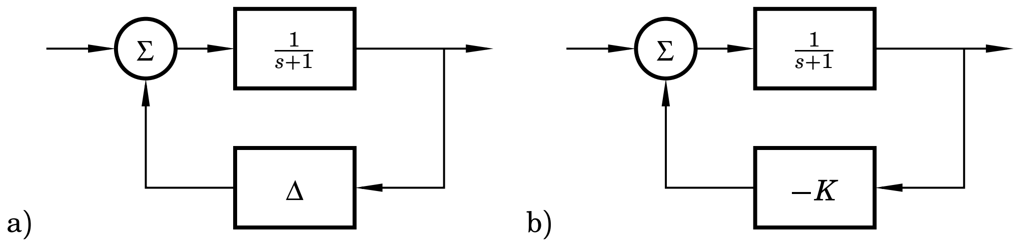

Figure 3.1

3.2 Consider the system in Figure 3.2.

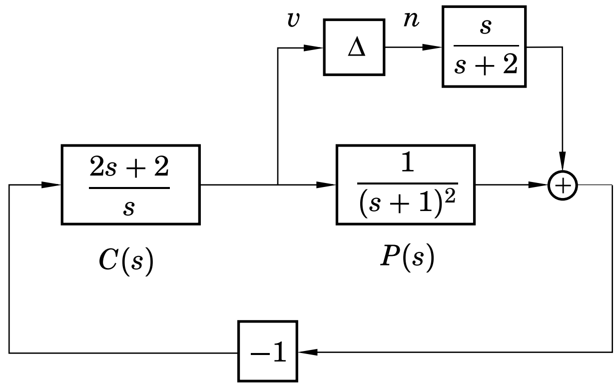

Figure 3.2 System for Problem 3.2.

a. Find the transfer function G_{vn} from n to v.

b. How large is the gain $\|G_{vn}\|_\infty$? Support the solution by a Matlab plot.

c. Using the small gain theorem, find the largest possible L_2-gain of Δ for which the closed loop system is stable.

d. The Δ block is used to account for uncertainty in the process model. Explain the role of the factor $\frac{s}{s+2}$ multiplying Δ.

3.3 A feedback system is shown in Figure 3.3.

a. Compute the poles of the closed loop system. You may use Matlab.

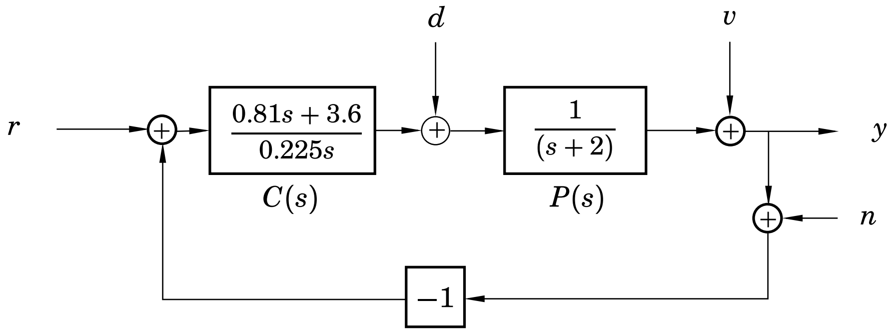

Figure 3.3 System in Problem 3.3.

b. Derive the transfer functions from r, d, v and n to y. Identify the sensitivity function S and the complementary sensitivity function T. Plot them in the same Bode plot.

c. If we have a disturbance $v = \sin(0.5t)$ acting on the system and all other input signals are zero, what amplitude would the oscillations in the output signal have when the transients have disappeared?

d. If we have sinusoidal measurement disturbances $n(t)$ with a frequency of 50 Hz and unit amplitude and all other input signals are zero, what amplitude would the oscillations in the output signal have when the transients have disappeared?

3.4 (*) Consider the system in Figure 3.4.

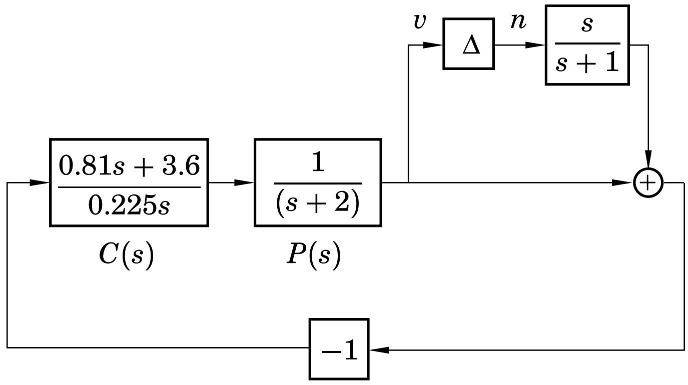

Figure 3.4 System for Problem 3.4.

a. What is the largest possible bound on the L_2-gain of Δ, for which the closed loop system is stable, by the small gain theorem? Support the solution by a Matlab plot.

b. If $\Delta(s)$ is replaced by a real parameter δ, then for what values of δ is the closed loop stable? Compare with the gain bound in (a).

c. What is the difference between the uncertainty model in this problem and the one in problem **3.2?**

3.5 A continuous-time stochastic process $u(t)$ has the power spectrum $\Phi_u(\omega)$. The process can be represented by a linear filter that has white noise as input. Determine the linear filter when

a.

$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}$$

b.

$$\Phi_u(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}$$

3.6 Consider a missile travelling in the air. It is propelled forward by a jet force u along a horizontal path. The coordinate along the path is z. We assume that there is no gravitational force. The aerodynamic friction force is described by a simple model as

$$f = k_1 \cdot \dot{z} + v,$$

where v are random variations due to wind and pressure changes. Combining this with Newton's second law, $m\ddot{z} = u - f$, where m is the mass of the missile, gives the input-output relation

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v).$$

a. Express the input-output relation in state-space form.

b. The disturbance v has been determined to have the spectral density

$$\Phi_v(\omega) = k_0 \cdot \frac{1}{\omega^2 + a^2}$$

Expand your state-space description so that the disturbance input can be expressed as white noise. What is the new input-output relation?

3.7 **(*)** This problem builds on problem 3.6.

a. Assume that the position measurement is distorted by an additive error $n(t)$,

$$y(t) = z(t) + n(t)$$

Write down the state-space equations for the system, assuming that $n(t)$ is white noise with intensity 0.1, i.e. $\Phi_n(\omega) \equiv 0.1$.

b. Solve the same problem, this time with

$$\Phi_n(\omega) = 0.1\frac{\omega^2}{\omega^2 + b^2}.$$

c. Solve the problem with

$$\Phi_n(\omega) = 0.1\frac{1}{\omega^2 + b^2}.$$

3.8 (*) A system is given by

$$\dot{x} = Ax + Bu + Nw$$
$$y = Cx + n$$

The load disturbance w is piecewise constant, that is, it changes in steps. The measurement noise n is periodic with a frequency of 2 Hz.

Expand the system model to include the known characteristics of the disturbances.

3.9 (*) Consider a swing (swe: gunga) that is hanging outside in the wind. The swing is described by the transfer function

$$Y(s) = \frac{1}{s^2 + s + 1} U(s)$$

where the output signal $y(t)$ is the angle of the swing (relative to the vertical axis) and the input signal $u(t)$ is the moment around the pivotal point. The influence of wind can be described as

$$u(t) = Kv(t)$$

where $v(t)$ is a normally distributed disturbance with spectrum

$$\Phi_v(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}, \qquad \alpha > 0.$$

K is a measure of the wind strength and α is a measure of the occurrence of wind gusts (swe: vindbyar).

a. Does α increase or decrease when there are more wind gusts (i.e., when the wind changes strength and direction more often)?

b. Determine the variance of $y(t)$. What is your interpretation?

Hint:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|b_2(i\omega)^2 + b_1 i\omega + b_0|^2}{|(i\omega)^3 + a_2(i\omega)^2 + a_1 i\omega + a_0|^2} \, d\omega = \frac{b_2^2 a_0 a_1 + (b_1^2 - 2b_0 b_2)a_0 + b_0^2 a_2}{2a_0(-a_0 + a_1 a_2)}$$

3.10 (*) Consider an electric motor with the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

from input current to output angle.

There are two different disturbance scenarios:

(i) $y(t) = G(p)(u(t) + w(t))$

(ii) $y(t) = G(p)u(t) + w(t)$

In both cases, $\dot{w}(t) = v(t)$, where $v(t)$ is a unit disturbance, e.g. an impulse.

a. Draw a block diagram of the two cases.

b. Put both cases in state-space form. It is assumed in the second case that the disturbance does not give cause to any common states with the engine.

c. Give a physical interpretation of $w(t)$ is, in the two cases.

Exercise 4. Loop Shaping

Exercises 4.1–4.4 are preparatory exercises for laboratory exercise 1. In these exercises, we will design a controller, step by step, for the process given by the transfer function

$$P(s) = \frac{1}{s^2 + 0.7s + 1}$$

4.1 Create a transfer function object in Matlab, and take a look at the Bode and Nyquist diagrams of the process. In the following exercises you will use a number of different controllers to shape the Bode diagram of the open loop system.

The structure of the control system is given in Figure 4.1. As you may have already heard, several transfer functions should be studied in a design. Besides a nice step response from r to y, also a fast recovery from, e.g., load disturbances, d is required. Furthermore, it is important to see how the control signal responds to the different input signals.

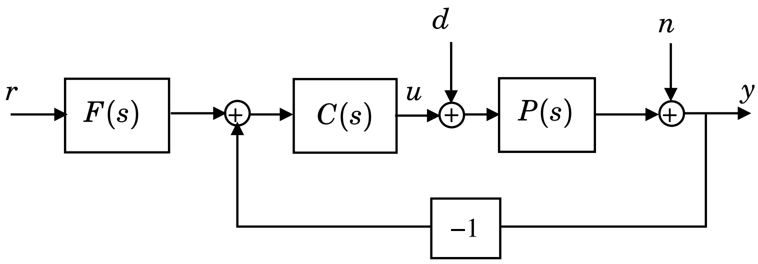

Figure 4.1 Our control loop with reference signal r, load disturbance d, measurement noise n and output y.

We have **two** degrees of freedom in designing our controller; the feedback part $C(s)$ and the prefilter $F(s)$.

We start by designing $C(s)$. For evaluation, we can look at the effect of a step load disturbance d (as this is only affected by the feedback loop). A good load step disturbance response goes quickly to zero. What is the closed-loop transfer function from d to y?

a. We will first try to control the system using a simple P-controller. Simulate **load step responses** for K=0.1, 1.0, 5.0, 10.0. Does the output go to zero? How much stationary error is left for different K's?

