Lecture 9 — Nonlinear Control Design

- Exact-linearization
- Lyapunov-based design
  - Lab 2
  - Adaptive control
- Sliding modes control

Literature: [Khalil, ch.s 13, 14.1,14.2] and [Glad-Ljung,ch.17]

Course Outline

Lecture 1-3	Modelling and basic phenomena (linearization, phase plane, limit cycles)
Lecture 4-6	Analysis methods (Lyapunov, circle criterion, describing functions)
Lecture 7-8	Common nonlinearities (Saturation, friction, backlash, quantization)
Lecture 9-13	Design methods (Lyapunov methods, optimal control)
Lecture 14	Summary

Exact Feedback Linearization

Idea:

Find state feedback $u = u(x, v)$ so that the nonlinear system

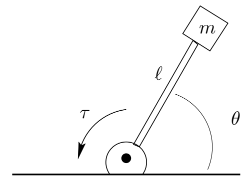
$$\dot{x} = f(x) + g(x)u$$

turns into the linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

Exact linearization: example [one-link robot]



$$m\ell^2\ddot{\theta} + d\dot{\theta} + m\ell g\cos\theta = u$$

where d is the viscous damping.

The control $u = \tau$ is the applied torque

Design state feedback controller $u = u(x)$ with $x = (\theta, \dot{\theta})^T$

Introduce new control variable v and let

$$u = m\ell^2 v + d\dot{\theta} + m\ell g\cos\theta$$

Then

$$\ddot{\theta} = v$$

Choose e.g. a PD-controller

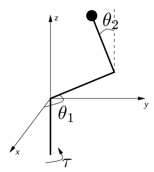
$$v = v(\theta, \dot{\theta}) = k_p(\theta_{\text{ref}} - \theta) - k_d\dot{\theta}$$

This gives the closed-loop system:

$$\ddot{\theta} + k_d\dot{\theta} + k_p\theta = k_p\theta_{\text{ref}}$$

Hence, $u = m\ell^2[k_p(\theta - \theta_{\text{ref}}) - k_d\dot{\theta}] + d\dot{\theta} + m\ell g\cos\theta$

Multi-link robot (n-joints)



General form

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

M	$n \times n$ inertia matrix, $M = M^T > 0$
$C\dot{\theta}$	$n \times 1$ vector of centrifugal and Coriolis forces
G	$n \times 1$ vector of gravitation terms

Computed torque

The computed torque
(also known as "Exact linearization", "dynamic inversion" , etc.)

$$\begin{aligned} u &= M(\theta)v + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) \\ v &= K_p(\theta_{ref} - \theta) - K_d\dot{\theta}, \end{aligned} \tag{1}$$

gives closed-loop system

$$\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_{Ref}$$

The matrices K_d and K_p can be chosen diagonal (no cross-terms) and then this decouples into n independent second-order equations.

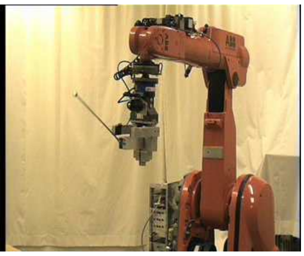
Lyapunov-Based Control Design Methods

$$\dot{x} = f(x, u)$$

- Select Lyapunov function $V(x)$ for stability verification
- Find state feedback $u = u(x)$ that makes V decreasing
- Method depends on structure of f

Examples are energy shaping as in Lab 2 and, e.g., **Back-stepping control design**, which require certain f discussed later.

Lab 2 : Energy shaping for swing-up control

[movie]

Use Lyapunov-based design for swing-up control.

Lab 2 : Energy shaping for swing-up control

Rough outline of method to get the pendulum to the upright position

- Find expression for total energy E of the pendulum (potential energy + kinetic energy)
- Let E_n be energy in upright position.
- Look at deviation $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find "swing strategy" of control torque u such that $\dot{V} \leq 0$

Example of Lyapunov-based design

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2,\end{aligned} \tag{2}$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

We try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \; \forall (x_1,\, x_2) \neq (0,\, 0)$.

