Lecture 11: More on LQG

- Example: Lab servo revisited
- Connections to loop shaping
- Example: LQG design for DC-servo

The purpose of this lecture is not to introduce new results, but to explain the use of previous theory. The DC-servo example is from section 10.2 in Glad/Ljung.

Recall the main result of LQG

Given white noise v and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad \mathbf{E}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\hat{x}$ with $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K[y - C\hat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

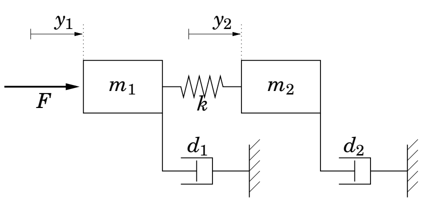
is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$
$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

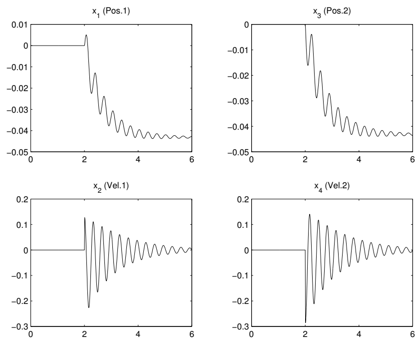
$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^TSB + Q_2)L]$$

Example: Flexible servo



$$m_1\frac{d^2y_1}{dt^2} = -d_1\frac{dy_1}{dt} - k(y_1 - y_2) + F(t)$$
$$m_2\frac{d^2y_2}{dt^2} = -d_2\frac{dy_2}{dt} + k(y_1 - y_2)$$

Open loop response



Choice of minimization criterion

How choose Q_1, Q_2, Q_{12} in the cost function

$$x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u$$

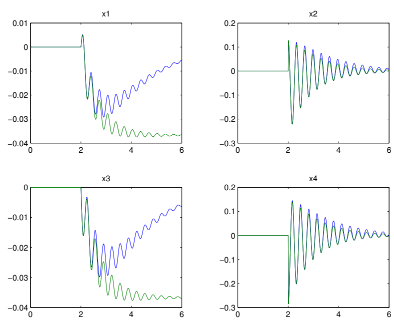
Rules of thumb:

- Put $Q_{12} = 0$ and make Q_1, Q_2 diagonal
- Make the diagonal elements equal to the inverse value of the square of the allowed deviation:

$$x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)$$
$$= \left(\frac{x_1(t)}{x_1^{\max}}\right)^2 + \cdots + \left(\frac{x_n(t)}{x_n^{\max}}\right)^2 + \left(\frac{u_1(t)}{u_1^{\max}}\right)^2 + \cdots + \left(\frac{u_m(t)}{u_m^{\max}}\right)^2$$

Velocity error or position error?

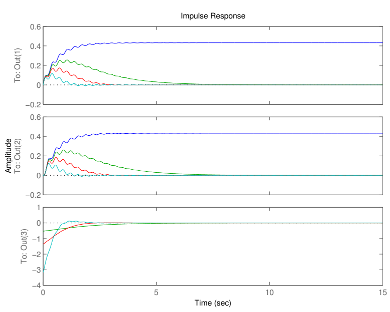
Minimize $\quad \mathbf{E}[x_2(k)^2 + x_4(k)^2 + u(k)^2] \quad$ or $\quad \mathbf{E}[x_1(k)^2 + x_3(k)^2 + u(k)^2] \quad$?



When only velocity is penalized, a static position error remains

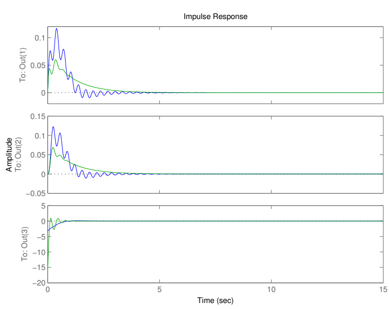
Position error control

Response of $x_1(k), x_3(k), u(k) = -Lx(k)$ on impulse disturbance in F. $Q_1 = \text{diag}\{\rho, 0, \rho, 0\}$ ($\rho = 0, 1, 10, 100$), $Q_{12} = 0$, $Q_2 = 1$. Large $\rho \Rightarrow$ fast response but large control signal.

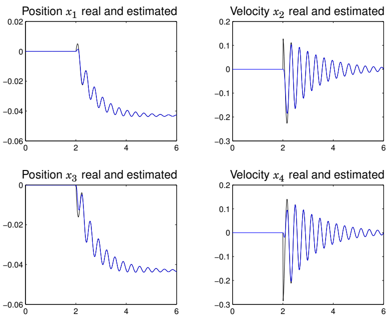


Position+velocity error control

To reduce oscillations, penalize also velocity error. Comparision between $Q_1 = \text{diag}\{100, 0, 100, 0\}$ and $Q_1 = \text{diag}\{100, 100, 100, 100\}$



Real and estimated states



A Kalman filter estimates the states using measured positions.