Tip: Use the Matlab command `figure(n)` to draw several plots. E.g:

```
>> figure(1)
>> bode(P*C, P)     % Plot both the process and the
>>                  % compensated open loop process
>> figure(2)
>> step(P/(1+C*P)) % Plot step load disturbance
>> figure(3)
>> bode(P/(1+C*P)) % Bode-plot of closed loop from
>>                  % load disturbance
```

b. To remove the stationary error in the response to load disturbances, we need to add integral action to the controller. The transfer function of a PI-controller is given as

$$C(s) = K\left(1 + \frac{1}{sT_i}\right) = K\left(\frac{sT_i + 1}{sT_i}\right)$$

Try some different values of K and T_i, and plot the step load response. Study the Bode diagram of the open-loop system. What effect does the integrator have on the phase curve? Try to find a controller that gives good performance. The error should vanish fast without too much oscillation.

4.2 To create a more advanced controller, we need to know the effect of adding additional poles and zeros to the controller, $C(s)$.

a. Let $C_a(s) = \frac{1}{s/a+1}$. How are the magnitude and the phase affected by a?. When and why would we add this kind of system to our controller?

b. Let $C_b(s) = \frac{s/b+1}{1}$. Again, how are magnitude and phase affected by b? When/why would we add this kind of system to our controller?

4.3 By combining a pole and a zero, we get a compensator on the form

$$C(s) = K\frac{(s/b + 1)}{(s/a + 1)}$$

A compensator where $b < a$ is called a *lead-compensator*, and a compensator having $b > a$ is called a *lag-compensator*. Plot the Bode diagrams for the two cases, and recall the properties of the lead- and lag-compensators from the basic course.

Now create a feedback controller

$$C(s) = K\frac{(sT_i + 1)}{sT_i} \cdot \frac{(s/b + 1)}{(s/a + 1)}$$

for the process $P(s)$, by adding a pole and a zero to the PI-controller in Problem 4.1. Note that the added compensator will allow you to adjust the parameters, K and T_i, of the PI-controller.

Requirements:

- The disturbance step response should settle within about 5 seconds. Specifically, $|y(t)| < 4 \cdot 10^{-3}$ for $t > 5$.
- No more than 20% overshoot in the step response from r to y.

Hints:

- Faster response is often tightly connected to a higher cut-off frequency ω_c.
- Oscillations are due to bad margins (being close to the -1 point in the Nichols or Nyquist diagrams).

If needed you can add more poles and zeros to your controller, but make sure that you keep the number of poles at least as many as the zeros. This will ensure that the system is *proper*, i.e. is not containing a pure derivative.

4.4

a. Calculate the closed-loop transfer function $G_{yr}(s)$ from reference r to output y with your controller in the loop. What would be the ideal frequency response for this transfer function?

b. The control signal $u(t)$ to the process is physically limited by

$$-10 \leq u(t) \leq 10,$$

which must be taken into account in the design. This limits how fast we can change the process.

Simulate the step response. Is the constraint on $u(t)$ satisfied? Improve $G_{yr}(s)$ by tuning the prefilter $F(s)$ so that the step response behaves nicely. How should $F(s)$ compensate $G_{yr}(s)$ in the frequency domain?

4.5 **(*)** A servo system has the transfer function:

$$G_o(s) = \frac{2.0}{s(s+0.5)(s+3)}$$

The closed system has a step response according to Figure 4.2. It is clear that the system is poorly damped and has a large overshoot. It is, however, fast enough. Create a lead controller that stabilizes the system by increasing the phase margin to $\phi_m = 50°$, without changing the cut-off frequency. ($\phi_m = 50°$ gives a relative damping of $\zeta \approx 0.5$ which achieves an overshoot of $M \approx 17\%$). The stationary error for the closed system is 0.75 for a ramp-shaped input signal. The error for a ramp function at the compensated system must not exceed 1.5.

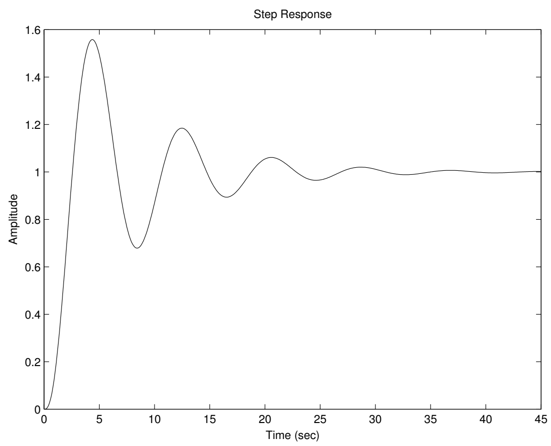

Figure 4.2 Step response from the closed servo system in 4.5.

4.6 **(*)** Consider the control system in Figure 4.1, where the plant is described by

$$P(s) = \frac{1}{(s+1)(s+0.02)}$$

and $F(s) = 1$. A non-control experienced engineer has designed the controller

$$C(s) = \frac{(s+r)}{s}$$

with $r = 0.02$, but the resulting control system reacts extremely slowly to step disturbances in d. The reason is that the slow pole in -0.02 is canceled by the controller zero.

The Bode diagrams of the plant, the controller, and the open-loop system are shown in Figure 4.3.

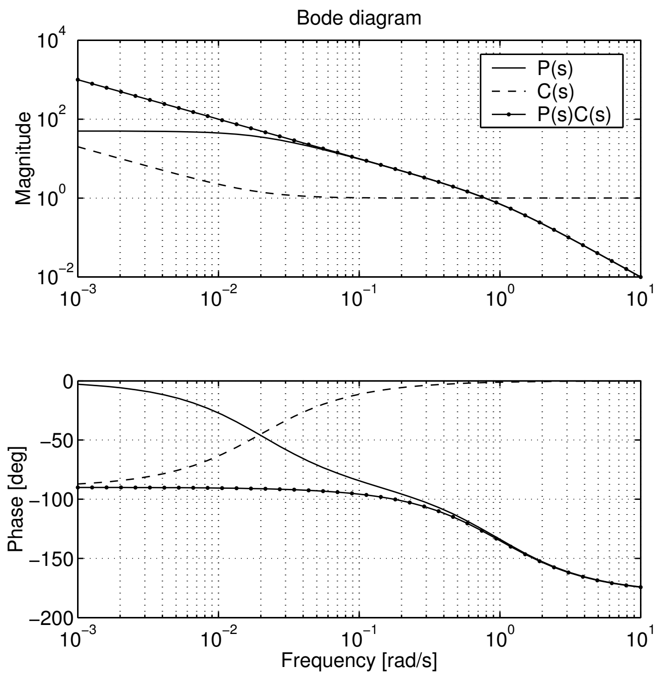

Figure 4.3 The Bode diagrams of $P(s)$, $C(s)$ and the open loop $P(s)C(s)$ when $r = 0.02$.

a. The load disturbance d is typically most significant at low frequencies, so we are interested in keeping the magnitude of the transfer function G_{yd} from d to y significantly smaller than 1 in a frequency range $[0, \omega_b]$. What is (approximately) ω_b if you use the given controller? Use the Bode diagram in Figure 4.3.

b. To reject the disturbance d faster, ω_b should be increased. For noise reasons, we want the cross-over frequency of the system to be the same.

How should the value of r in the controller be changed to achieve this? Motivate your design by showing that:

- The range $[0, \omega_b]$ where you get good disturbance rejection of d is increased.
- The cross-over frequency of the system is still approximately the same.

Exact proofs are not required; some Bode-diagram reasoning will do.

Exercise 5. Multivariable Zeros, Singular Values and Controllability/Observability

5.1 Consider the following system

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x$$

a. Show that the system is neither controllable nor observable. Also determine the uncontrollable and unobservable modes.

b. Determine the transfer function of the system and the order of a minimal state space realisation. How can this be related to the controllable and observable states of the system?

5.2 The following model of a heat exchanger was presented in the course book (see Example 2.2)

$$\dot{x} = \begin{pmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{pmatrix} x + \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x,$$

where the first state represents the temperature of the cold water and the second state is the temperature of the warm water.

a. Use Matlab to calculate the controllability Gramian.

b. What state direction is hardest to control?

5.3 (*) In the first exercise session we were given a rough model of the pitch dynamics of JAS 39 Gripen

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 & -1/2 & 0 \\ 4 & -1 & 0 & -25 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 3/2 & 1/2 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{pmatrix} u. \qquad (5.1)$$

Using Matlab:

a. Show that there is no scalar output signal that makes the system observable.

Hint: Use symbolic toolbox to determine a general C matrix and calculate the observability matrix. For instance, the following lines of Matlab code may help you:

```
>> syms c1 c2 c3 c4 c5
>> C = [c1 c2 c3 c4 c5]
>> Wo = ...
```

b. Let the output be

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} x(t).$$

Which are the non-observable modes?

5.4 In Figure 5.1 and Figure 5.2 you see an interconnection of two systems $P_1 = \dfrac{s+3}{s+2}$ and $P_2 = \dfrac{s+1}{(s+3)(s+4)(s-2)}$. One can notice that after multiplying the two systems we can cancel a pole and a zero in $s_0 = -3$. Usually it means that the whole system is not observable, or not controllable. Which of these two situations are depicted in the Figure 5.1 and Figure 5.2?

Figure 5.1 Block diagram for problem 5.4.

Figure 5.2 Block diagram for problem 5.4.

5.5 Consider the following transfer function matrix

$$G(s) = \begin{pmatrix} \dfrac{1}{s+2} & -\dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{s+1}{s+2} \end{pmatrix}$$

a. Determine the pole and zero polynomials for this system. What is the least order needed to realize the system on state space form?

b. Find a state-space realization of the system.

c. Use Matlab to draw a singular value plot for the system. What is the $\mathcal{L}_2$-gain of the system?

5.6 Consider the system

$$G(s) = \begin{pmatrix} 1 & 1/s \end{pmatrix}$$

with two inputs and one output.

a. Use Matlab to determine the singular values of the system at $\omega = 1$ rad/s, together with the input directions giving the maximum and minimum output gains respectively.

b. The derived input directions are complex. What does this mean? Explain why it's logical that these received input directions are those giving the smallest and highest system gains respectively for this particular system.

