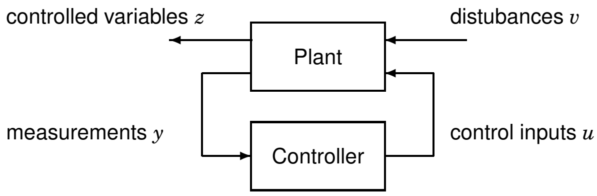
Lecture 10: Optimal Kalman Filtering

- Observer Based Feedback
- The Optimal Kalman filter
- LQG by Separation
- Stochastic interpretations

Textbook sections 9.1-9.4 and 5.7

Linear Quadratic Gaussian Control (LQG)

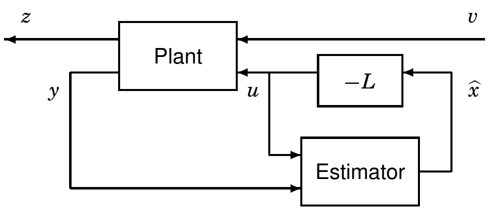


For a linear plant, minimize a quadratic function of the map from disturbance v to controlled variable z

$$\text{Minimize trace} \int_{-\infty}^{\infty} QG_{zv}(i\omega)G_{zv}(i\omega)^* d\omega$$

Last week: State feedback solution.

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$

Closed loop dynamics

Eliminate u and y:

$$\frac{d}{dt}x(t) = Ax(t) - BL\widehat{x}(t) + v_1(t)$$
$$\frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) - BL\widehat{x}(t) + K[Cx(t) - C\widehat{x}(t)] + Kv_2(t)$$

Introduce $\widetilde{x} = x - \widehat{x}$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \widetilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \widetilde{x}(k) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_1(t) - Kv_2(t) \end{bmatrix}$$

Two kinds of closed loop poles

Process poles:	$0 = \det(sI - A + BL)$
Observer poles:	$0 = \det(sI - A + KC)$

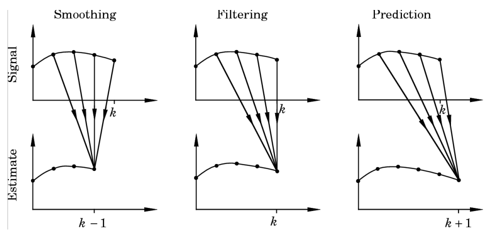
Rudolf Kalman, (born 1930)

Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering "for the devlopment and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products."

Prediction and filtering

* Wiener (1949) Stationary I/O case
* Kalman and Bucy (1960) Time-varying state-space

Estimate $x(k + m)$ given $\{y(i), u(i) \mid i \leq k\}$

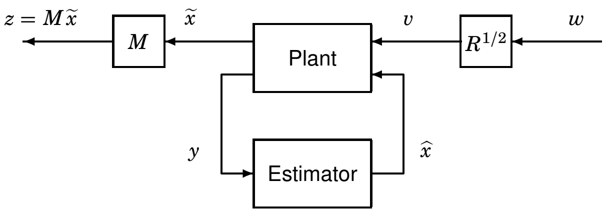


Examples

Smoothing	To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Thursday
Filtering	To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Wednesday (helps to reduce measurement error)
Prediction	To predict the Wednesday temperature based on temperature measurements from Sunday, Monday and Tuesday

Norbert Wiener, 1894–1964

The Kalman Filter Optimization Problem



Minimize error variance when v is white noise with intensity R:

$$\mathbf{E}|z|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} M\, G_{\tilde{x}v}(i\omega)\, R\, G_{\tilde{x}v}(i\omega)^* M^T d\omega$$

Equivalent reformulations

The time domain version of the optimization problem can be written

$$\text{Minimize } \int_0^\infty M\, g_{\tilde{x}v}(t)\, R\, g_{\tilde{x}v}(t)^T M^T dt$$

Given the error dynamics

$$\frac{d}{dt}\tilde{x}(t) = [A - KC]\tilde{x}(t) + v_1(t) - K v_2(t)$$

the impulse response from v to $\tilde{x}$ is

$$g_{\tilde{x}v}(t) = e^{(A-KC)t}[I \quad -K]$$

so K should be chosen to

$$\text{Minimize } \int_0^\infty M\, e^{(A-KC)t}[I \quad -K]R[I \quad -K]^T e^{(A-KC)^T t} M^T dt$$

Recall lecture 9: Linear Quadratic Optimal Control

For the system $\dot{x} = Ax(t) + Bu(t)$, $x(0) = x_0$ with control law $u = -Lx$ consider the cost

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt = \int_0^\infty x_0^T e^{(A-BL)^T t} \begin{bmatrix} I \\ -L \end{bmatrix}^T Q \begin{bmatrix} I \\ -L \end{bmatrix} e^{(A-BL)t} x_0 dt$$

The minimal cost is achieved by $L = Q_2^{-1}(SB + Q_{12})^T$, where $S > 0$ solves

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

The minimal value of the integral is $x_0^T S x_0$.

