Lecture 4 — Lyapunov Stability

Material

- Glad & Ljung Ch. 12.2
- Khalil Ch. 4.1-4.3
- Lecture notes

Today's Goal

To be able to

- *prove local and global stability of an equilibrium point using Lyapunov's method*
- *show stability of a set (e.g., an equilibrium, or a limit cycle) using La Salle's invariant set theorem.*

Alexandr Mihailovich Lyapunov (1857–1918)

Master thesis "On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis "The general problem of the stability of motion," 1892.

Main idea

Lyapunov formalized the idea:

If the total energy is dissipated, then the system must be stable.

Main benefit: By looking at **how** an energy-like function **V** (a so called *Lyapunov function*) **changes over time**, we might **conclude** that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.

Analysis: Check if V is decreasing with time:

- Continuous time: $\frac{dV}{dt} < 0$
- Discrete time: $V(k+1) - V(k) < 0$

Main question: **How to find** a Lyapunov function?

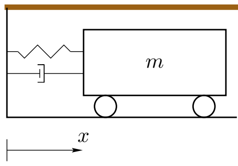
Examples

Start with a Lyapunov *candidate* V to measure e.g.,

- "size"[1] of state and/or output error,
- "size" of deviation from true parameters,
- energy difference from desired equilibrium,
- weighted combination of above
- ...

[1] Often a magnitude measure or (squared) norm like $|e|_2^2, \ldots$

A Motivating Example



$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}$$

$$b, k_0, k_1 > 0$$

Total energy = kinetic + pot. energy: $V = \frac{mv^2}{2} + \int_0^x F_{spring}\, ds \;\Rightarrow$

$$V(x, \dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

$$\frac{d}{dt}V(x,\dot{x}) = m\ddot{x}\dot{x} + k_0 x\dot{x} + k_1 x^3 \dot{x} = \{\text{plug in system dynamics}^2\}$$
$$= -b|\dot{x}|^3 \quad < 0, \text{ for } \dot{x} \neq 0$$

What does this mean?

[2] Also referred to evaluate "along system trajectories".

Stability Definitions

An equilibrium point x^* of $\dot{x} = f(x)$ (i.e., $f(x^*) = 0$) is

- **locally stable**, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0) - x^*\| < r \quad \Rightarrow \quad \|x(t) - x^*\| < R, \quad t \geq 0$$

- **locally asymptotically stable**, if locally stable and

$$\|x(0) - x^*\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = x^*$$

- **globally asymptotically stable**, if asymptotically stable for all $x(0) \in \mathbb{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(x^*) = 0$ where x^* is in the interior of $\Omega \subset \mathbb{R}^n$. Assume that $V : \Omega \to \mathbb{R}$ is a $\mathcal{C}^1$ function. If

(1) $V(x^*) = 0$
(2) $V(x) > 0$, for all $x \in \Omega$, $x \neq x^*$
(3) $\dot{V}(x) \leq 0$ along all trajectories of **the system** in Ω

then x^* is locally stable.

If also

(4) $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq x^*$

then x^* is locally asymptotically stable.

Furthermore, if $\Omega = \mathbb{R}^n$ and also

(5) $V(x) \to \infty$ as $\|x\| \to \infty$

then x^* is globally asymptotically stable.

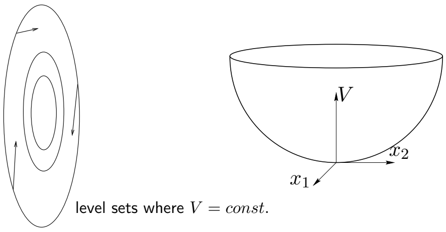
Lyapunov Functions (≈ Energy Functions)

A function V that fulfills (1)–(3) is called a *Lyapunov function*.