5.7 The following is an idealized dynamic model of a distillation column:

$$G(s) = \frac{1}{75s+1}\begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}$$

a. Using Matlab, plot the singular values of the process.

b. For the frequencies $\omega = 0$, 0.1 rad/s, calculate the gains of the system in the input directions $d_1 = [0.6713 \quad 0.7412]^T$ and $d_2 = [1 \quad 0]^T$, i.e. the amplification of the input $d_i \cdot sin(\omega)$ by the transfer matrix $G(s)$.

c. Determine the minimum and maximum output gains respectively at $\omega = 0$ rad/s as well as the input directions associated with them. Will the directions depend on frequency for this particular system? Explain your answer.

Exercise 6. Fundamental Limitations

6.1 Consider the ball in the hoop in Figure 6.1. This process consists of a cylinder rotating with the angular velocity ω. Inside the cylinder, a ball is rolling. The position of the ball is given by the angle θ and the linearized dynamics can be written as $\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$. Let $k = 1$ and $c = 2$.

a. What is the transfer function from cylinder velocity ω to the position θ of the ball? Where is the zero located?

b. What limitation on the sensitivity function for a stable closed loop system is imposed by the process zero?

c. What consequence does the process zero have on the static error when a reference signal $r(t)$, e.g. a step with the magnitude a, is to be followed. Let the control signal $\omega(t)$ be determined from the error signal $r(t) - \theta(t)$ via the controller transfer function $C(s)$. Give a physical interpretation.

d. Usually a controller integrator is introduced in order to remove static errors. How would the ball/hoop-system behave with a PI controller and a non-zero reference position for the ball?

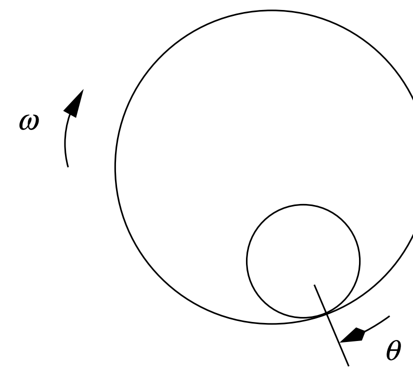

Figure 6.1 The ball in the hoop.

6.2 A resonant mechanical system has the pole-zero configuration shown in Figure 6.2. The controller structure is given by Figure 6.3.

a. What constraint does a purely imaginary process pole in $i\omega_p$ impose on the Bode diagram of the sensitivity function?

b. What consequence does this give for the control error e, in presence of a sinusoidal measurement disturbance n with frequency ω_p?

c. What effect does the controller $C(s)$ have on this response?

d. What constraint does a purely imaginary process zero in $i\omega_z$ impose on the sensitivity function?

e. What consequence does this give for the response x, in presence of a sinusoidal measurement disturbance n with a frequency ω_z?

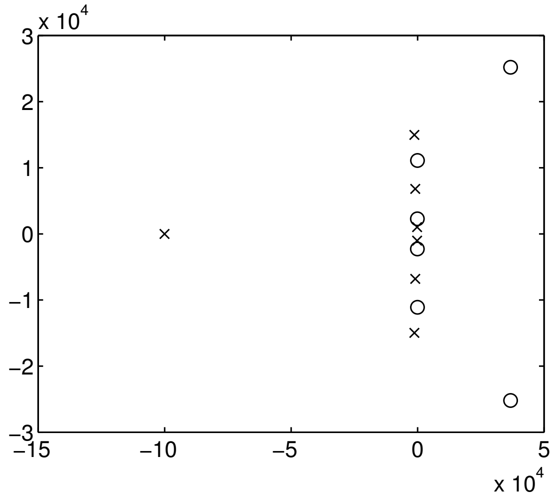

Figure 6.2 Pole-zero configuration of a resonant mechanical system.

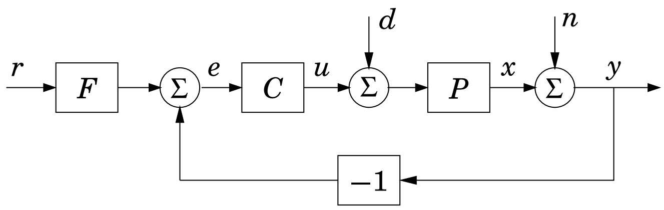

Figure 6.3 Two degree of freedom controller structure.

6.3 Consider the setup in Figure 6.3 with $P(s) = (3-s)/(s+1)^2$

a. Does there exist a stabilizing controller $C(s)$ such that the transfer function from n to x becomes $5/(s+5)$? (Note: All transfer functions in the gang of four must be stable.)

b. Show that the specification

$$|S(i\omega)| \leq \frac{2\omega}{\sqrt{\omega^2+36}} \qquad \omega \in \mathbb{R}$$

is equivalent to

$$\sup_{\omega} |W_a(i\omega)S(i\omega)| \leq 1$$

with $a = 6$ and

$$W_a(s) = \frac{s+a}{2s}$$

Is this specification possible to satisfy ?

c. Use Matlab to find a stabilizing controller $C(s)$ such that

$$\left|\frac{1}{1+PC}(i\omega)\right| \leq \frac{2\omega}{\sqrt{\omega^2+1}} \qquad \text{for } \omega \in [0,2]$$

Hint: Use a PI controller on the form:

$$C(s) = K\frac{s/b+1}{s}$$

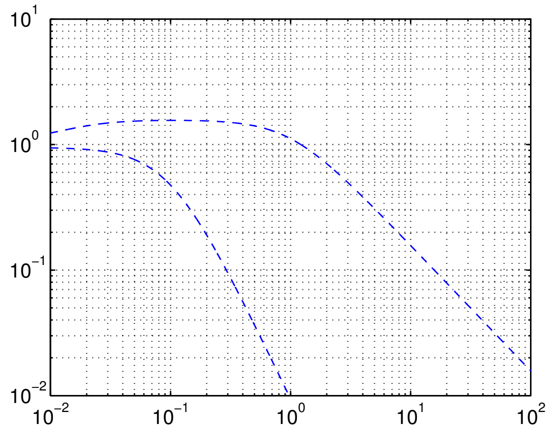

Figure 6.4 Gain specification for the closed-loop transfer function T in problem 6.4.

6.4 For each of the following three design problems, state if it is possible to construct a controller that can achieve the given specification. Motivate your answers! (Hint: It is *not* possible in *at least* two of the cases.)

System:	Specification:
$P_1(s) = \dfrac{e^{-2s}}{s+2}$	The step response must reach 0.9 before $t = 1$.
$P_2(s) = 3\dfrac{(s+40)(s-20)}{s^2(s-10)}$	The gain curve of the closed-loop transfer function T should lie between the two gain curves depicted in Figure 6.4.
$P_3(s) = \dfrac{1}{s-3}$	The step response must stay in the interval $[0, 2]$ for all t.

6.5 **(*)** The specifications

$$\sup_\omega |W_S(i\omega)S(i\omega)| \le 1 \qquad \sup_\omega |W_T(i\omega)T(i\omega)| \le 1$$

can be used to make sure that the sensitivity is small in a low frequency range and measurement noise is rejected in a high frequency range.

a. Show that the two specifications are incompatible if

$$|W_S(s)| = |W_T(s)| > 2$$

for some right half plane s. (Hint: Use that $S + T = 1$.)

b. Show that the two specifications are incompatible if

$$W_S(s) = \left(\frac{s+0.1}{s}\right)^n \qquad W_T(s) = \left(\frac{s+10}{10}\right)^n$$

and $n \ge 8$.

Hint: Use the result of **a**.

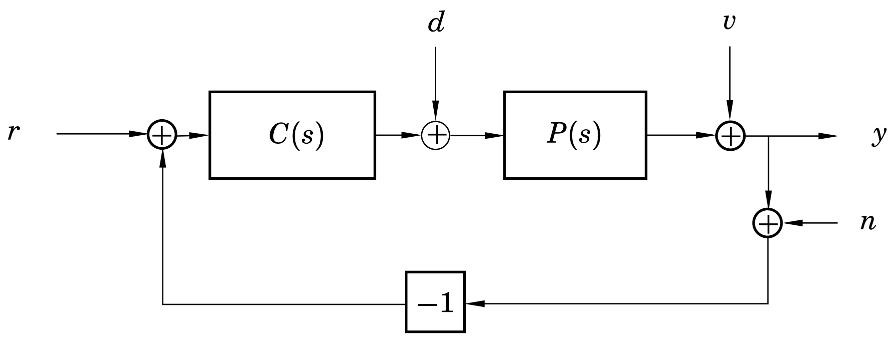

Figure 6.5 System in Problem 6.6

6.6 A multi-variable system (block diagram in Figure 6.5) is supposed to attenuate all output load disturbances (v) with at least a factor 10 for frequencies *below* 0.1 rad/sec. Constant output load disturbances should be attenuated by at least a factor 100 in stationarity.

Furthermore, the system should also attenuate measurement disturbances (n) with at least a factor 10 for frequencies *above* 2 rad/sec.

a. Formulate specifications on (the singular values of) S and T that guarantee the above requirements.

b. Re-formulate the specifications in **a** using $\|\cdot\|_\infty$ and the weighting functions W_S and W_T.

c. (*) Give conditions on the open-loop gain $|L(i\omega)| = |P(i\omega)C(i\omega)|$ that are sufficient to fulfill these specifications.

d. (*) What cut-off frequency and what phase margin could have been expected, following your answer in **c**, if the system had been SISO? What lower bound on $\|T\|_\infty$ does this give?

6.7 Consider the setup in Figure 6.3 with

$$P(s) = \frac{6-s}{s^2+5s+6}$$

Give an upper bound for how fast the closed loop system can be made. That is, give a value a such that the following specification is impossible to satisfy if $c > a$.

$$|S(i\omega)| \leq \frac{2\omega}{\sqrt{\omega^2+c^2}} \qquad \omega \in \mathbb{R}$$

Exercise 7. Controller Structures and Preparations for Laboratory Exercise 2

Note: Exercises 7.1-7.3 serve as preparation for Laboratory Excercise 2.

7.1 **a.** Give the definition of RGA for a complex valued, not necessarily square, matrix A. How do you apply it to a process $G(s)$ and what information can be extracted in an automatic control perspective?

b. Let

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix}.$$

Compute RGA($G(0)$).

c. What input-output pairing would you recommend be used in a decentralised control structure?

7.2 Consider the MIMO process

$$P(s) = \begin{pmatrix} \dfrac{1}{s+1} & 0 & 0 \\ 0 & \dfrac{0.1}{s+10} & \dfrac{1}{s+10} \\ \dfrac{0.1}{s+1} & \dfrac{1}{s+1} & 0 \end{pmatrix}.$$

Compute the relative gain array, RGA, of $P(0)$ and suggest an input-output pairing for the system based on this.