Example - cont'd

$$\begin{aligned}\dot{V} &= \dot{x}_1x_1 + \dot{x}_2x_2 = (-3x_1 + 2x_1x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4\end{aligned}$$

We would like to have

$$\dot{V} < 0 \qquad \forall (x_1,\, x_2) \neq (0,\, 0)$$

Inserting the control law, $u = -2x_1x_2^2$, we get

$$\dot{V} = -3x_1^2 - x_2^2 \underbrace{-2x_1^2x_2^2 + 2x_1^2x_2^2}_{=0} - x_2^4 = -3x_1^2 - x_2^2 - x_2^4 < 0, \quad \forall x \neq 0$$

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned} \tag{3}$$

Find a globally asymptotically stabilizing control law $u = u(x)$.

Attempt 1: Try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \; \forall (x_1,\, x_2) \neq (0,\, 0)$.

$$\dot{V} = \dot{x}_1x_1 + \dot{x}_2x_2 = x_2^3 \cdot x_1 + u \cdot x_2 = x_2\underbrace{(x_2^2x_1 + u)}_{-x_2} = -x_2^2 \leq 0$$

where we chose

$$u = -x_2 - x_2^2x_1$$

However $\dot{V} = 0$ as soon as $x_2 = 0$ (Note: x_1 could be anything).

According to LaSalle's theorem the set $E = \{x | \dot{V} = 0\} = \{(x_1,\, 0)\} \, \forall x_1$

What is the largest invariant subset $M \subseteq E$?

Plugging in the control law $u = -x_2 - x_2^2x_1$, we get

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_2 - x_2^2x_1\end{aligned} \tag{4}$$

Observe that if we start anywhere on the line $\{(x_1,\, 0)\}$ we will stay in the same point as both $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, thus M=E and we will not converge to the origin, but get stuck on the line $x_2 = 0$.

Draw phase-plot with e.g., `pplane and study the behaviour`.

Attempt 2:

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned} \tag{5}$$

Try the Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4,$$

which satisfies

- $V(0, 0) = 0$
- $V(x_1, x_2) > 0, \quad \forall (x_1,\, x_2) \neq (0,\, 0).$
- radially unbounded,
- compute

$$\dot{V} = \dot{x}_1x_1 + \dot{x}_2x_2^3 = x_2^3(x_1 + u) = -x_2^4 \leq 0$$

↑

if we use $u = -x_1 - x_2$

With

$$u = -x_1 - x_2$$

we get the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2\end{aligned} \tag{6}$$

$\dot{V} = 0$ if $x_2 = 0$, thus

$$E = \{x | \dot{V} = 0\} = \{(x_1,\, 0) \, \forall x_1\}$$

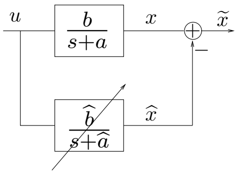
However, now the only possibility to stay on $x_2 = 0$ is if $x_1 = 0$, (else $\dot{x}_2 \neq 0$ and we will leave the line $x_2 = 0$).
Thus, the largest invariant set

$$M = (0, 0)$$

According to the Invariant Set Theorem (LaSalle) all solutions will end up in M and so the origin is GAS.

Draw phase-plot with e.g., `pplane and study the behaviour`.

Adaptive Noise Cancellation Revisited



$$
\begin{aligned}
\dot{x} + ax &= bu \\
\dot{\widehat{x}} + \widehat{a}\widehat{x} &= \widehat{b}u
\end{aligned}
$$

Introduce $\widetilde{x} = x - \widehat{x}$, $\widetilde{a} = a - \widehat{a}$, $\widetilde{b} = b - \widehat{b}$.

Want to design adaptation law so that $\widetilde{x} \to 0$

Let us try the Lyapunov function

$$
\begin{aligned}
V &= \frac{1}{2}(\widetilde{x}^2 + \gamma_a \widetilde{a}^2 + \gamma_b \widetilde{b}^2) \\
\dot{V} &= \widetilde{x}\dot{\widetilde{x}} + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} = \\
&= \widetilde{x}(-a\widetilde{x} - \widetilde{a}\widehat{x} + \widetilde{b}u) + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} = -a\widetilde{x}^2
\end{aligned}
$$

where the last equality follows if we choose

$$
\dot{\widetilde{a}} = -\dot{\widehat{a}} = \frac{1}{\gamma_a}\widetilde{x}\widehat{x} \qquad \dot{\widetilde{b}} = -\dot{\widehat{b}} = -\frac{1}{\gamma_b}\widetilde{x}u
$$

Invariant set: $\widetilde{x} = 0$.

