Lecture 3:

Todays lecture: Disturbance models and Robustness

Continuing from lecture 2...

- Stability / poles
  - The closed-loop system
  - Look at all transfer functions in the loop! (Gang of Four / Gang of six)
  - Robustness

New today

- disturbances and scalings

[Glad & Ljung] Ch. 5.1–5.6, 6.1–6.3

Poles of a system

- The eigenvalues of the system matrix for a *minimal realization*.
- The points $p \in C$ where $G(p) = \infty$ are called *poles of* G.
- For a given G(s):

  The "pole polynomial" is the *least common denominator* to all sub-determinants of $G(s)$. NOTE: Always a pole if in denominator of any transfer element, but use pole polynomial to find out multiplicity. (see Ch 3.3 and the exercises).

2 minute problem (Poles for Multi-input-multi output (MIMO)

[Example 3.1, Glad&Ljung]

Consider the 2-input-2-output system with *transfer matrix/ transfer function matrix* $G(s)$

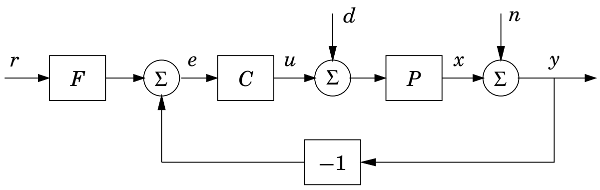
$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}}_{G(s)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{1}$$

- Is it a second or a third order system?
- How many states do we need for a minimal realization? How many poles does the system have?

Draw a block scheme of the system with first order transfer functions.

See also solution to Example 3.1 in [G&L].

A Basic Control System



Ingredients:

- Controller: feedback C, feedforward F
- Load disturbance d: Drives the system from desired state
- Measurement noise n: Corrupts information about x
- Process variable x should follow reference r

Key Issues

Find a controller that

A: Reduces effects of load disturbances

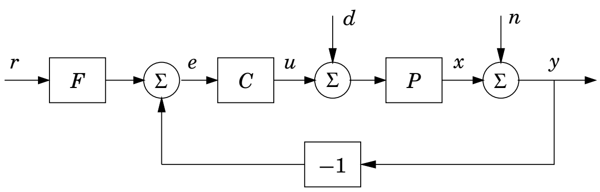
B: Does not inject to much measurement noise into the system

C: Makes the closed loop insensitive to variations in the process

D: Makes output follow command signals

Convenient to use a controller with two degrees of freedom, i.e. separate signal transmission from y to u and from r to u. This gives a complete separation of the problem: Use feedback to deal with A, B, and C. Use feedforward to deal with D!

System with Two Degrees of Freedom



The controller has two degrees of freedom (2DOF) because the transfer function from reference r to control u is different from the transfer function from y to u.

We have already encountered this in *e.g.,* PID control

$$u(t) = k(br(t) - y(t)) + \int_0^t (r(\tau) - y(\tau))d\tau + \frac{d}{dt}\{0 \cdot r - y\}$$

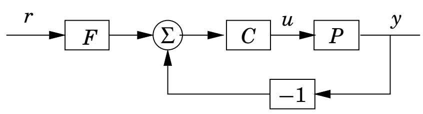
Designing System with Two Degrees of Freedom

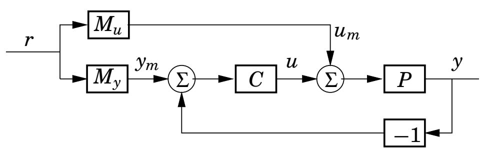
Design procedure:

- Design the feedback C to achieve
  - Small sensitivity to load disturbances d
  - Low injection of measurement noise n
  - High robustness to process variations
- Then design the feedforward F to achieve desired response to command signals r

For many problems in process control the load disturbance response is much more important than the set point response. The set point response is more important in motion control. Few textbooks and papers show more than set point responses.

Many Versions of 2DOF

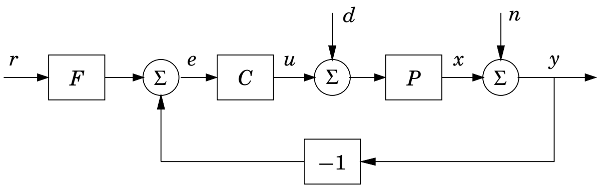




For linear systems all 2DOF configurations have the same properties. For the systems above we have

$$CF = M_u + CM_y$$

3. Relations between signals



$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$
$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Some Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to only show properties of a few systems for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

Some Observations cont'd

Important Remark:

If C, P and F are general MIMO-systems, so called *transfer function matrices* (will be used later on in the course), the **order of multiplication matters** and

$$PC \neq CP$$

and thus we need to multiply with the inverse from the correct side as in general

$$(1+L)^{-1}M \neq M(1+L)^{-1}$$

Note, however that

$$(1+PC)^{-1}PC = PC(1+PC)^{-1}$$

A Possible Choice

Six transfer functions are required to show the properties of a basic feedback loop. Four characterize the response to load disturbances and measurement noise.