Hint: The inverse of $P(s)$ is given by

$$P(s)^{-1} = \begin{pmatrix} s+1 & 0 & 0 \\ -0.1(s+1) & 0 & s+1 \\ 0.01(s+1) & s+10 & -0.1(s+1) \end{pmatrix}.$$

7.3 Figure 7.1 shows the quadruple-tank process that will be used in Lab 2. The goal is to control the levels in the lower tanks (y_1, y_2) using the pumps (u_1, u_2). For each tank $i = 1 \ldots 4$, mass balance and Bernoulli's law give that

$$A_i \frac{dh_i}{dt} = -a_i\sqrt{2gh_i} + q_{in} \tag{7.1}$$

where A_i is the cross-section of the tank, h_i is the water level, a_i is the cross-section of the outlet hole, g is the acceleration of gravity, and q_{in} is the inflow to the tank. The non-linear equation (7.1) can be linearized around a stationary point (h_i^0, q_{in}^0), giving the linear equation

$$A_i \frac{d\Delta h_i}{dt} = -a_i\sqrt{\frac{g}{2h_i^0}}\,\Delta h_i + \Delta q_{in} \tag{7.2}$$

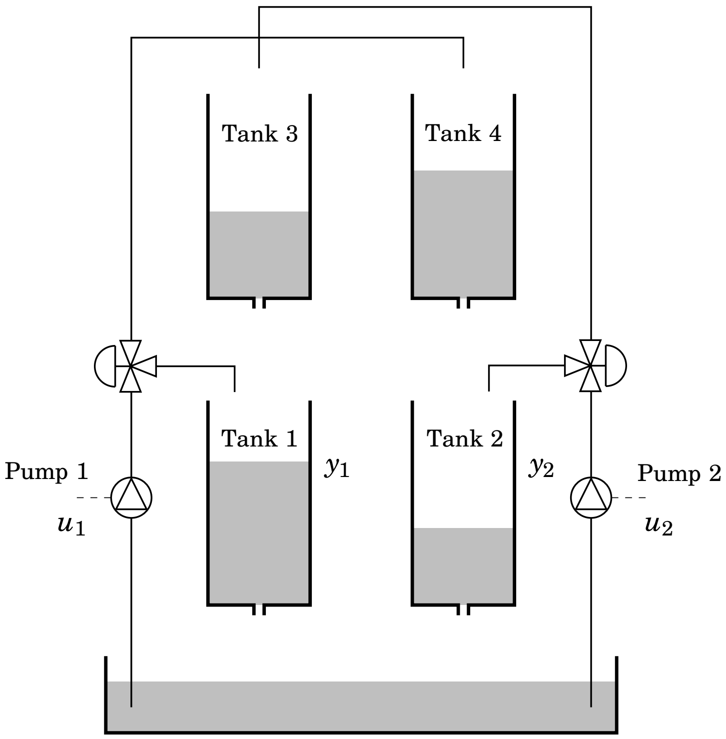

Figure 7.1 The quadruple-tank process.

where $\Delta h_i = h_i - h_i^0$, and $\Delta q_{in} = q_{in} - q_{in}^0$ denote deviations around the stationary point.

The flows from the pumps are divided according to two parameters $\gamma_1, \gamma_2 \in (0,1)$. The flow to Tank 1 is $\gamma_1 k_1 u_1$ and the flow to Tank 4 is $(1-\gamma_1)k_1 u_1$. Symmetrically, the flow to Tank 2 is $\gamma_2 k_2 u_2$ and the flow to Tank 3 is $(1-\gamma_2)k_2 u_2$.

a. Let $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$, and $\Delta y_i = y_i - y_i^0$. Verify that the linearized dynamics of the complete quadruple-tank system are given by

$$\begin{aligned}
\frac{d\Delta h_1}{dt} &= -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_1^0}}\,\Delta h_1 + \frac{a_3}{A_1}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{\gamma_1 k_1}{A_1}\,\Delta u_1 \\
\frac{d\Delta h_2}{dt} &= -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_2^0}}\,\Delta h_2 + \frac{a_4}{A_2}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{\gamma_2 k_2}{A_2}\,\Delta u_2 \\
\frac{d\Delta h_3}{dt} &= -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{(1-\gamma_2)k_2}{A_3}\,\Delta u_2 \\
\frac{d\Delta h_4}{dt} &= -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{(1-\gamma_1)k_1}{A_4}\,\Delta u_1
\end{aligned}$$

Introduce the input vector, u, output vector, y, and state vector, x, as

$$u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \quad x = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix}, \quad y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

Verify that the linearized system can be written on state-space form as

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{pmatrix} x + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix} u,$$

$$y = \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} x,$$

where $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$, and k_c is a measurement constant.

b. Show that the transfer matrix from u to y is given by

$$P(s) = \begin{pmatrix} \frac{\gamma_1 c_1}{1+sT_1} & \frac{k_2}{k_1}\cdot\frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \frac{k_1}{k_2}\cdot\frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

where $c_1 = T_1k_1k_c/A_1$ and $c_2 = T_2k_2k_c/A_2$.

Hint: Use the fact that

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & -\frac{b}{ae} & 0 \\ 0 & \frac{1}{c} & 0 & -\frac{d}{cf} \\ 0 & 0 & \frac{1}{e} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix}$$

c. The zeros are given by the equation

$$\det P(s) = \frac{c_1c_2(\gamma_1\gamma_2(1+sT_3)(1+sT_4) - (1-\gamma_1)(1-\gamma_2))}{(1+sT_1)(1+sT_2)(1+sT_3)(1+sT_4)} = 0$$

which is simplified to

$$(1+sT_3)(1+sT_4) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2} = 0.$$

Show that the system is minimum phase (i.e., that both zeros are stable) if $1 < \gamma_1 + \gamma_2 < 2$, and that the system is non-minimum phase (i.e., that at least one zero is unstable) if $0 < \gamma_1 + \gamma_2 < 1$.

Hint: A second-order polynomial has all of its roots in the left half plane if and only if all coefficients have the same sign.

In the lab, we will first study the case $\gamma_1 = \gamma_2 \approx 0.7$, and then the case $\gamma_1 = \gamma_2 \approx 0.3$. In which case will the process be more difficult to control?

d. Show that the RGA for $P(0)$ is given by

$$\begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$$

where $\lambda = \gamma_1\gamma_2/(\gamma_1+\gamma_2-1)$.

Based on this RGA matrix, suggest an input-output pairing in the two cases $\gamma_1 = \gamma_2 \approx 0.7$ and $\gamma_1 = \gamma_2 \approx 0.3$.

7.4 Consider the following multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10s+1} & \frac{-2}{2s+1} \\ \frac{1}{10s+1} & \frac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

a. By using RGA at $\omega = 0$ rad/s, decide the input-output pairing that should be used in a decentralized control structure.

b. Another approach is to use decentralized control, that is, we want to use a controller that can be described by

$$F^{\text{diag}}(s) = \begin{pmatrix} F_{11}(s) & 0 \\ 0 & F_{22}(s) \end{pmatrix}.$$

Also, we want the control loops to be decoupled in stationarity. Give the structure of such a controller $F(s)$ expressed in $F^{\text{diag}}(s)$ that is capable to do so. *Hint: Use a suitable decoupling matrix.*

7.5 (*) In this exercise we will try to design controllers for a 2x2-process, that is, a process that has 2 inputs and 2 outputs. The process is described by the transfer function matrix

$$G(s) = \begin{pmatrix} \frac{4}{s+1} & \frac{3}{3s+1} \\ \frac{1}{3s+1} & \frac{2}{s+0.5} \end{pmatrix}.$$

Design two different decentralized controllers for the process.

1. Decentralized control, using the RGA of the process.
2. Decentralized control, using decoupling with respect to stationarity

In both cases, use ordinary PI-controllers. Use the step responses to evaluate the performance of the loop.

Exercise 8. Linear Quadratic Optimal Control

8.1 Consider the first order unstable process

$$\begin{aligned} \dot{x}(t) &= ax(t) + u(t) \\ y(t) &= x(t) \end{aligned}$$

where the state is measured without any noise.

a. Design, analytically, an LQ-controller that minimizes the criterion

$$J = \int_0^\infty \left(x^2(t) + Ru^2(t) \right) dt.$$

We want a stationary gain of 1 from the reference to the output. Design therefore a feedforward gain L_r such that the control signal is given by

$$u(t) = -Lx(t) + L_r r(t),$$

and achieves the performance specification.

b. Do the design for different R using Matlab when assuming $a = 1$, and plot the position of the closed loop pole as a function of R. Conclusion?

8.2 Consider the second order system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \end{aligned}$$

Design an LQ controller, with equal weight on output and control signal, by

1. Solving the algebraic Riccati equation in Matlab using `care`.
2. Using `lqry` in Matlab. Simulate the closed loop system from the initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

8.3 Consider a process

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that

$$u(t) = -\begin{pmatrix} 2 & -3 \end{pmatrix} x(t)$$

can *not* be an optimal state feed-back designed using LQ-technique with the cost function

$$J = \int_0^\infty (x^T(t) Q_1 x(t) + Q_2 u^2(t))\, dt$$

where Q_1 and Q_2 are positive definite matrices.

Hint: Look at the Nyquist plot of the loop gain.

8.4 Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u \\ y &= (1 \quad 1)\, x \end{aligned}$$

One wishes to minimize the criterion

$$V(T) = \int_0^T x^T(t) Q_1 x(t) + Q_2 u^2(t) dt$$

Is it possible to find positive definite weights Q_1 and Q_2 such that the cost function $V(T) < \infty$ as $T \to \infty$?

8.5 We would like to control the following process with linear quadratic optimal control, that is, LQ-technique.

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} u(t) \\ z(t) &= (0 \quad 1)\, x(t) \end{aligned}$$

The weight on $x_1(t)^2$ should be 1, on $x_2(t)^2$ we want 2. On the control signal $u(t)^2$ we will try different values, $R = 0.01,\ 10,\ 1000$.

a. Determine the cost function matrices for the three different cases.

b. In Matlab, calculate the three different resulting controllers, calculate the resulting closed loop poles and do step responses. Make sure that there is no static error in the step responses!

8.6 Consider the double integrator

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

a. Design an LQ-controller $u(t) = -Lx(t) + L_r r(t)$ that minimizes the criterion

$$J = \int_0^\infty x^T(t) Q_1 x(t) + Q_2 u^2(t) dt$$

with

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = 0.1$$

The stationary gain from reference $r(t)$ to output $z(t)$ should be equal to 1.

b. What measurements are needed by the controller?