Condition (3) means that V is non-increasing along all trajectories in Ω:

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \sum_i \frac{\partial V}{\partial x_i} f_i(x) \le 0$$

where $\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots \frac{\partial V}{\partial x_n}\right]$



Conservation and Dissipation

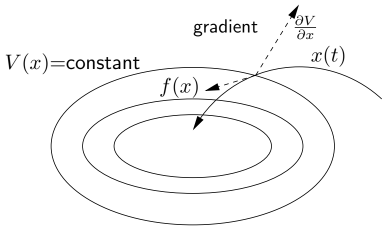
Conservation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 0$, i.e., the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \le 0$, i.e., the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $\{z : V(z) = c\}$ make an obtuse angle (Sw. "trubbig vinkel").

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

Geometric interpretation



Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

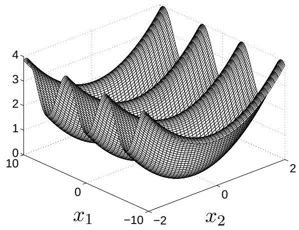
2 min exercise—Pendulum

Show that the origin is locally stable for a mathematical pendulum.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Use as a Lyapunov function candidate

$$V(x) = (1 - \cos x_1)g\ell + \ell^2 x_2^2/2$$



Lyapunov's Linearization Method

Consider

$$\dot{x} = f(x)$$

Assume that $f(0) = 0$.

$$\dot{x} = Ax + g(x), \qquad ||g(x)|| = o(||x||) \text{ as } x \to 0.$$

Find Lyapunov function for linearization $\dot{x} = Ax$. Use it for original system (near the equilibrium)!

Positive Definite Matrices

Definition: Symmetric matrix $M = M^T$ is

- **positive definite** $(M > 0)$ if $x^T M x > 0, \ \forall x \neq 0$
- **positive semidefinite** $(M \ge 0)$ if $x^T M x \ge 0, \ \forall x$

Lemma:

- $M = M^T > 0 \iff \lambda_i(M) > 0, \ \forall i$
- $M = M^T \ge 0 \iff \lambda_i(M) \ge 0, \ \forall i$

$$M = M^T > 0 \qquad V(x) := x^T M x$$

$$\Downarrow$$

$$V(0) = 0, \quad V(x) > 0, \forall x \neq 0$$

$V(x)$ candidate Lyapunov function

More matrix results

- for symmetric matrix $M = M^T$

  $$\lambda_{\min}(M)\|x\|^2 \ \le \ x^T M x \ \le \ \lambda_{\max}(M)\|x\|^2, \quad \forall x$$

  Proof idea: factorize $M = U\Lambda U^T$, unitary U (i.e., $||Ux|| = ||x|| \ \forall x$), $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$

- for any matrix M

  $$\|Mx\| \ \le \ \sqrt{\lambda_{\max}(M^T M)}\|x\|, \quad \forall x$$

Example - Lyapunov function for linear system

$$\dot{x} = Ax \ = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

Eigenvalues of A : $\{-1, -3\}$ ⇒ (global) asymptotic stability.

Find a quadratic Lyapunov function for the system (1):

$$V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad P = P^T > 0$$

Take any $Q = Q^T > 0$, say $Q = I_{2\times 2}$. Solve $A^T P + PA = -Q$.

Example cont'd

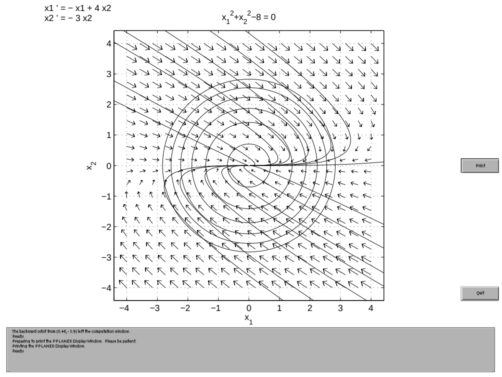
$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (2)$$