This proves that $\widetilde{x} \to 0$.

(The parameters $\widetilde{a}$ and $\widetilde{b}$ do not necessarily converge: $u \equiv 0$.)

A Control Design Idea, and a Problem

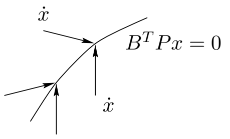
Assume $V(x) = x^T P x$, $P > 0$, represents the energy of

$$
\dot{x} = Ax + Bu, \qquad u \in [-1, 1]
$$

Idea: Choose u such that V decays as fast as possible

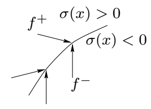
$$
\begin{aligned}
\dot{V} &= x^T(A^T P + AP)x + 2B^T P x \cdot u \\
u &= -\text{sgn}(B^T P x)
\end{aligned}
$$

The following situation might then occur ("system is not Lipschitz")



Sliding Modes

$$
\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}
$$



The **switching set/ sliding set** is where $\sigma(x) = 0$ and f^+ and f^- point towards $\sigma(x) = 0$.

The **switching set/ sliding set** is given by x such that

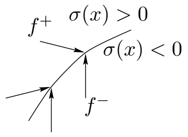
$$
\begin{aligned}
\sigma(x) &= 0 \\
\frac{\partial \sigma}{\partial x} f^+ &= (\nabla \sigma) f^+ < 0 \\
\frac{\partial \sigma}{\partial x} f^- &= (\nabla \sigma) f^- > 0
\end{aligned}
$$

Note: If f^+ and f^- point "in the same direction" on both sides of the set $\sigma(x) = 0$ then the solution curves will just pass through and this region will not belong to the sliding set.

Sliding Mode

If f^+ and f^- both points towards $\sigma(x) = 0$, what will happen then?

The sliding dynamics are $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$, where α is obtained from $\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial x} \cdot \dot{x} = 0$ on $\{\sigma(x) = 0\}$.



More precisely, find α such that the components of f^+ and f^- perpendicular to the switching surface cancel: $\alpha f^+_\perp + (1 - \alpha) f^-_\perp = 0$ The resulting dynamics is then the sum of the corresponding components along the surface.

Example

$$
\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u = Ax + Bu
$$
$$
u = -\text{sgn}\sigma(x) = -\text{sgn}x_2 = -\text{sgn}(Cx)
$$

which means that

$$
\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}
$$

Determine the *sliding set* and the *sliding dynamics*.

$$
\begin{aligned}
\dot{x}_1 &= -x_2 + u = -x_2 - \text{sgn}(x_2) \\
\dot{x}_2 &= x_1 - x_2 + u = x_1 - x_2 - \text{sgn}(x_2)
\end{aligned}
$$

$$
f^+ = \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} \qquad f^- = \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix}
$$

$$
\begin{aligned}
\sigma(x) = x_2 = 0 \ &\Rightarrow \underline{x_2 = 0} \\
\frac{\partial \sigma}{\partial x} f^+ = \begin{bmatrix} 0 & 1 \end{bmatrix} f^+ = x_1 - x_2 - 1 < 0 \ &\Rightarrow \underline{x_1 < 1} \\
\frac{\partial \sigma}{\partial x} f^- = \begin{bmatrix} 0 & 1 \end{bmatrix} f^- = x_1 - x_2 + 1 > 0 \ &\Rightarrow \underline{x_1 > -1}
\end{aligned}
$$

We will thus have a sliding set for $\{-1 < x_1 < 1, \ x_2 = 0\}$

The normal projections of (f^+, f^-) to $\sigma(x) = x_2 = 0$ are

$$
f^+_\perp = \begin{bmatrix} 0 \\ x_1 - x_2 - 1 \end{bmatrix} \qquad f^-_\perp = \begin{bmatrix} 0 \\ x_1 - x_2 + 1 \end{bmatrix}
$$