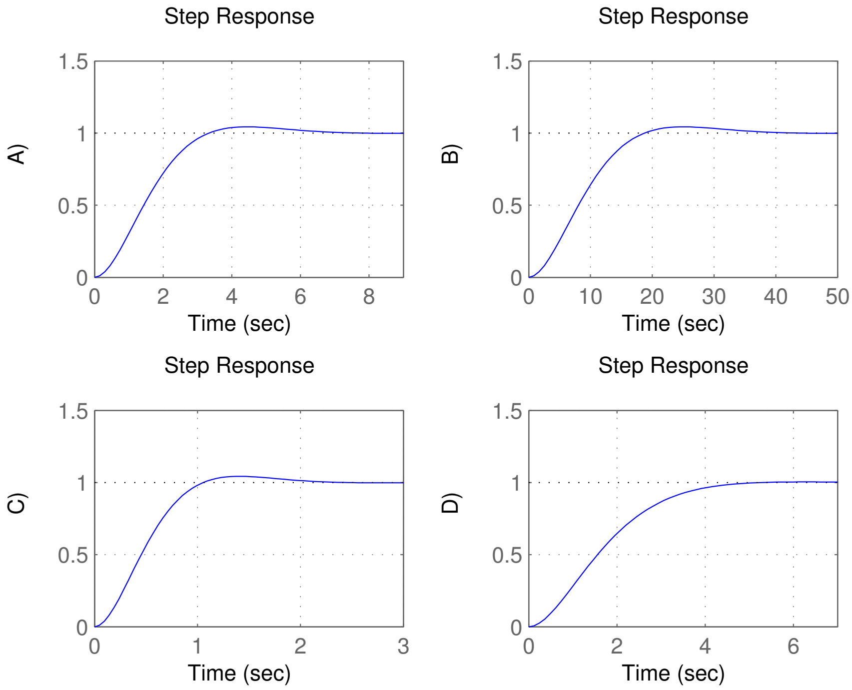

Figure 8.1 Step responses for LQ-control of the system in Problem 8.6 with different weights on Q_1, Q_2.

c. The four plots in Figure 8.1 show the step responses of the closed loop system for four different combinations of weights, Q_1, Q_2. Pair the combinations of weights given below with the step responses in Figure 8.1.

1.
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.01$$

2.
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1$$

3.
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

4.
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1000$$

8.7 **(*)** Consider the double integrator

$$\ddot{\xi}(t) = u(t).$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

where $y = (\xi(t), \dot{\xi}(t))$, i.e., both states are measured. You would like to design a controller using the criterion

$$\int_0^\infty (\xi^2(t) + \eta \cdot u^2(t))\, dt$$

for some $\eta > 0$.

a. Show that $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ with

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. What are the closed loop poles of the system when using this optimal state feedback? What happens with the control signal if η is reduced?

Exercise 9. Kalman Filtering/LQG

9.1 Consider the first order unstable system

$$G(s) = \frac{1}{s-1}$$

with the state space representation with additive noise

$$\begin{aligned}\dot{x}(t) &= x(t) + u(t) + v_1(t)\\ z(t) &= x(t)\\ y(t) &= x(t) + v_2(t)\end{aligned}$$

The noise signals $v_i(t)$ are white with intensities R_i. We are about to investigate how the optimal Kalman filter depends on the R_i's.

a. Show that the optimal Kalman filter only depends on the ratio $\beta = R_1/R_2$.

b. Find the error dynamics, i.e., the dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$.

c. How does the error dynamics depend on the ratio $\beta = R_1/R_2$? Interpret the result for large β (process noise much larger than measurement noise), and for small β (measurement noise much larger than process noise).

9.2 A Kalman filter should be designed for the second order system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1(t)\\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t)\end{aligned}$$

where v_i are white noise with intensity 1.

Design the Kalman filter by

a. solving the algebraic Riccati equation by using `care` in Matlab.

b. using `lqe` in Matlab.

9.3 Consider the first order stable system

$$G(s) = \frac{1}{s+1}$$

with the state space representation with additive noise

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t) + v_1(t)\\ z(t) &= x(t)\\ y(t) &= x(t) + v_2(t)\end{aligned}$$

The noise signals $v_i(t)$ are white with intensities 1. Often, we have load disturbances acting on the system, hence there is a need for integral action for acceptable control. Using LQ-techniques in designing a state-feedback controller do not automatically give integral action. One way to introduce integral action is to model the disturbance as filtered white noise and use a Kalman filter to estimate the disturbance.

The load disturbance is then modelled as a signal w that influences y and z

$$z(t) = x(t) + w(t)$$
$$y(t) = x(t) + w(t) + v_2(t)$$

In order to model the static error in z and y, $w(t)$ should have large low-frequncy content. To use a Kalman filter to estimate the error, we need to find a filter $H(s)$ that generates the signal w from a white noise process n

$$w = H(s)n.$$

For true integral action we want $H(s) = 1/s$, but with this model the noise state will be neither controllable nor stable, and we will not be able to design an LQG controller for the extended system. To get around this problem, we replace the pure integrator by a first order system

$$H(s) = \frac{1}{s+\delta}$$

for some small δ.

a. Find a state-space realization of the extended system, including the noise model

$$\dot{x}_e = A_e x_e + B_e u_e + N_e v_{1e}$$
$$y = C_e x_e + v_2$$
$$z = M_e x_e$$

where $v_{1e} = \begin{pmatrix} v_1 \\ n \end{pmatrix}$

b. Design the full LQG-controller in Matlab using the extended model. Be sure to have small weight on $u(t)$. Why?

c. Examine the Bode plot of the controller. How does the (almost) integral action in the controller change when changing δ and the noise variance corresponding to the added state?

d. Will the response to constant load disturbances have a static error?

9.4 Consider control of a DC-motor,

$$G(s) = \frac{1}{s(s+1)}$$

White process noise is active on both states with intensity 1 and with input vector $(0.1 \quad 0.1)^T$. There is also noise on the output with intensity 0.1. Let the states be $x_1 = y$, $x_2 = \dot{y}$. This gives the following state space model

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} v_1(t) \\ y(t) &= \underbrace{(1 \quad 0)}_{C} x(t) + v_2(t)\end{aligned}$$

with $R_1 = 1$, $R_2 = 0.1$ and $R_{12} = 0$

One wishes to use the motor together with a system that might be oscillatory at the frequency 0.5 rad/s, but there is not much knowledge about its properties.

a. How can you change the model such that the LQG-controller will have good robustness at this frequency (a small complementary sensitivity function)? Derive this extended model and determine the intensity matrices needed to solve for the Kalman filter gain.

b. Compute the Kalman filter using `lqe` in Matlab. Plot the transfer function from $y(t)$ to $\hat{y}(t) = C\hat{x}(t)$. Can you see the implication of the noise modelling?

9.5 Consider the problem of estimating the states of a double integrator where noise with variance 1 effects the input only and we have measurement noise of variance 1.

a. Determine the optimal Kalman filter.

b. What are the Kalman filter poles?

Exercise 10. LQG and Preparations for Laboratory Exercise 3

10.1 Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix} u + v_1 \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x + v_2 \\ z &= \begin{pmatrix} 1 & 1 \end{pmatrix} x\end{aligned}$$

a. Design an LQG controller for the system, assume initially that process and measurement noise are independent and have intensity 1, and that we should weight the control signals u and output z exactly the same.

Useful commands: `lqry, kalman`

b. Using the states x and $\hat{x}$, write the closed loop system in state-space form using letters. Use L for state-feedback gain and K for Kalman filter gain.

c. Simulate the system without noise with the initial state $x = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Plot both process states and estimated states. The Kalman filter begins with its estimates in 0. Try different noise intensities, any conclusions?

Useful commands: `lqgreg, feedback, initial`

10.2 Consider the problem of controlling a double integrator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1$$

where the white noise v_1 has intensity I. We can only measure x_1, unfortunately with added white noise also of intensity 1. We want to minimize the cost

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2 \right) dt.$$

Solve the control problem by hand (not using Matlab) and give the controller on state-space form.

10.3 Do preparatory exercises for Laboratory 3 – Crane with rotating load. The lab manual is found on the course homepage.

Exercise 11. Youla Parametrization and Dead Time Compensation

11.1 Consider the control system in Figure 11.1, designed around the SISO system P_0.

We first want to rewrite the system to the more general form presented in Figure 11.2. In this figure: w are the external inputs to the system (e.g. disturbances and reference), z gathers all signals that we are interested in controlling, u are the control signals from K, and y contains all signals used by the controller (e.g. reference and measurements).

a. Choose

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad z = \begin{pmatrix} x \\ v \end{pmatrix}$$

P in Figure 11.2 consists of a number of different subsystems as

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

shows. Derive P and determine P_{zw}, P_{zu}, P_{yw} and P_{yu} as transfer function matrices.

b. Call the closed loop system (from w to z) H. Determine this transfer function matrix and rewrite it in terms of the sensitivity function S and complementary sensitivity function T. Use the formula $H = P_{zw} + P_{zu}C(1 - P_{yu}C)^{-1}P_{yw}$. Note that we normally have a -1 in the feedback loop. Here it is assumed that this sign is part of the controller instead, hence the minus sign in $(1 - P_{yu}C)$.

c. Rewrite H using the Q parameterization $Q = C(1 - P_{yu}C)^{-1}$. Show that all the elements in H are linear in Q.

d. Using the Q above, show that C will be an IMC controller.

11.2 *Note: It is recommended that you solve this problem before you start on Exercise 12.*

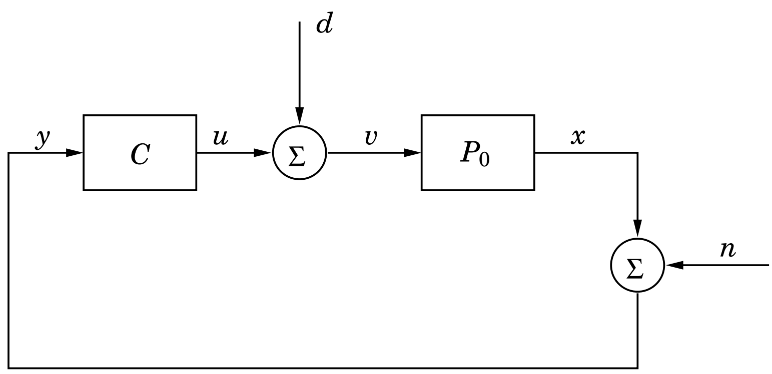

Figure 11.1 The block diagram of the closed loop system in 11.1.