The solution can be reused to get the optimal Kalman filter!

Optimal Kalman Filtering — The Solution

The Kalman filter $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$ gives the error covariance

$$\mathbf{E}|M\tilde{x}|^2 = \int_0^\infty M\, e^{(A-KC)t} \begin{bmatrix} I & -K \end{bmatrix} R \begin{bmatrix} I & -K \end{bmatrix}^T e^{(A-KC)^T t} M^T dt$$

The minimal error covariance is achieved by $K = (PC^T + R_{12})R_2^{-1}$ where $P > 0$ solves

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T$$

Remark: Notice that K is independent of M. Hence the same filter is optimal regardless of which state we want to estimate! The minimal error covariance is $\mathbf{E}\tilde{x}\tilde{x}^T = P$.

Example 1 – Kalman filter

$$\dot{x}(t) = v(t) \qquad \mathbf{E}v^2 = R_1$$
$$y(t) = x(t) + e(t) \qquad \mathbf{E}e^2 = R_2$$
$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

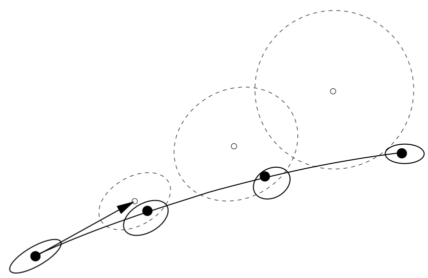
Riccati equation $\quad 0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1 R_2}$

Filter gain $\quad K = P/R_2 = \sqrt{R_1/R_2}$

Error dynamics $\quad \dfrac{d\tilde{x}}{dt} = -\sqrt{R_1/R_2}\,\tilde{x}$

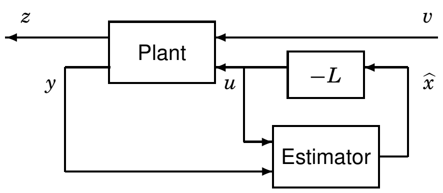
Error covariance $\quad \mathbf{E}\tilde{x}^2 = P = \sqrt{R_1 R_2}$

Example 2 – Tracking of a moving object



Dotted ellipses show estimates based on only a model with known initial state. Solid ellipses show Kalman filter estimates based on noisy measurements.

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$

Minimize $\quad \mathbf{E}|z|^2 = \mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$

when v is white noise of intensity R

The idea of separation

The state feedback control law is independent of R

The Kalman filter minimizes $\mathbf{E}|M\tilde{x}|^2$ independently of M

This makes it possible to optimize the control law $u(t) = -L\hat{x}(t)$ and the estimator separately.

Linear Quadratic Optimal Control (LQG)

Given the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \\ z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{cases} \qquad Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \quad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\hat{x}$ with $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K[y - C\hat{x}]$. The frequency integral

$$\text{trace} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} QG_{zv}(i\omega)RG_{zv}(i\omega)^* d\omega$$

is minimized when K and L satisfy

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T \qquad K = (PC^T + NR_{12})R_2^{-1}$$

The minimal value of the integral is

$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^TSB + Q_2)L]$$

Stochastic Interpretation of LQG Control

Given white noise v and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad \mathbf{E} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\hat{x}$ with $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K[y - C\hat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$
$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^TSB + Q_2)L]$$

Duality between control and estimation

Optimal control	State estimation
A	A^T
B	C^T
Q_1	R_1
Q_2	R_2
Q_{12}	R_{12}
S	P
L	K^T

Example

Consider the problem to minimize $\mathbf{E}(Q_1x^2 + Q_2u^2)$ for

$$\begin{cases} \dot{x}(t) = u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad \mathbf{E} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

The observer based controller

$$\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$$

is optimal for K and L computed as follows:

$$0 = Q_1 - S^2/Q_2 \quad \Rightarrow \quad S = \sqrt{Q_1Q_2} \quad \Rightarrow \quad L = S/Q_2 = \sqrt{Q_1/Q_2}$$
$$0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1R_2} \quad \Rightarrow \quad K = P/R_2 = \sqrt{R_1/R_2}$$