Find $\alpha \in [0, 1]$ such that $\alpha f^+_\perp + (1 - \alpha) f^-_\perp = 0$ on $\{x_2 = 0\}$

$$
\alpha(x_1 - x_2 - 1) + (1 - \alpha)(x_1 - x_2 + 1) = 0
$$
$$
\Rightarrow
$$
$$
\alpha = \frac{x_1 + 1}{2} \text{ as } x_2 = 0
$$

Note: $\alpha \in [0, 1] \Leftrightarrow x_1 \in [-1, 1]$

The *sliding dynamics* are the given by

$$
\begin{aligned}
\dot{x} &= \alpha f^{+} + (1-\alpha) f^{-} \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \alpha \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -2\alpha - x_2 - 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

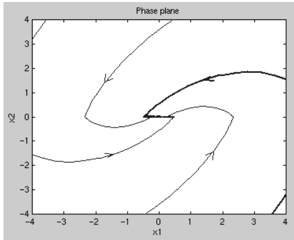
where we inserted $x_2 = 0$ and $\alpha = \frac{x_1+1}{2}$

We see that on the sliding set $\{-1 < x < 1,\, x_2 = 0\}$ we have

$$
\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 0
\end{aligned}
$$

For any initial condition starting on the sliding set, there will be exponential convergence to $x_1 = x_2 = 0$.

Sliding Mode Dynamics



The dynamics along the sliding set in $\sigma(x) = 0$ can also be obtained by finding $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.

Equivalent Control

Assume

$$
\begin{aligned}
\dot{x} &= f(x) + g(x) u \\
u &= -\text{sgn}\sigma(x)
\end{aligned}
$$

has a sliding set on $\sigma(x) = 0$. Then, for $x(t)$ staying on the sliding set we should have

$$
0 = \dot{\sigma}(x) = \frac{\partial \sigma}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial \sigma}{\partial x}\Big(f(x) + g(x)u\Big)
$$

The equivalent control is thus given by solving

$$
u_{\text{eq}} = -\left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x} f(x)
$$

for all those x such that $\sigma(x) = 0$ and $\frac{\partial \sigma}{\partial x} g(x) \neq 0$.

Equivalent Control for Linear System

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
u &= -\text{sgn}\sigma(x) = -\text{sgn}(Cx)
\end{aligned}
$$

Assume CB invertible. The sliding set lies in $\sigma(x) = Cx = 0$.

$$
0 = \dot{\sigma}(x) = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big) = C\big(Ax + Bu_{\text{eq}}\big)
$$

gives $CBu_{\text{eq}} = -CAx$.

Example (cont'd) For the previous system

$$
u_{\text{eq}} = -(CB)^{-1}CAx = -(x_1 - x_2)/1 = -x_1,
$$

because $\sigma(x) = x_2 = 0$. Same result as above.

More on the Sliding Dynamics

If $CB > 0$ then the dynamics along a sliding set in $Cx = 0$ is

$$
\dot{x} = Ax + Bu_{\text{eq}} = \Big(I - (CB)^{-1}BC\Big)Ax,
$$

One can show that the eigenvalues of $(I - (CB)^{-1}BC)A$ equals the zeros of $G(s) = C(sI - A)^{-1}B$. (exercise for PhD students)

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$. Choose $\sigma(x)$ such that the sliding mode tends to the origin.

Assume system has form

$$
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = f(x) + g(x)u
$$

Choose control law

$$
u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)}\text{sgn}\sigma(x),
$$

where $\mu > 0$ is a design parameter, $\sigma(x) = p^T x$, and $p^T = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix}$ represents a stable polynomial.