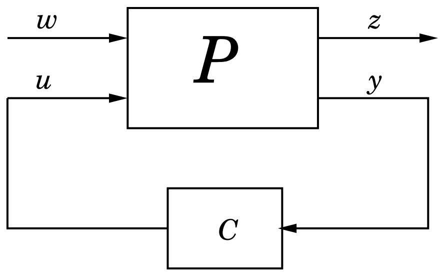

Figure 11.2 General form of a closed loop system.

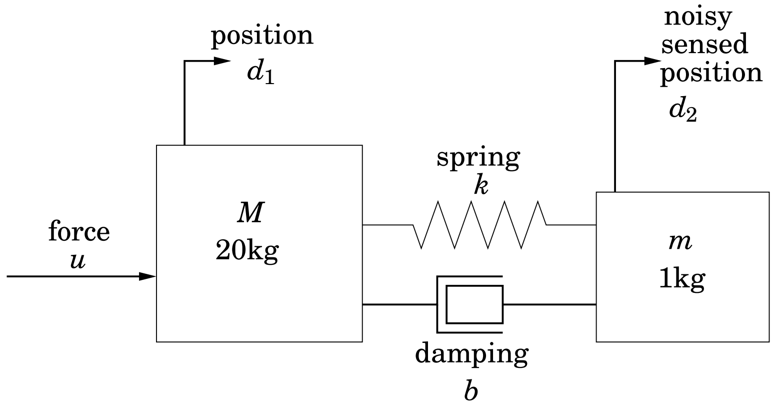

Figure 11.3 Mass spring system in Exercise 11.2.

Let us consider the physical system shown in Figure 11.3, showing two masses, lightly coupled through a spring with spring constant k and damping b. The only sensor signal we have is the noisy measurement $d_2 + n$ of the position, d_2, for the small mass, m. The purpose of the controller is to make the position of the large mass, d_1, follow a reference input, r, such that the control error e becomes small. This is in turn weighted against controller effort in a quadratic cost function (the objective)

$$J = \int_0^\infty \gamma e^2(t) + \rho u^2(t) dt$$

Minimization of this function will be subject to constraints on:

- maximum magnitude of the control signal $\|u(t)\| < u_{max}$ (the force acting on the large mass) during a reference step
- step response overshoot, rise time and settling time from r to the position d_1 (performance constraint)
- the maximum norm of the sensitivity function, $\|S(i\omega)\|_\infty \leq M_s$ (robustness constraint)

The system can be described by the equations of motion

$$M\ddot{d}_1 + b(\dot{d}_1 - \dot{d}_2) + k(d_1 - d_2) = u$$
$$m\ddot{d}_2 + b(\dot{d}_2 - \dot{d}_1) + k(d_2 - d_1) = 0$$

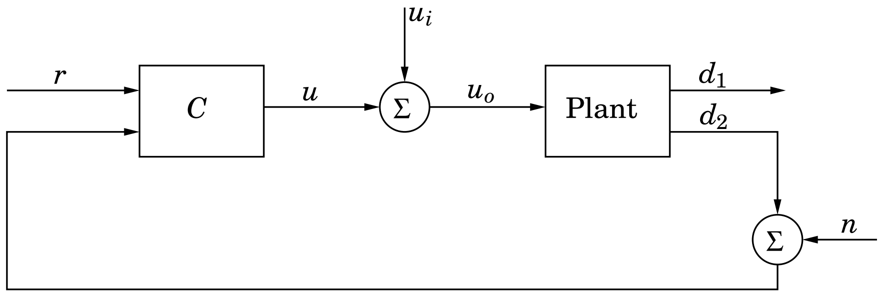

Figure 11.4 The block diagram of the closed loop system in 11.2.

Setting the plant states to

$$x = \begin{pmatrix} \dot{d}_1 \\ d_1 \\ \dot{d}_2 \\ d_2 \end{pmatrix}$$

we can rewrite the system on state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ d_1 &= C_1 x \\ d_2 &= C_2 x \end{aligned}$$

where

$$\begin{aligned} A &= \begin{pmatrix} -b/M & -k/M & b/M & k/M \\ 1 & 0 & 0 & 0 \\ b/m & k/m & -b/m & -k/m \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 1/M \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ C_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\ C_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $M = 20$kg, $m = 1$kg, $k = 32$ N/m and $b = 0.3$ Ns/m.

Now, consider the problem to set up this system on a form such that we can optimize over the Q parametrization. Have Figure 11.4 as a reference. Then, the exogenous signals, w, of the system are

- the reference r,
- the noise input n,
- a loop input u_i (used for the robustness constraint).

The exogenous outputs, z, are

- the position, d_1, of the large mass M,
- the actuator input to the plant, u_o,
- the control error $e = r - d_1$.

The control signal, u, to the plant is

- the force u on the large mass M.

The sensed (measured) outputs y (i.e. those accessible to the controller) are

- the reference r,
- the noisy measurement $d_2 + n$.

In other words, we have

$$w = \begin{pmatrix} r \\ n \\ u_i \end{pmatrix}, \quad z = \begin{pmatrix} d_1 \\ u_o \\ e \end{pmatrix}, \quad y = \begin{pmatrix} r \\ d_2 + n \end{pmatrix}.$$

With these variables, we can rewrite the system on the more general form shown in Figure 11.2. On state space form, this becomes

$$\dot{x} = Ax + B_w w + Bu \tag{11.1}$$

$$z = C_z x + D_{zw} w + D_{zu} u \tag{11.2}$$

$$y = C_y x + D_{yw} w + D_{yu} u \tag{11.3}$$

a. Determine all matrices in equations (11.1)-(11.3).

b. On the next exercise session we will use software that solves the minimization problem. This software will need to know the general process P, determined by (11.1)-(11.3), and the element indices of the closed-loop transfer function H corresponding to the constraints and cost function specified for the control design problem. For instance, the step response overshoot, rise time and settling time will correspond to $H_{d_1 r}$ which has index $(1, 1)$. Determine the rest of these indices.

c. How many inputs and outputs will the Q parametrization filter have?

11.3 Derive a controller using the IMC method on the following system

$$P(s) = \frac{6 - 3s}{s^2 + 5s + 6}.$$

Show that the controller has the form of a PID controller and a first order filter, i.e.

$$K\left(1 + \frac{1}{T_i s} + T_d s\right)\frac{1}{sT + 1}$$

11.4 Processes in industry often have time delays that give phase lags with the result of limiting the achievable performance, resulting in a *fundamental* *limitation*. Model based control structures that give good performance for such processes are available. Consider the simple process

$$P(s) = \frac{1}{s+1}\, e^{-4s},$$

which is clearly delay dominant (time delay larger than time constant). Use IMC to design a delay compensating controller for this process. Draw the Nyquist diagram for the loop transfer function and conclude if the closed loop system is stable.

Exercise 12. Synthesis by Convex Optimization

12.1 *Note: It is recommended that you do problem 11.2 before you start this exercise session. If you have done 11.2, but don't remember it, you should read through the problem text again.*

Last exercise session we were introduced to the mass spring system shown in Figure 11.3. We will now use the general system we derived there in order to find an optimal controller based on the Q parametrization. The Matlab tool we will use to find this controller can be found on `http://www.control.lth.se/course/FRTN10/exercises.html` and was developed here at the department. In short, the tool needs the user to give it the general system, P (see Figure 11.2) and point at the indices in the closed loop system to which the constraints and objective function correspond to (this has already been done in advance this time, so you will not need to worry about it). The user will also need to specify these constraints and objective to the tool. The tool then uses the software packages `yalmip` and `sedumi` in order to carry out the optimization of the specified problem. By editing the m-file called `spring_mass_problem.m`, you will be able to modify the given objective and constraints.

In the course we have only worked in continuous time. The Matlab tool that we will use does however work in discrete time, but could just as well have been written for continuous time. This exercise have been written such that you will get as little contact with discrete time as possible. You should merely look at the discrete time part of the tool as an approximation of the continuous time system, which is used in order to solve your optimization problem numerically. What you may need to know however, is that the constraints will be divided into a finite number of points. For instance, the constraints on the signals (in the time domain) are split into 170 time points with an interval of 0.2 seconds. These constraints will therefore only be active from 0 to 34 seconds and it is important that this interval has been chosen to be wide enough for the system. The M_s-constraint, which is in the frequency domain, has been divided into a fixed number of frequency points in a similar manner.

a. Go through `spring_mass_problem.m` and try to understand it. Fill in the blanks in the program such that:

- You use the system matrices from the previous exercise session.
- The order of the two Q-filters are 10 each.
- The unit step response in d_1 from r has a maximum overshoot of 2 % from the final value.
- The time constant of step response should not be greater than 3.6 seconds.
- The settling time constraints become active after 19 seconds.
- $p = 0.8$ (gives the decay rate 20 %). p is related to settling time constraints by the functions $1 + kp^t$ and $1 - kp^t$. k is set automatically by the program to match the overshoot constraint.

- The maximum magnitude of the control signal is $u_{max} = 6$.
- The maximum amplitude on the sensitivity function is $M_s = 1.4$ (gives at least phase margin 41.8° and gain margin 3.5).
- We get the weights $\gamma = 0.5$ and $\rho = 1.0$ on the cost function J.

The constraints on the unit step response from r to d_1 are visualized in Figure 12.1.

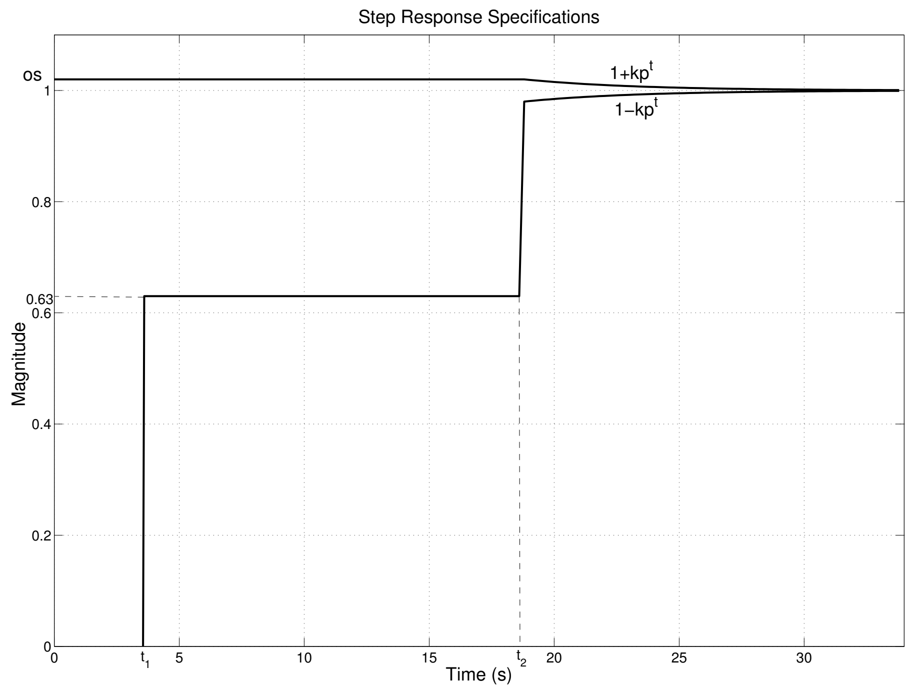

Figure 12.1 Constraints on the closed loop step response from the reference, r, to the position of the first mass, d_1. *os* is the maximum allowed overshoot.

b. The mass-spring system has two poles in $s = 0$ and is thus unstable. In order to get the Q optimization working, it is therefore necessary to have a so called nominal controller that stabilizes the plant. The final controller will then be optimized with respect to this stabilized closed loop system. In the given tool for Q optimization, this nominal controller is set by the software (LQG controller) and will therefore not need any of our attention. The plots given by the program will however show two solutions, one of which correspond to the nominal controller and one to our optimal controller.