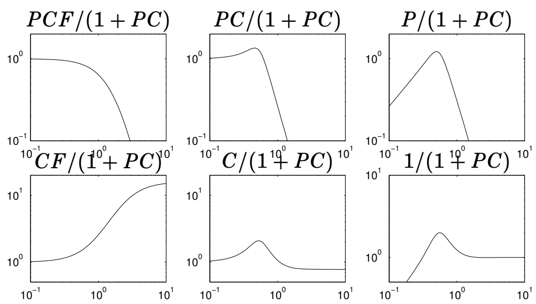
$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC}$$
$$\frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

Two more are required to describe the response to set point changes.

$$\frac{PCF}{I+PC} \qquad \frac{CF}{1+PC}$$

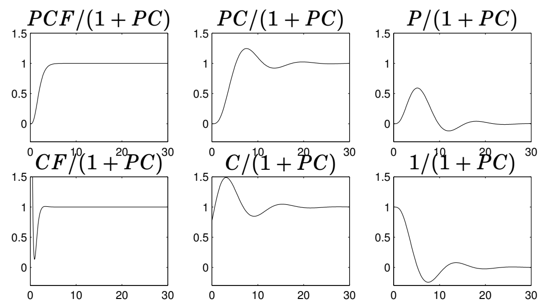
Amplitude Curves of Frequency Responses

PI control $k = 0.775$, $T_i = 2.05$ of $P(s) = (s+1)^{-4}$ with $M(s) = (0.5s+1)^{-4}$



Step Responses

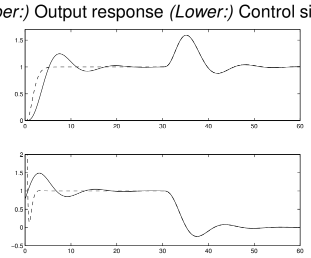
PI control $k = 0.775$, $T_i = 2.05$ of $P(s) = (s+1)^{-4}$ with $M(s) = (0.5s+1)^{-4}$



An Alternative

Show the responses in the output and the control signal to a step change in the reference signal for system with pure error feedback and with feedforward. Keep the reference signal constant and make a unit step in the process input.

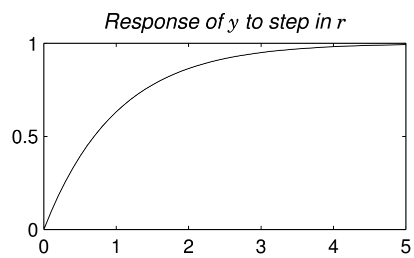
(Upper:) Output response *(Lower:)* Control signal.



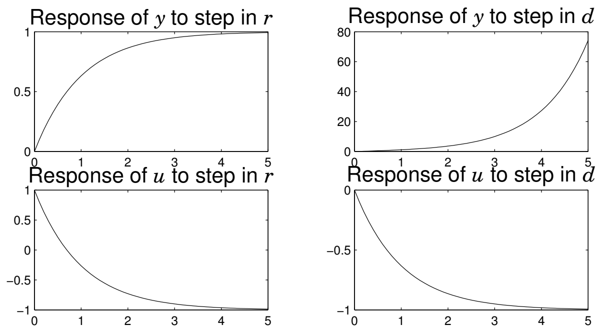
step responce load disturbance

A Warning!

*Please remember to always look at **all** responses when you are dealing with control systems. The step response below looks fine but ...*



Four Responses



What is going on?

The System

Process $P(s) = \frac{1}{s-1}$

Controller $C(s) = \frac{s-1}{s}$

Response of y to reference r

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{1}{s+1}$$

Response of y to step in disturbance d

$$\frac{Y(s)}{D(s)} = \frac{P}{1+PC} = \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)}$$

Scaling

Warning: The norms used to measure signal size can be very misleading if we are using states with very different magnitudes!

Common to scale/normalize variables for state representations

$$x_i = x_i^p / d_i$$

where

- x_i^p corresponds to physical units
- d_i corresponds to (expected) max size of variable (absolute value).

Can also introduce weighted quadratic norms such as

$$|x|_P^2 = x^T P x$$

where $P = P^T > 0$

Scaling cont'd

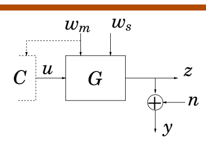
[Skogestad]
Remark:

- It is particularly important for the sensitivity function $S = (I + GC)^{-1}$of a MIMO system that outputs or output errors are of the same magnitude for correct comparisons.
- If operating around a set-point where the expected or allowed variation is not symmetric (e.g. if only positive values allowed) then it may be better to introduce deviations and scale these instead.

