Lecture 13

Control Synthesis by Convex Optimization[1]

This chapter is devoted to numerical optimization of controllers using the Q-parametrization (Youla). In the previous lecture, we saw that a closed loop map $G_{zw}(s)$ from w to z in the diagram of Figure 13.1 is achievable by a stabilizing controller $C(s)$ if and only if it has the form

$$G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s)$$

Hence a control design problem can be viewed as a search for $Q(s)$, to get desirable properties of $G_{zw}(s)$. Once $Q(s)$ is determined, a corresponding controller is derived by the formula $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$.

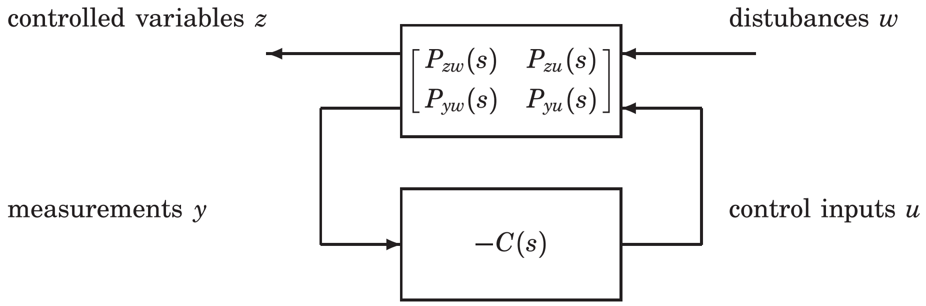

Figure 13.1 The controller $C(s)$ is computed to optimize the closed loop map from w to z.

Many natural specifications on the closed loop system can be stated as norm constraints on $G_{zw}(s)$. This, together with the fact that $Q(s)$ appears linearly in the expression for $G_{zw}(s)$, makes it possible to do controller design using *convex optimization*. This is a special kind of optimization that allows for fast algorithms and guaranteed convergence. The basics will be described next.

13.1 Basics of Convex Optimization

We consider optimization problems of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i \quad i = 1, \dots, m \end{array} \tag{13.1}$$

where x is the optimization variable, f_0 is the objective function and $f_1, \dots, f_m$ are constraints functions. This is a convex optimization problem if $f_0, \dots, f_m$ are

[1]Much of this lecture is based on source material kindly provided by Stephen Boyd. See http://www.control.lth.se/course/FRTN10/lectures.html.

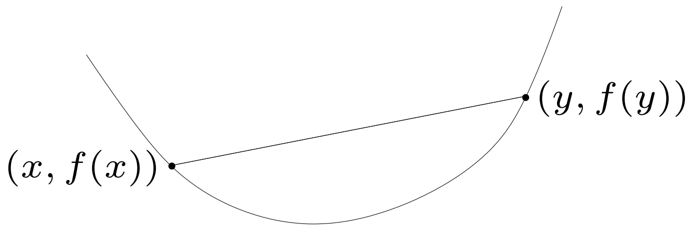

Figure 13.2 A straight line connecting two point on the graph of a convex function always stays above the graph

convex, that is if

$$f_i(\theta x + (1-\theta)y) \le \theta f_i(x) + (1-\theta) f_i(y)$$

for all x, y and for $0 \le \theta \le 1$. See Figure 13.2. Convex optimizations problems are particularly easy to solve, since

Examples of convex functions are the following:

- affine functions: $a^T x + b$ where $x, a \in \mathbf{R}^n$, $b \in \mathbf{R}$
- exponentials: e^{ax} for $x, a \in \mathbf{R}$
- powers: x^a for $x, a > 0$
- norms: $\|x\|$

The most common convex optimization problem is the least-squares problem, where f_0 is quadratic and no constraints exist.

$$\text{minimize} \quad \|Ax - b\|_2$$

This was used in earlier lectures to solve linear-quadratic control problems.