Run the program, with the given setup, and identify which plots correspond to the nominal and optimal controller respectively. What constraints are active? A constraint is active if the solution touches this constraint in any points.

c. Plot the Bode diagram of the controller. Can you recognize any resemblance with any other type of controller? Can you give some intuition to any of the dips in the magnitude plot?

d. Every time you run the program you should be given some text similar to this:

```
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 510 free variables in a quadratic cone
eqs m = 531, order n = 826, dim = 1725, blocks = 53
nnz(A) = 12452 + 1, nnz(ADA) = 20671, nnz(L) = 10601
Handling 2 + 2 dense columns.
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            4.52E+03 0.000
  1 :  -4.12E+02 2.98E+02 0.000 0.0660 0.9900 0.9900   1.13  1  1  1.4E+00
  2 :  -5.40E+00 2.08E+02 0.000 0.6984 0.9000 0.9000   6.00  1  1  3.5E-01
...
 48 :  -1.40E+02 3.90E-08 0.000 0.1878 0.9000 0.9000   0.97 43 35  1.4E-06
Run into numerical problems.
```

Lets not go into detail what all columns stand for, but rather look only at the `feas` one. The number you end up with should ideally be 1, or at least close by, in order for you to have a fully feasible solution (a solution that satisfies all constraints you have posed). If you end up with a value of -1, then you can know for sure that your optimization did not succeed. The higher you have chosen the order of your Q-filters, the more likely it is that the optimization problem will succeed if there are any feasible solutions to be found. If we do not find a feasible solution, even though the order of the Q-filters is large, this tells us that we should try to loosen our constraints a bit if we want to find a usable controller.

Take your code and decrease `NQ=n_q1=n_q2` until your problem is no longer feasible. What constraint will fail first? What is the least order `NQ` you need your Q-filters to have in order to get feasibility?

e. Increase the order of Q_1 and Q_2 simultaneously (`NQ=n_q1=n_q2`) from the value you received in the previous subproblem and plot `NQ` against cost function value. Explain the shape of this plot and comment on how the max value of the control signal changes when `NQ` increases.

f. Change the weights on the objective function and see how this alters the solution. Also play around with the constraints to see if you can achieve an extremely fast step response. Explain your results.

g. Go back to your original setup of constraints and objective function. Change your maximum allowed overshoot on the step response to see how tight this constraint can be made before the solution becomes infeasible. Can you play around with some of the other constraints and the order of the Q-filters to make the problem feasible again? Is it possible to have no overshoot at all?

h. (*) Take a look in the other m-files and see if you can find the closed loop transfer function matrix (*Note: It will be in discrete time*). Plot all 9 (Why 9?) Bode diagrams. Point out at least one of these that you would have wished had a different appearance. How would you have prefered it to look and why?

i. (*) Change the system and see how this affects the solution. For instance, you can try to make both k and b small to see if you can make the system very hard to control with the constraints we have have. Play around with different constraints, weights on the objective and order of the Q-filters to see if you can find a decent controller for the new system.

Exercise 13. Controller simplification

13.1 Consider a SISO system for which the pole-zero map is given in figure 13.1.

a. Determine the transfer function of the system. You can assume that the static gain is $G(0) = 1$.

b. By studying the pole-zero map, it is possible to get a hint that the system is a candidate for model order reduction. How?

c. Use a computer to calculate a balanced realization and the Hankel singular values of the system. Perform a model reduction by eliminating the state corresponding to the smallest singular value.

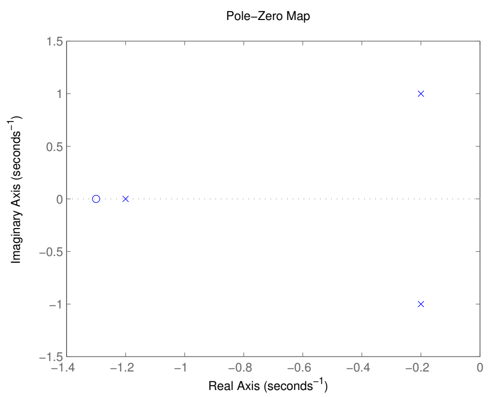

Figure 13.1 Pole-zero map of the system in problem 13.1

13.2 For the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 10u$$

solve the following problems by hand:

a. Verify that the controllability gramian is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ while $\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ is the observability gramian.

b. Determine the Hankel singular values.

c. Find a coordinate change that gives a balanced realization.

d. Find a reduced system $G_1(s)$ by truncating the state corresponding to the smallest Hankel singular value.

13.3 For the same system and notation as in the previous problem, use a computer for the following:

a. Find the transfer function $G(s)$ from u to y.

b. Compare the error $\max_\omega |G(i\omega) - G_1(i\omega)|$ with the error bound for balanced truncation.

c. Find a reduced system G_2 by truncating both states and keeping just a constant gain.

d. Compare the error $\max_\omega |G(i\omega) - G_2(i\omega)|$ with the error bound for balanced truncation.

13.4 Find a reduced order approximation of

$$\frac{2s^2 + 2.99s + 1}{s(s+1)^2}$$

by writing the transfer function as the sum of an integrator and a stable transfer function, then applying balanced truncation to the stable part. You may use a computer.

Exercise 14. Old Exam

14.1 Write down a state-space realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+6s+8)} \end{bmatrix}$$

14.2 A system has the transfer function $G(s) = \frac{1}{s+a}$, where $a > 0$. The input to the system is white noise with spectral density $\Phi_n = 1$. What is the spectral density of the output?

14.3 Steve is working with a process and he wants to design a controller for it. After identifying the transfer function $P(s)$ he decides to try a PI-controller $C_{PI}(s) = 0.88(1 + \frac{1}{s})$. With the PI-controller, the disturbance response is very poorly damped, so he adds a lead filter. The controller is now:

$$C(s) = 0.88\left(1 + \frac{1}{s}\right)\frac{s/1.79+1}{s/8.94+1}.$$

The bode diagrams of the controller and the loop transfer function can be seen in Figure 14.1. The disturbance response is still poorly damped. You realize that Steve has made a serious mistake when calculating his lead filter parameters.

a. What is Steve's mistake?

b. Can you improve the disturbance response by adjusting the lead filter (including the gain K)? You have to follow the specifications:

- The cutoff frequency ω_c must not change
- The high frequency gain of the controller must not increase

14.4 A linear model of an inverted pendulum on a cart is given by

$$Y(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = G(s)U(s) = \begin{pmatrix} \dfrac{\omega_0^2}{s^2 - \omega_0^2} \\ \dfrac{1}{s} \end{pmatrix} U(s)$$

where Y_1 is the pendulum angle, Y_2 is the cart velocity and U (which is the control signal) is the acceleration of the cart. Consider the control system shown in Figure 14.2. The controller C_1 is used to stabilize the pendulum. Assume that a stabilizing controller has been designed and is given by

$$C_1(s) = \frac{B_1(s)}{A_1(s)}.$$

a. Let us now consider design of the cart velocity loop with input r and output y_2. In order to evaluate different control designs, it is useful to analyze the loop transfer function, $G_o(s) = C_2(s)G_{y_2,m}$, where $G_{y_2,m}$ is the transfer function from m to y_2. Calculate $G_{y_2,m}$.

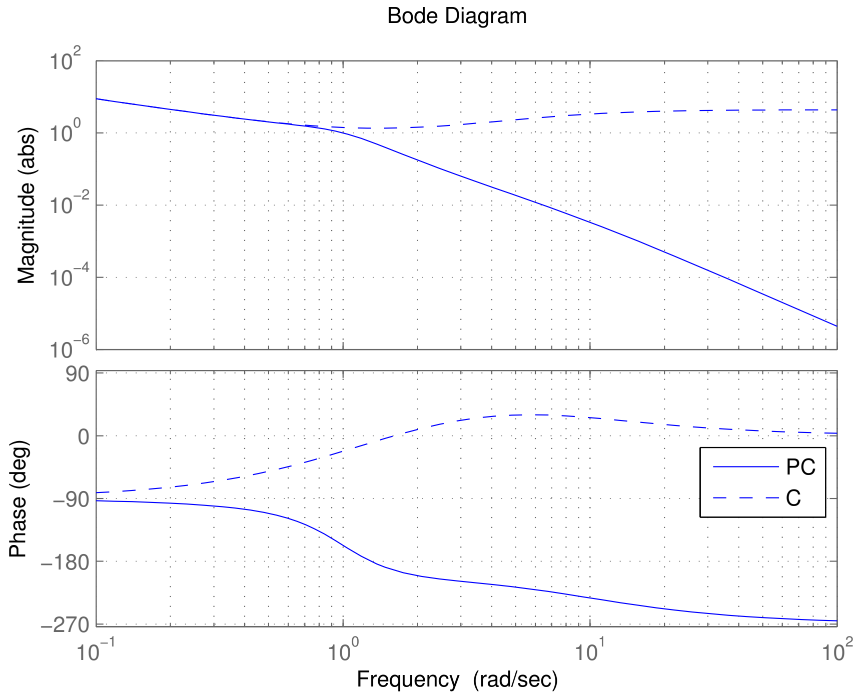

Figure 14.1 Loop transfer function PC and controller C in problem 14.3

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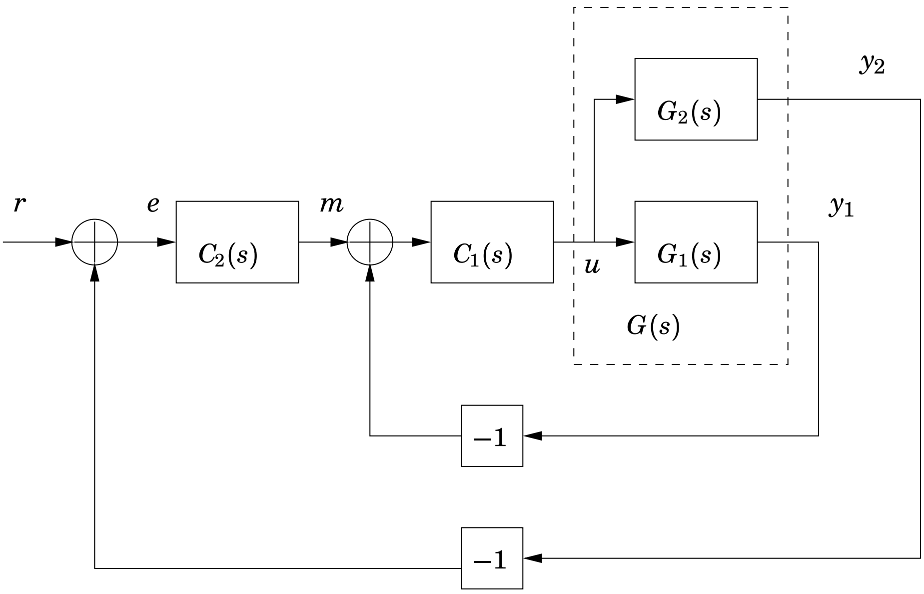