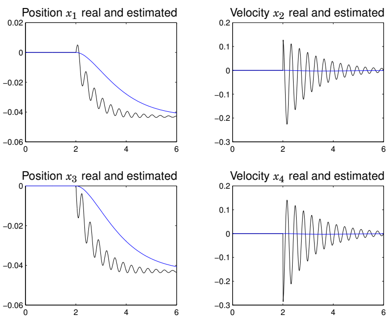
Why is the transient error bigger in the right plots?

Miniproblem

What happens if

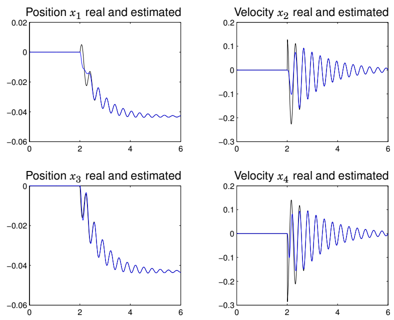
- we reduce R_1 by 10000?
- we increase the upper left corner of R_2 by 10000?
- we increase the lower right corner of R_2 by 10000?

Reduced R_1



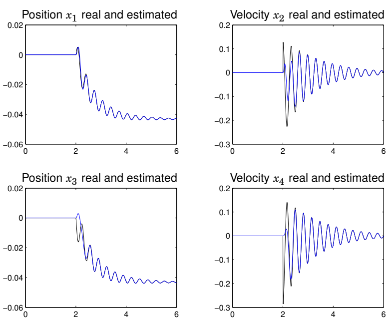
When the expected process perturbations are small, the observer will be slower.

Increased the upper left corner of R_2



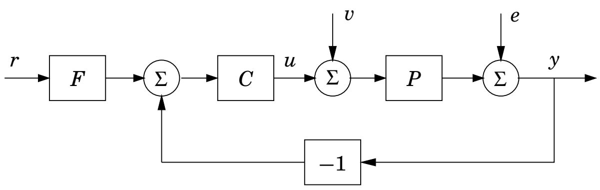
The measurement y_1 is not trusted, so the estimate of x_1 slows down.

Increased lower right corner of R_2



The measurement y_2 is not trusted, so the estimate of x_3 slows down.

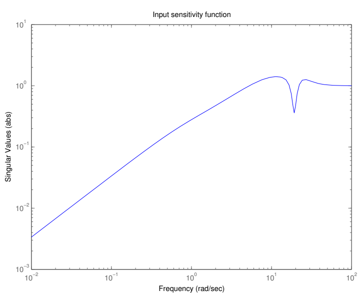
Recall the simple control loop



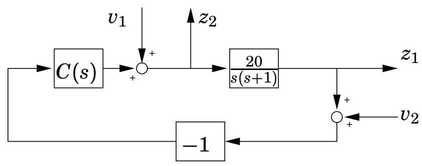
- Reduce the effects of load disturbances
- Control the effects of measurement noise
- Reduce sensitivity to process variations
- Make output follow command signals

Don't forget "The Gang of Four"!

Check all relevant transfer functions for robustness and signal sizes. The input sensitivity $|(I + CP)^{-1}(i\omega)|$ is plotted below. No large peaks, maximum=1.4.



Example — DC-servo



With $P(s) = \frac{20}{s(s+1)}$, the transfer matrix from (v_1, v_2) to (z_1, z_2) is

$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$

As a first (preliminary) design, we choose $C(s)$ to minimize

$$\text{trace} \int_{-\infty}^{\infty} G_{zv}(i\omega) G_{zv}(i\omega)^* d\omega$$

This minimizes $\mathbf{E}(|z_1|^2 + |z_2|^2)$ when (v_1, v_2) is white noise.

Example — DC-motor

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}}^{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \overbrace{\begin{bmatrix}20\\ 0\end{bmatrix}}^{B} u + \overbrace{\begin{bmatrix}20\\ 0\end{bmatrix}}^{N} v_1$$

$$y = x_2 + v_2 \qquad z_1 = x_2 \qquad z_2 = u + v_1$$

Minimization of $\mathbf{E}(|z_1|^2 + |z_2|^2)$ is the LQG problem defined by

$$Q_1 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad Q_2 = 1 \qquad \begin{cases} R_1 = \mathbf{E}|v_1|^2 = 1 \\ R_2 = \mathbf{E}|v_2|^2 = 1\end{cases}$$

Solving the Riccati equations gives the optimal controller

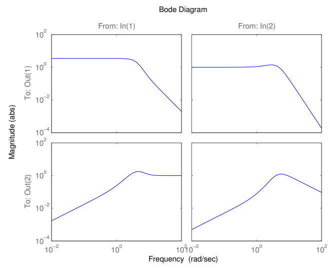
$$\frac{d}{dt}\widehat{x} = (A - BL)\widehat{x} + K[y - C\widehat{x}] \qquad u = -L\widehat{x}$$

where

$$L = \begin{bmatrix}0.2702 & 0.7298\end{bmatrix} \qquad K = \begin{bmatrix}20.0000\\ 5.4031\end{bmatrix}$$