Another important class of convex optimization problems is *linear programming*, where the functions $f_0, \ldots, f_m$ are all affine:

$$\begin{array}{lll} \text{minimize} & c^T x & x \in \mathbf{R}^n \\ \text{subject to} & a_i^T x \le b_i & i = 1, \ldots, m \end{array}$$

Geometrically, the linear functions define a polyhedron, and the optimum is achieved at a corner of the polyhedron. See Figure 13.3. Linear programs can be solved

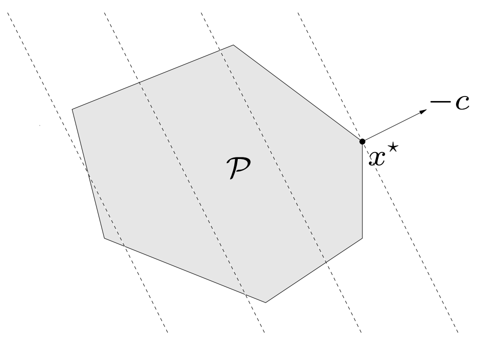

Figure 13.3 A linear program finds a point as far as possible in the direction $-c$ within a polyhedron defined by the constraints $a_i^T x \le b_i$

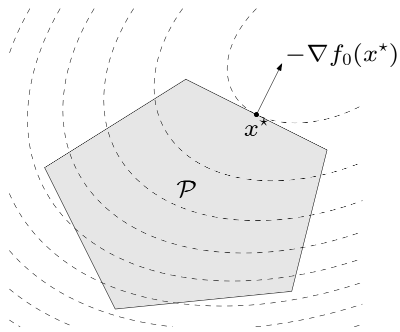

Figure 13.4 A quadratic program finds the smallest ellipsoid that touches a polyhedron defined by the constraints $a_i^T x \leq b_i$

efficiently for problems with hundreds of thousands of variables and they are used in a wide range of applications. The complexity grows as $n^2 m$ when $m \geq n$.

Combining a quadratic objective with linear constraints, we get another well known class of convex optimization problems, known as *quadratic programming*.

$$\begin{array}{lll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + r & x \in \mathbf{R}^n \\ \text{subject to} & a_i^T x \leq b_i & i = 1, \dots, m \end{array}$$

The geometric picture is again optimization over a polyhedron, but the quadratic objective need not necessarily achieve its optimum at a corner. See Figure 13.4

If instead the constraints are defined by convex quadratic functions, the convex optimization problem is called *second order cone programming*:

$$\begin{array}{lll} \text{minimize} & c^T x & x \in \mathbf{R}^n \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i & i = 1, \dots, m \end{array}$$

All the previous problem classes can be restated into this form and solved using reliable and efficient algorithms. In fact, many modern algorithms for convex programming are based on Newton's method:

$$x^+ = x - t[\nabla^2 f(x)]^{-1} \nabla f(x)$$

where t is chosen by line search. The iteration above can be used to find the minimum of $f(x)$ when there are no constraints. See Figure 13.5.

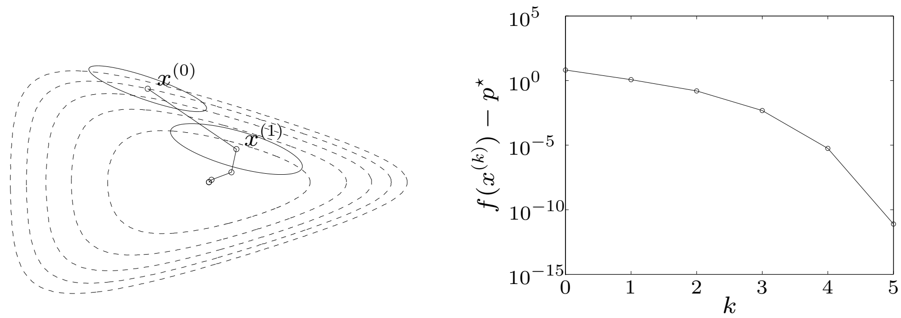

Figure 13.5 A few steps of Newton iteration are illustrated to the left, together with dotted level curves of the objective function. The ellipsoids illustrate level curves of the local second order approximation of f. The right plot shows the values of the objective function, illustrating quadratic convergence near the optimum.