Disturbances

- Description/representations of disturbances
- State-space form of system with disturbances
- ...

Disturbances cont.



Load disturbances

- disturbances which really affect the system
  - w_m measurable — use e.g., in feedforward compensation
  - w_s non-measurable — controller need to suppress these

Measurement disturbances n

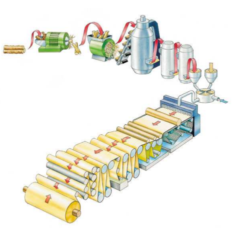
- Controller should not be "fooled" by measurement disturbances

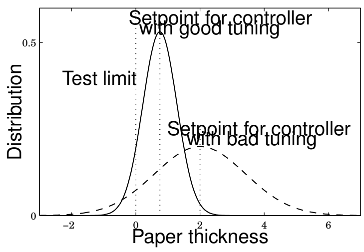
Common case: $z = \mathcal{S}(u, w_m, w_s)$, $\quad y = z + n$ where

z is the control objective, $\quad y$ is the measured output

Motivation

Example: Paper thickness — want to keep down variation in output!





All paper production below the test limit is wasted.
Good control allows for lower setpoint with the same waste.
The average thickness is lower, which saves significant costs.

Motivation cont'd - LQG control

System with state-disturbances w and measurement-disturbances v.

Minimize $\int \left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \right) dt$

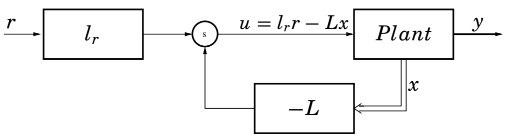
subject to
$$\dot{x} = Ax + Bu + w$$
$$y = Cx + Du + v$$

where v and w is white noise with $\mathbf{E}(ww^T) = R_1$, $\mathbf{E}(vv^T) = R_2$

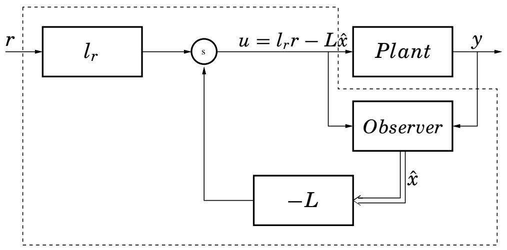
Can solve two separate problems thanks to

Separation principle:
Controller design for full state information
Optimal estimation of states
$\Longrightarrow$ Output feedback using observer

State-feeadback



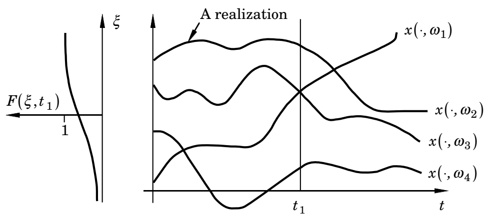
Observer feedback



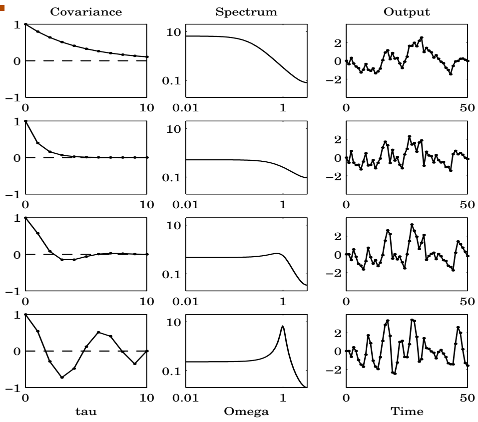
A **stochastic process** (random process, random function) is a family of stochastic variables $\{x(t), t \in T\}$
Index set $T = \{\ldots, -h, 0, h, \ldots\}$, or $h = 1$

A function of two variables $x(t, \omega)$

Fixed $\omega = \omega_0$ gives a time function $x(\cdot, \omega_0)$ (realization)
Fixed $t = t_1$ gives a random variable $x(t_1, \cdot)$

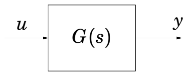


Covariance, spectral density, and realization



Error-correction: The spectra should be divided by 2π

Spectral density and transfer functions



What is the output spectral density for y if the input u has spectral density $\Phi_u(\omega)$?

$$Y(i\omega) = G(i\omega)U(i\omega)$$

where $Y = \mathcal{F}\{y\}$, $U = \mathcal{F}\{u\}$ are the Fourier transforms.

$$\Phi_y(\omega) \hat{=} \Phi_{yy}(\omega) = Y(i\omega)Y(i\omega)^* = G(i\omega)U(i\omega)U(i\omega)^*G(i\omega)^*$$