Bode magnitude plots after optimization

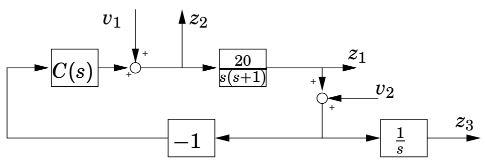
$$G_{zv}(s) = \begin{bmatrix}\frac{P}{1+PC} & \frac{-PC}{1+PC}\\ \frac{1}{1+PC} & \frac{-C}{1+PC}\end{bmatrix}$$



Nonzero static gain in $\frac{P}{1+PC}$ indicates poor disturbance rejection

Example — DC-motor

To remove static errors in the output we penalize also z_3:



The transfer matrix from (v_1, v_2) to (z_1, z_2, z_3) is

$$G_{zv}(s) = \begin{bmatrix}\frac{P}{1+PC} & \frac{-PC}{1+PC}\\ \frac{1}{1+PC} & \frac{-C}{1+PC}\\ \frac{P}{s(1+PC)} & \frac{-PC}{s(1+PC)}\end{bmatrix}$$

Extended DC-motor model

With the model

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 0\\ 0 & 1 & 0\end{bmatrix}}^{A_e}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \overbrace{\begin{bmatrix}20\\ 0\\ 0\end{bmatrix}}^{B_e} u + \overbrace{\begin{bmatrix}20 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}}^{N_e}\overbrace{\begin{bmatrix}v_1\\ v_2\end{bmatrix}}^{v_{1e}}$$

$$y = x_2 + v_2$$

minimization of $|x_2|^2 + |x_3|^2 + |u|^2$ gives the optimal controller

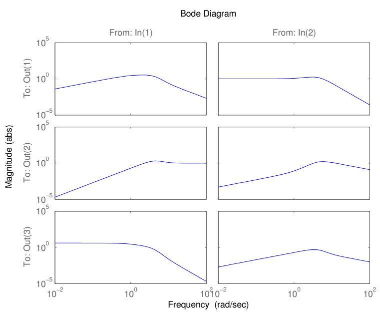
$$\frac{d}{dt}\widehat{x}_e = (A_e - B_e L_e)\widehat{x}_e + K_e[y - C_e\widehat{x}_e] \qquad u = -L\widehat{x}$$

where

$$C_e = \begin{bmatrix}0.0000 & 1.0000 & 0.0000\end{bmatrix} \qquad L_e = \begin{bmatrix}0.3162 & 1.0000 & 1.0000\end{bmatrix} \qquad K_e = \begin{bmatrix}20.0000\\ 5.4031\\ 1.0000\end{bmatrix}$$

Bode magnitude plots after optimization

$$G_{zv}(s) = \begin{bmatrix}\frac{P}{1+PC} & \frac{-PC}{1+PC}\\ \frac{1}{1+PC} & \frac{-C}{1+PC}\\ \frac{P}{s(1+PC)} & \frac{-PC}{s(1+PC)}\end{bmatrix}$$



Summary of LQG

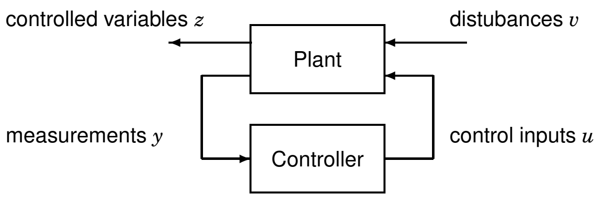
Advantages

- Works fine with multi-variable models
- Observer structure ties to reality
- Always stabilizing
- Guaranteed robustness in state feeback case
- Well developed theory

Disadvantages

- High order controllers
- Sometimes hard to choose weights

Alternative norms for optimization



LQG optimal control:

$$\text{Minimize} \quad \int_{-\infty}^{\infty} G_{zv}(i\omega)G_{zv}(i\omega)^* d\omega$$

H_∞ optimal control:

$$\text{Minimize} \quad \max_{\omega} \|G_{zv}(i\omega)\|$$

Linear Quadratic Game Problems

Notice that $\max_\omega \|G_{zv}(i\omega)\| \le \gamma$ if and only if

$$|z|^2 - \gamma^2|v|^2 \le 0$$

for all solutions to the system equations.

The H_∞ optimal control problem with $|z|^2 = x^T Q_1 x + u^T Q_2 u$ can be restated in terms of linear quadratic games of the form

$$\min_u \max_v (x^T Q_1 x + u^T Q_2 u - \gamma^2 |v|^2)$$

These can be solved using Riccati equations, just like LQG.