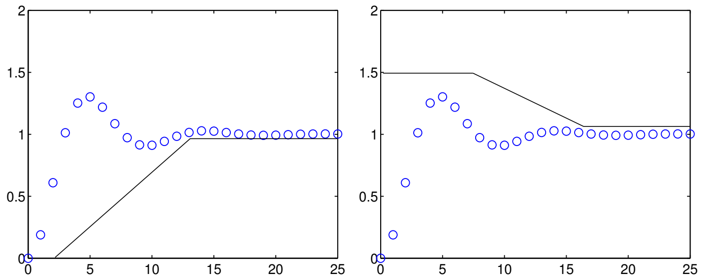

Figure 13.6 Both lower bounds and upper bounds are convex, because intermediate transfer functions have intermediate step responses.

For problems with constraints, many algorithms use so-called barrier functions to enforce the constraints. For example, the modified objective function

$$f_0(x) - (1/\gamma)\sum_{i=1}^{m} \log(-f_i(x))$$

is similar to $f_0(x)$ when $f_i(x) > 0$ and γ is small, but the function grows to infinity when x approaches the constraint boundary $f_i(x) = 0$. The minimum of the modified objective approaches the minimum of (13.1) as $\gamma \to \infty$.

13.2 Convex Specifications on Feedback Systems

Several important specifications on control systems can be stated as convex constraints on the closed loop transfer function $G_{zw}(s)$. Alternatively, because of the linear relationship between $G_{zw}(s)$ and $Q(s)$, the same specifications can be viewed as convex constraints on $Q(s)$:

- Stability of the closed loop system
- Lower and upper bounds on step response from w_i to z_j at time t_i
- Upper bound on Bode amplitude from w_i to z_j at frequency ω_i
- Interval bound on Bode phase from w_i to z_j at frequency ω_i

In each case the convexity must be verified according to the definition: If $G^1_{zw}(s)$ and $G^2_{zw}(s)$ are stable, then $\theta G^1_{zw}(s) + (1-\theta)G^2_{zw}(s)$ is stable for all $\theta \in [0, 1]$. Similarly, if the step responses of $G^1_{z_jw_i}(s)$ and $G^2_{z_jw_i}(s)$ stay within given lower and upper bounds at time t_i, then the same is true for the intermediate transfer functions $\theta G^1_{zw}(s) + (1-\theta)G^2_{zw}(s)$. See Figure 13.6.

Upper bounds on the Bode amplitude at a certain frequency are convex constraints (see Figure 13.7), because

$$|G^1_{z_jw_i}(i\omega)| \le \gamma, \quad |G^2_{z_jw_i}(i\omega)| \le \gamma \qquad \Rightarrow \qquad |G^1_{z_jw_i}(i\omega) + (1-\theta)G^2_{z_jw_i}(i\omega)| \le \gamma$$

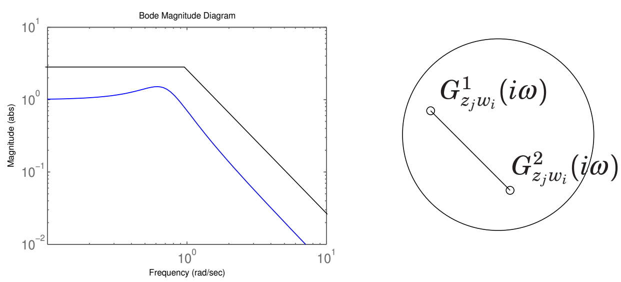

Figure 13.7 An upper bound on the Bode amplitude is a convex quadratic constraint.