Spectral density $\Phi_{yy}(\omega) = G(i\omega)\Phi_{uu}(\omega)G(i\omega)^*$

Linear Quadratic Control (LQ)
Find state-feedback gain $L = [l_1 \ \ l_2 \ \ \ldots \ \ l_n]$ for the control $u = -Lx$, being the solution to the optimization problem

Minimize	$\int (x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u)\, dt$
subject to	$\dot{x} = Ax + Bu$
	$y = Cx + Du$

Stochastic Linear Quadratic Control (LQG)
Based on information of the noise v and w find the optimal observer/Kalman gain K and use control $u = -L\hat{x}$

Minimize	$\int (x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u)\, dt$
subject to	$\dot{x} = Ax + Bu + w$
	$y = Cx + Du + v$
where v and w is	
white noise with	$\mathbf{E}(ww^T) = R_1$, $\mathbf{E}(vv^T) = R_2$

Zero mean stationary stochastic processes

The distribution is independent of t

Mean-value function

$$\mathbf{E}x(t) \equiv 0$$

Covariance function

$$r_{xx}(\tau) = \mathbf{E}x(t+\tau)x(t)^T$$

Cross-covariance function

$$r_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T$$

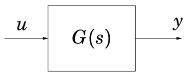
A zero mean Gaussian process x is completely determined by its covariance function.

Main Problems

1. Determine covariance function and spectral density of y when white noise u is filtered through the linear system

$$\dot{x} = Ax + u(k)$$
$$y = Cx$$

2. Conversely, find filter parameters A and C to give y a desired spectral density.



In similar way we find

cross-spectral density $\Phi_{yu}(\omega) = G(i\omega)\Phi_{uu}(\omega)$

"Everything" can be generated by filtering white noise.

How do we go "backwards"?

Disturbance representations– Spectral factorization

- Assume that the disturbance w has spectrum $\Phi_w(\omega)$
- *(Spectral factorization)* Assume that we can find a transfer function $G(s)$ such that $G(i\omega)RG(i\omega)^* = \Phi_w(\omega)$ for a constant R.

In that case we can consider w as an output from the linear system G with white noise as input, $\Phi_v(\omega) = R$ (equal energy for all frequencies/flat spectrum).
If v and w are *scalar valued* and $\Phi_w(\omega)$ is a rational function of ω^2 this is easy to do and furthermore G can always be chosen to have stable poles.

Remark: If the characteristic polynomial for $G(i\omega)$ is $\Pi_{k=1}^n(s-\lambda_k)$ then G^* will have its poles as the mirrored in the the imaginary axis.

State-space model

State-space model with disturbances

$$\dot{x}(t) = Ax(t) + Bu(t) + Nw_1(t)$$
$$z(t) = Mx(t) + D_z u(t)$$
$$y(t) = Cx(t) + D_y u(t) + w_2(t)$$

where

- w_1 is called state- or system noise
- w_2 is called measurement- or output noise

How to handle colored noise?

If w_1 and w_2 is colored noise then re-write w_1 and w_2 as output signals from linear systems with *white noise inputs* v_1 and v_2 .

$$w_1 = G_1(p)v_1, \qquad w_2 = G_2(p)v_2$$

Make a state space realization of G_1 and G_2 and extend the system description with these states

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}\overline{u}(t) + \overline{N}v_1(t)$$
$$z(t) = \overline{M}\overline{x}(t) + D_z u(t)$$
$$y(t) = \overline{C}\overline{x}(t) + D_y u(t) + v_2(t)$$

where the *extended state* $\overline{x}$ consists of the state x and the states from the state-space realizations of G_1 and G_2.

$\overline{A}$ is the corresponding system matrix for the extended system etc.

Linear system with white noise input

Consider the linear system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

The transfer function

$$G_{v\to x}(s) = (sI - A)^{-1}B$$

and the spectrum for x will be

$$\Phi_x(\omega) = (i\omega I - A)^{-1}BR\underbrace{B^*(-i\omega I - A)^{-T}}_{((i\omega I - A)^{-1}B)^*}$$

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

Covariance matrix for state x:

$$\Pi_x = R_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_x(\omega)d\omega$$

Alternative way of calculating Π_x

Theorem [G&L 5.3]

If all eigenvalues of A are strictly in the left half plane (i.e. $Re\{\lambda_k\} < 0$) then there exists a unique matrix $\Pi_x = \Pi_x^T > 0$ which is the solution to the matrix equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

Example: Consider the system

$$\dot{x} = Ax + Bv \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

where v is white noise with variance 1.

What is the covariance for x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP.$ OK!

Solve the Lyapunov equation $A\Pi_x + \Pi_x A^T + BRB^T = 0_{2,2}$.

Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12} + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0],[1 ; 0]*[1 0])`