Sliding Mode Control gives Closed-Loop Stability

Consider $\mathcal{V}(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$
\dot{\mathcal{V}} = \sigma(x)\dot{\sigma}(x) = x^T p\big(p^T f(x) + p^T g(x) u\big)
$$

With the chosen control law, we get

$$
\dot{\mathcal{V}} = -\mu\sigma(x)\text{sgn}\sigma(x) \leq 0
$$

so $\sigma(x) \to 0$ as $t \to +\infty$. In fact, one can prove that this occurs in finite time.

$$
\begin{aligned}
0 = \sigma(x) &= p_1 x_1 + \dots + p_{n-1}x_{n-1} + p_n x_n \\
&= p_1 x_n^{(n-1)} + \dots + p_{n-1}x_n^{(1)} + p_n x_n^{(0)}
\end{aligned}
$$

where $x^{(k)}$ denote time derivative. P stable gives that $x(t) \to 0$.

Note: $\mathcal{V}$ by itself does not guarantee stability. It only guarantees convergence to the line $\{\sigma(x) = 0\} = \{p^T x = 0\}$.

Time to Switch

Consider an initial point x such that $\sigma_0 = \sigma(x) > 0$. Then

$$
\dot{\sigma} = p^T \dot{x} = p^T(f + gu) = -\mu\text{sgn}(\sigma) = -\mu
$$

Hence, the time to the first switch is

$$
t_s = \frac{\sigma_0}{\mu} < \infty
$$

Note that $t_s \to 0$ as $\mu \to \infty$.

Example—Sliding Mode Controller

Design state-feedback controller for

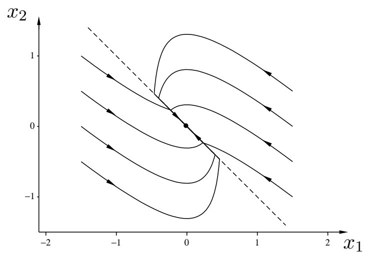
$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$u = -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \mathrm{sgn}\sigma(x)$$
$$= -2x_1 - \mu \mathrm{sgn}(x_1 + x_2)$$

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding set is in $\sigma(x) = x_1 + x_2$.



Time Plots

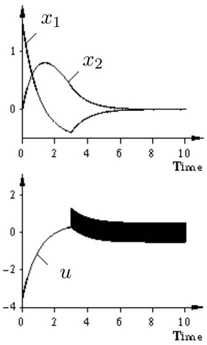
Initial condition $x(0) = \begin{bmatrix} 1.5 & 0 \end{bmatrix}^T$.

Simulation agrees well with time to switch

$$t_s = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \widehat{f}(x) + \widehat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{V} = \sigma(x) \left[\frac{p^T (f \widehat{g}^T - \widehat{f} g^T) p}{p^T \widehat{g}} - \mu \frac{p^T g}{p^T \widehat{g}} \mathrm{sgn}\sigma(x) \right] < 0$$

if $\mathrm{sgn}(p^T g) = \mathrm{sgn}(p^T \widehat{g})$ and $\mu > 0$ is sufficiently large.

Closed-loop system is quite robust against model errors!

(High gain control with stable open loop zeros)

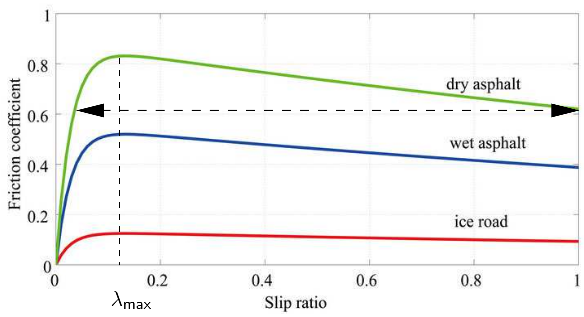
Implementation

A relay with hysteresis or a smooth (e.g. linear) region is often used in practice.

Choice of hysteresis or smoothing parameter can be critical for performance

More complicated structures with several relays possible. Harder to design and analyze.

ABS Breaking



By moving along the dashed arrow, an ABS controller attains higher average friction than what is obtained by locked wheels.

Ideally, the slip ratio should be kept at the maximizing value $\lambda_{\max}$.

Next Lectures

- L10–L12: Optimal control methods
- L13: Other synthesis methods
- L14: Course summary

Next Lecture

- Optimal control

Read chapter 18 in [Glad & Ljung] for preparation.