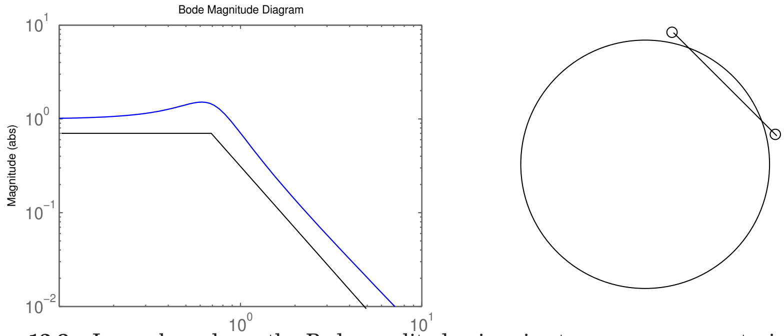

Figure 13.8 Lower bounds on the Bode amplitude give rise to non-convex constraints and should be avoided

However, the implication does not hold for lower bounds on the amplitude function, because two points outside the circular disc defined by the amplitude bound may very well have a convex combination that is inside the disc. See Figure 13.8. Hence this type of specifications are not easily treated in the context of convex optimization.

13.3 Optimization of Controllers

By using the convex specifications discussed in the previous section, a typical convex optimization problem for control synthesis could be stated as follows:

$$\text{Minimize}_Q \quad \int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega)Q(i\omega)P_{yw}(i\omega)|^2 d\omega$$

$$\text{subject to} \quad \begin{array}{l} \text{step response } w_i \to z_j \text{ is smaller than } f_{ijk} \text{ at time } t_k \\ \text{step response } w_i \to z_j \text{ is bigger than } g_{ijk} \text{ at time } t_k \\ \text{Bode amplitude } w_i \to z_j \text{ is smaller than } h_{ijk} \text{ at } \omega_k \end{array}$$

Here the optimization variable is Q, which could be any stable transfer matrix of the right dimension. In order to solve the problem numerically, we need to restrict the optimization to a finite number of parameters. Hence we will consider a fixed set of basis function $\phi_0(s), \dots, \phi_N(s)$ and search numerically for matrices $Q_0, \dots, Q_N$ such that the closed loop transfer matrix $G_{zw}(s)$ satisfies given specifications when

$$G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s) \qquad \text{and} \qquad Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s)$$

An intuitively simple parametrization of $Q(s)$ is obtained by letting each parameter Q_k represent a point on the corresponding impulse response in time domain:

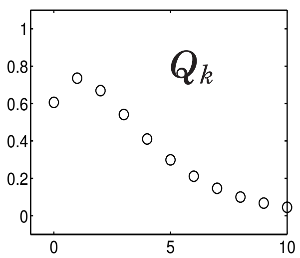

Figure 13.9 The transfer function $Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s)$ can be parametrized by letting each parameter Q_k represent a point on the corresponding impulse response.

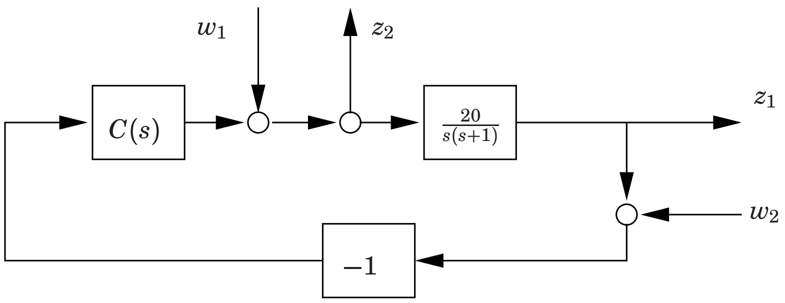

Figure 13.10 Feedback control of a DC servo.

This gives a second-order-cone programming problem in the coefficients of the Q_k-matrices:

$$\text{Minimize}_{Q_k} \quad \int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} Q(i\omega) P_{yw}(i\omega)|^2 d\omega \quad \Big\} \text{ quadratic objective}$$

subject to
- step response $w_i \to z_j$ is smaller than f_{ijk} at time t_k } linear constraints
- step response $w_i \to z_j$ is bigger than g_{ijk} at time t_k } linear constraints
- Bode amplitude $w_i \to z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Once $Q(s)$ has been determined, we will recover the desired controller from the formula

$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s)$$

This controller may be of very high order and unsuitable for implementation. However, the computation is still useful, for two reasons:

1. There are techniques for model reduction, which can be used to approximate the high order controller with low order controllers. This will be described in detail in the next lecture.
2. It is useful to know the limits of what is achievable by a linear time-invariant controller. Studying the behavior of the optimal high order controller can give a better understanding for the implications of the closed loop system specifications.