Figure 14.2 A block diagram for the inverted pendulum control system.

b. What can be said about performance limitations of the closed loop system from r to y_2? *Notice:* You do not have to design any controllers!

c. Figure 14.3 shows four plots, where one of the plots shows the sensitivity and complementary sensitivity function of a closed loop system discussed in problem **b**, with $\omega_0 = 1$ and a particular choise of $C_2(s)$. Which plot?

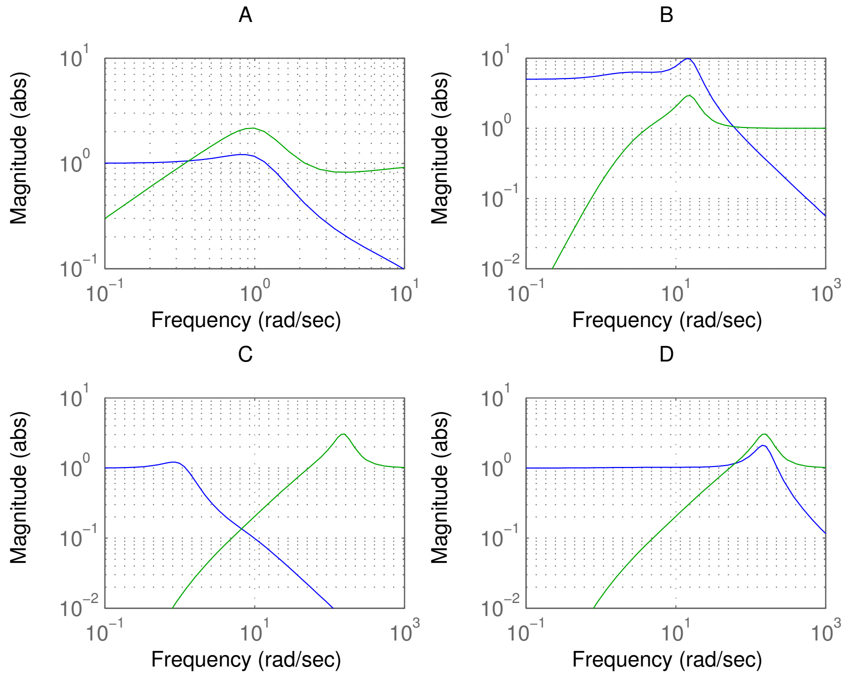

Figure 14.3 Magnitude plots of S and T in problem 6.

Motivate your answer!

14.5 Solve the following problems:

a. Consider the system

$$G_2(s) = \begin{pmatrix} \dfrac{s-1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{-6}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \end{pmatrix}.$$

Calculate the poles and zeros of $G_2(s)$. Are there any limitations on the achievable bandwidth?

b. Determine the RGA of $G_2(s)$ at $\omega = 0$ rad/s and choose reasonable input/output pairs for decentralized control. Can we expect decentralized control to work well for low frequencies?

14.6 Assume that we can model a physical process with the following transfer function

$$G(s) = \frac{(s+a)^m}{(s+b)^n},$$

where $m = 1 < n$ and $a, b > 0$. The IMC method was used to find a controller for this system, namely a PID controller with a lowpass filter

$$C(s) = K\frac{(1 + \frac{1}{T_i s} + T_d s)}{(s\frac{T_d}{N} + 1)}.$$

Determine what n the process must have and express K, T_i, T_d and N in a, b and the design parameter λ. What PID parameters are adjustable?

14.7 Recall the the quadruple tank system that was examined in Laboratory Exercise 2. The transfer function from the two inputs (u_1, u_2) to the two outputs (y_1, y_2) was

$$G(s) = \begin{pmatrix} \dfrac{\gamma_1 c_1}{1+sT_1} & \dfrac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \\ \dfrac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \dfrac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

This time we will approach the problem by use of Q-optimization. The objective will be to keep the control errors ($e_1 = r_1 - y_1$, $e_2 = r_2 - y_2$) low. For this reason we will choose r_1 as well as r_2 to be exogenous inputs w (see Figure 14.4) while e_1 and e_2 will be our exogenous outputs z. The signals r_1, r_2, y_1 and y_2 will all be given to the controller. Hence

$$z = \begin{bmatrix} r_1 - y_1 \\ r_2 - y_2 \end{bmatrix} \qquad w = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ r_1 \\ r_2 \end{bmatrix}$$

Determine the transfer function matrix

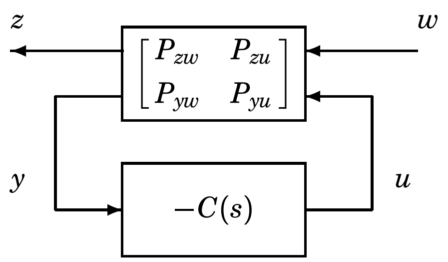

Figure 14.4 General form of a closed loop system.

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

(see Figure 14.4) for the quadruple tank system.

14.8 A MIMO system is described by the following transfer functions:

$$P(s) = \begin{bmatrix} \dfrac{1}{s+2} & -\dfrac{1}{s+2} \\ \dfrac{1}{s+1} & \dfrac{1}{s+4} \end{bmatrix}$$

a. Calculate the zero(s) of the process.

b. Suppose that we want to control the process by selecting input and output pairs for two SISO loops. How should we pair the input and outputs ?

14.9 Consider the following process

$$G(s) = \frac{4.2}{s^2 + 0.12s + 1}.$$

The control structure chosen is state feed-back design by minimizing the cost function

$$J = \int_0^\infty y^T Q_1 y + u^T Q_2 u \; dt$$

and a feed-forward gain L_r such that we have the control signal

$$u(t) = -Lx(t) + L_r r(t).$$

Four different cost functions have been used and step responses from r to y have been plotted for performance comparison. However, the plots have not come in the correct order. Help the designer by pairing the correct weights below and step responses in Figure 14.5. A cost function might suit several step responses, give all alternatives! Motivate!

A. $Q_1 = 1, \; Q_2 = 10$

B. $Q_1 = 1, \; Q_2 = 0.01$

C. $Q_1 = 100, \; Q_2 = 1000$

D. $Q_1 = 1, \; Q_2 = 100$

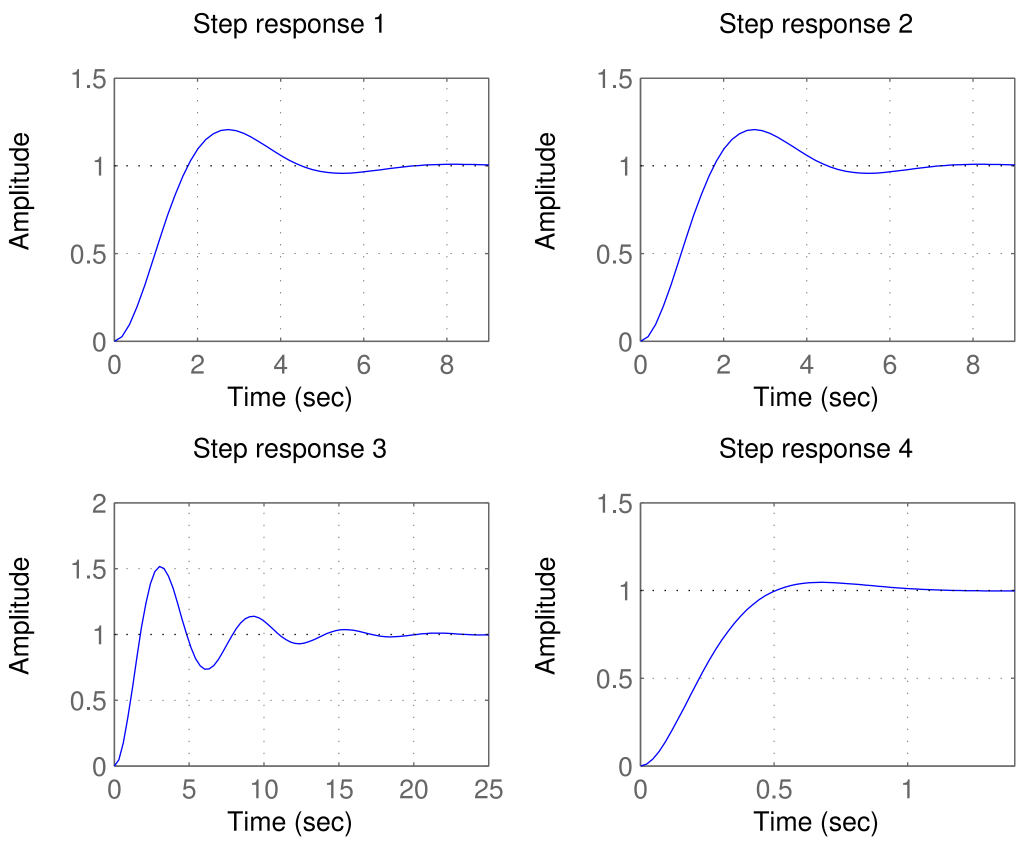

Figure 14.5 Step responses for Problem 14.9