Solving for p_{11}, p_{12} and p_{22} gives

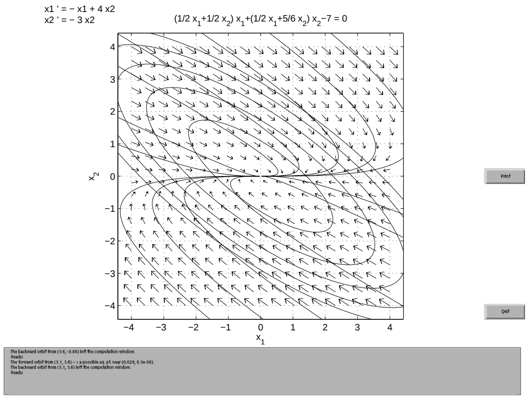
$$\begin{aligned} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$



Phase plot showing that

$V = \frac{1}{2}(x_1^2 + x_2^2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ does NOT work.



Phase plot with level curves $x^T P x = c$ for P found in example.

Lyapunov Stability for Linear Systems

Linear system: $\dot{x} = Ax$

Lyapunov equation: Let $Q = Q^T > 0$. Solve

$$PA + A^T P = -Q$$

with respect to the symmetric matrix P.

Lyapunov function: $V(x) = x^T P x$, $\Rightarrow$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x < 0$$

Asymptotic Stability: If $P = P^T > 0$, then the Lyapunov Stability Theorem implies (local=global) asymptotic stability, hence the eigenvalues of A must satisfy $\operatorname{Re} \lambda_k(A) < 0, \ \forall k$

Converse Theorem for Linear Systems

If $\operatorname{Re} \lambda_k(A) < 0 \ \forall k$, then for every $Q = Q^T > 0$ there exists $P = P^T > 0$ such that $PA + A^T P = -Q$

Proof: Choose $P = \displaystyle\int_0^\infty e^{A^T t} Q e^{At} dt$. Then

$$\begin{aligned} A^T P + PA &= \lim_{t \to \infty} \int_0^t \left(A^T e^{A^T \tau} Q e^{A\tau} + e^{A^T \tau} Q A e^{A\tau} \right) d\tau \\ &= \lim_{t \to \infty} \left[e^{A^T \tau} Q e^{A\tau} \right]_0^t \\ &= -Q \end{aligned}$$

Interpretation

Assume $\dot{x} = Ax$, $x(0) = z$. Then

$$\int_0^\infty x^T(t) Q x(t) dt = z^T \left(\int_0^\infty e^{A^T t} Q e^{At} dt \right) z = z^T P z$$

Thus $V(z) = z^T P z$ is the cost-to-go from z (with no input) and integral quadratic cost function with weighting matrix Q.

Lyapunov's Linearization Method

Recall from Lecture 2:

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $f(0) = 0$. Linearization

$$\dot{x} = Ax + g(x), \qquad ||g(x)|| = o(||x||) \text{ as } x \to 0.$$

(1) $\operatorname{Re} \lambda_k(A) < 0, \forall k \quad \Rightarrow \quad x = 0$ locally asympt. stable

(2) $\exists k : \operatorname{Re} \lambda_k(A) > 0 \quad \Rightarrow \quad x = 0$ unstable

Proof of (1) in Lyapunov's Linearization Method

Put $V(x) := x^T P x$. Then, $V(0) = 0$, $V(x) > 0 \ \forall x \neq 0$, and

$$\begin{aligned} \dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P [Ax + g(x)] + [x^T A^T + g^T(x)] P x \\ &= x^T (PA + A^T P) x + 2x^T P g(x) = -x^T Q x + 2x^T P g(x) \end{aligned}$$

$$x^T Q x \geq \lambda_{\min}(Q) \|x\|^2$$

and for all $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\| < \gamma \|x\|, \qquad \forall \|x\| < r$$

Thus, choosing γ sufficiently small gives

$$\dot{V}(x) \leq -\big(\lambda_{\min}(Q) - 2\gamma \lambda_{\max}(P)\big) \|x\|^2 < 0$$