13.4 Example — DCservo revisited

Consider again control of a DC servo as in the previous lecture: The transfer matrix from (w_1, w_2) to (z_1, z_2) is

$$G_{zw}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$

with $P(s) = \frac{20}{s(s+1)}$. We will choose $C(s)$ to minimize

$$\text{trace} \int_{-\infty}^{\infty} G_{zw}(i\omega) G_{zw}(i\omega)^* d\omega$$

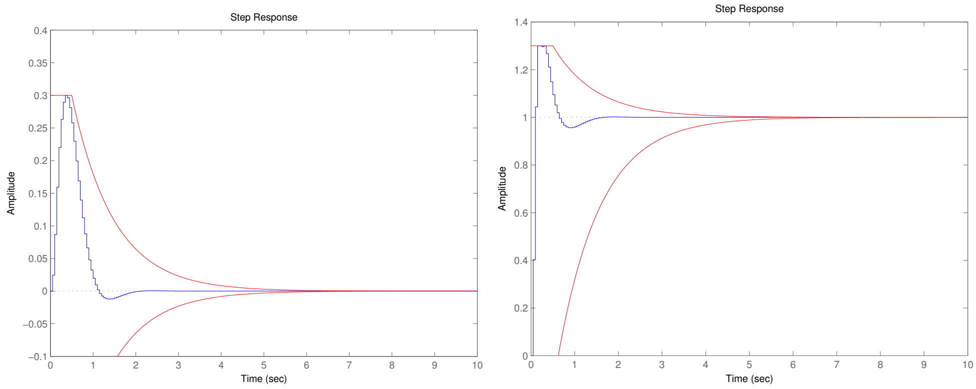

Figure 13.11 Time-domain responses for the optimized closed loop system (middle plot) plotted together with optimization bounds

subject to bounds on the time-domain response to a step disturbance w_1 and also bounds on the time-domain response to a reference step. Figure 13.11 shows the time-domain response of the optimized closed loop system together with the upper and lower bounds.

If the optimization is re-done without the upper bound on the input disturbance response, the controller drops the integral action and accepts a static error in the disturbance response. See Figure 13.12.

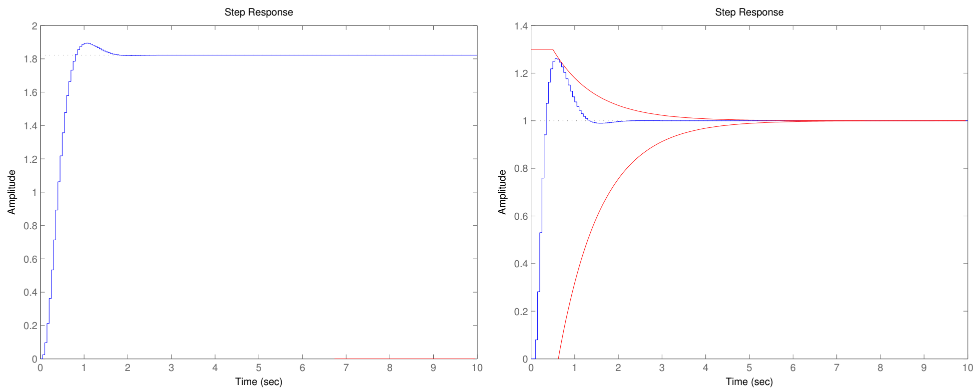

Figure 13.12 Time-domain responses for the optimized closed loop system with out upper bound on the response to the input disturbance. In this case we get a static error, so the controller has no longer any integral action.