Lyapunov Theorem for Global Asymptotic Stability

Theorem Let $\dot{x} = f(x)$ and $f(x^*) = 0$.
If there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(x^*) = 0$
(2) $V(x) > 0$, for all $x \neq x^*$
(3) $\dot{V}(x) < 0$ for all $x \neq x^*$
(4) $V(x) \to \infty$ as $\|x\| \to \infty$

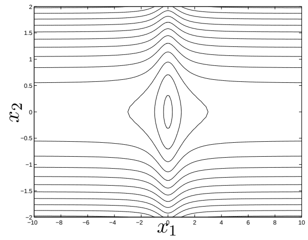
then x^* is a globally asymptotically stable equilibrium.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1 + x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



Example [Khalil]:

$$\dot{x}_1 = \frac{-6x_1}{(1 + x_1^2)^2} + 2x_2$$
$$\dot{x}_2 = \frac{-2(x_1 + x_2)}{(1 + x_1^2)^2}$$

Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(x^*) = 0$
(2) $V(x) > 0$ for all $x \neq x^*$
(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x
(4) $V(x) \to \infty$ as $\|x\| \to \infty$

then x^* is globally **exponentially** stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to t gives

$$\log V(x(t)) - \log V(x(0)) \leq -\alpha t \;\Rightarrow\; V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0$, $t \to \infty$.

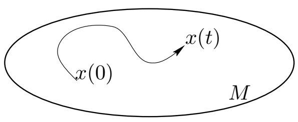
Using the properties of V it follows that $x(t) \to 0$, $t \to \infty$.

Invariant Sets

Definition: A set M is called **invariant** if for the system

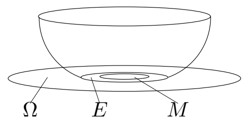
$$\dot{x} = f(x),$$

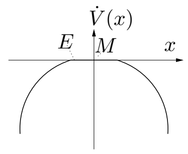
$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.



LaSalle's Invariant Set Theorem

Theorem Let $\Omega \subseteq \mathbb{R}^n$ compact invariant set for $\dot{x} = f(x)$.
Let $V : \Omega \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $\dot{V}(x) \leq 0, \ \forall x \in \Omega$,
$E := \{x \in \Omega : \dot{V}(x) = 0\}$, M :=largest invariant subset of E
$\Longrightarrow \quad \forall x(0) \in \Omega$, $x(t)$ approaches M as $t \to +\infty$





Note that V must **not** be a positive definite function in this case.

Special Case: Global Stability of Equilibrium

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$
(2) $\dot{V}(x) \leq 0$ for all x
(3) $V(x) \to \infty$ as $\|x\| \to \infty$
(4) The only solution of $\dot{x} = f(x)$, $\dot{V}(x) = 0$ is $x(t) = 0 \ \forall t$

$\Longrightarrow \qquad x = 0$ is globally asymptotically stable.

A Motivating Example (cont'd)

$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3$$
$$V(x) = (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0$$
$$\dot{V}(x) = -b|\dot{x}|^3$$

Assume that there is a trajectory with $\dot{x}(t) = 0$, $x(t) \neq 0$. Then

$$\frac{d}{dt}\dot{x}(t) = -\frac{k_0}{m}x(t) - \frac{k_1}{m}x^3(t) \neq 0,$$

which means that $\dot{x}(t)$ can not stay constant.

Hence, $\dot{V}(x) = 0 \iff x(t) \equiv 0$, and LaSalle's theorem gives global asymptotic stability.

Example—Stable Limit Cycle

Show that $M = \{x : \|x\| = 1\}$ is a asymptotically stable limit cycle for (almost globally, except for starting at x=0):

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

Let $V(x) = (x_1^2 + x_2^2 - 1)^2$.

$$\begin{aligned}\frac{dV}{dt} &= 2(x_1^2 + x_2^2 - 1)\frac{d}{dt}(x_1^2 + x_2^2 - 1) \\ &= -2(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \le 0 \quad \text{for } x \in \Omega\end{aligned}$$

$\Omega = \{0 < \|x\| \le R\}$ is invariant for $R = 1$.

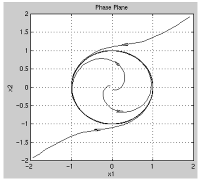
Example—Stable Limit Cycle

$$E = \{x \in \Omega : \dot{V}(x) = 0\} = \{x : \|x\| = 1\}$$

$M = E$ is an invariant set, because

$$\frac{d}{dt}V = -2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2) = 0 \quad \text{for } x \in M$$

We have shown that M is a asymtotically stable limit cycle (globally stable in $R - \{0\}$)



A Motivating Example (revisited)

$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3$$
$$V(x, \dot{x}) = (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0$$

$\dot{V}(x, \dot{x}) = -b|\dot{x}|^3$ gives $E = \{(x, \dot{x}) : \dot{x} = 0\}$.

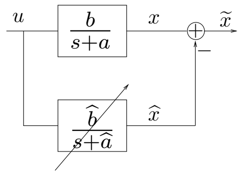
Assume there exists $(\bar{x}, \dot{\bar{x}}) \in M$ such that $\bar{x}(t_0) \neq 0$. Then

$$m\ddot{\bar{x}}(t_0) = -k_0\bar{x}(t_0) - k_1\bar{x}^3(t_0) \neq 0$$

so $\dot{\bar{x}}(t_0+) \neq 0$ so the trajectory will immediately leave M. A contradiction to that M is invariant.

Hence, $M = \{(0,0)\}$ so the origin is asymptotically stable.

Adaptive Noise Cancellation by Lyapunov Design



$$\dot{x} + ax = bu$$
$$\dot{\hat{x}} + \hat{a}\hat{x} = \hat{b}u$$

Introduce $\tilde{x} = x - \hat{x}$, $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$.

Want to design adaptation law so that $\tilde{x} \to 0$

Let us try the Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}(\tilde{x}^2 + \gamma_a\tilde{a}^2 + \gamma_b\tilde{b}^2) \\ \dot{V} &= \tilde{x}\dot{\tilde{x}} + \gamma_a\tilde{a}\dot{\tilde{a}} + \gamma_b\tilde{b}\dot{\tilde{b}} = \\ &= \tilde{x}(-a\tilde{x} - \tilde{a}\hat{x} + \tilde{b}u) + \gamma_a\tilde{a}\dot{\tilde{a}} + \gamma_b\tilde{b}\dot{\tilde{b}} = -a\tilde{x}^2\end{aligned}$$

where the last equality follows if we choose

$$\dot{\tilde{a}} = -\dot{\hat{a}} = \frac{1}{\gamma_a}\tilde{x}\hat{x} \qquad \dot{\tilde{b}} = -\dot{\hat{b}} = -\frac{1}{\gamma_b}\tilde{x}u$$

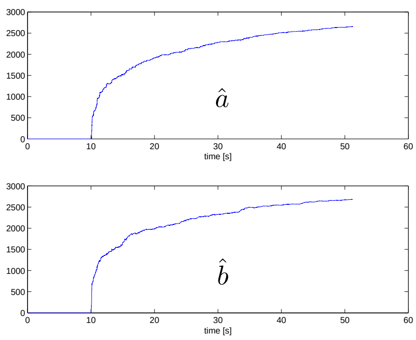
Invariant set: $\tilde{x} = 0$.

This proves that $\tilde{x} \to 0$.

(The parameters $\tilde{a}$ and $\tilde{b}$ do not necessarily converge: $u \equiv 0$.)

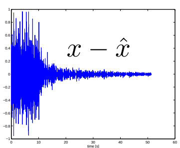
Demonstration if time permits

Results



Estimation of parameters starts at t=10 s.

Results



Estimation of parameters starts at t=10 s.

Next Lecture

- Stability analysis using input-output (frequency) methods