# Nonlinear Control and Servo systems

# Lecture 1

Giacomo Como, 2015

Dept. of Automatic Control
LTH, Lund University

November 2, 2015

# Overview Lecture 1

- Practical information
- Course contents
- Nonlinear control systems phenomena
- Nonlinear differential equations

# Course Goal

*To provide students with solid theoretical foundations of nonlinear control systems combined with good engineering ability*

You should after the course be able to

- recognize common nonlinear control problems,
- use some powerful analysis methods, and
- use some practical design methods.

# Today's Goal

- *Recognize some common nonlinear phenomena*
- *Transform differential equations to autonomous form, first-order form, and feedback form*
- *Describe saturation, dead-zone, relay with hysteresis, backlash*
- *Calculate equilibrium points*

# Course Material

- Textbook
  - Glad and Ljung, *Reglerteori, flervariabla och olinjra metoder*, 2003, Studentlitteratur,ISBN 9-14-403003-7 or the English translation *Control Theory*, 2000, Taylor & Francis Ltd, ISBN 0-74-840878-9. The course covers Chapters 11-16,18. (MPC and optimal control not covered in the other alternative textbooks.)
  - H. Khalil, *Nonlinear Systems* (3rd ed.), 2002, Prentice Hall, ISBN 0-13-122740-8. A good, a bit more advanced text.
  - `ALTERNATIVE:` Slotine and Li, *Applied Nonlinear Control*, Prentice Hall, 1991. The course covers chapters 1-3 and 5, and sections 4.7-4.8, 6.2, 7.1-7.3.

# Course Material, cont.

- Handouts (Lecture notes + extra material)
- Exercises (can be downloaded from the course home page)
- Lab PMs 1, 2 and 3
- Home page

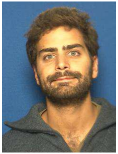
  `http://www.control.lth.se/course/FRTN05/`
- Matlab/Simulink other simulation software

  see home page

# Lectures and labs

The lectures (28 hours) are given as follows:

| | | |
|---|---|---|
| Mon 13–15, | E:C | Nov 2 – Dec 7 |
| Wed 8–10, | E:C | Nov 4 – Dec 9 |
| **Fri 8-10** | E:C | **Nov 6** |
| **Thu 8-10** | **M:2112B** | **Dec 10** |



Lectures are given in English.

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The three laboratory experiments are mandatory.

Sign-up lists are posted on the web at least one week before the first laboratory experiment. *The lists close one day before the first session.*

The Laboratory PMs are available at the course homepage.

Before the lab sessions some home assignments have to be done.

No reports after the labs.

# Exercise sessions and TAs

The exercises (28 hours) are offered twice a week

Tue 15:15-17:00 **M:2112B** Wed 15:15-17:00 **M:2112B**

NOTE: The exercises are held in the seminar room of the Automatic Control Department, M-building, second floor **see schedule on home page.**

EXCEPTIONS: (i) for the first two weeks only Wednesday exercise sessions are scheduled at 13:15–15:00 instead of 15:15-17:00.

(ii) on November 24 and 25 the exercise sessions will be held in a different room to be announced in due time

Christian Grussler

Olof Troeng

Mahdi Ghazei

# The Course

- 14 lectures
- 14 exercises
- 3 laboratories
- 5 hour exam: **January 13, 2015, 14:00-19:00, MA10 I-J**. Open-book exam: Lecture notes but no old exams or exercises allowed.

# Course Outline

| | |
|---|---|
| Lecture 1-3 | Modelling and basic phenomena (linearization, phase plane, limit cycles) |
| Lecture 2-6 | Analysis methods (Lyapunov, circle criterion, describing functions)) |
| Lecture 7-8 | Common nonlinearities (Saturation, friction, backlash, quantization)) |
| Lecture 9-13 | Design methods (Lyapunov methods, Backstepping, Optimal control) |
| Lecture 14 | Summary |

# Todays lecture

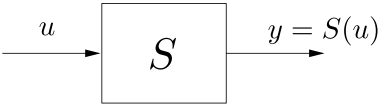
Common nonlinear phenomena

- Input-dependent stability
- Stable periodic solutions
- Jump resonances and subresonances

Nonlinear model structures

- Common nonlinear components
- State equations
- Feedback representation

# Linear Systems




**Definitions:** The system $S$ is *linear* if

$$\begin{aligned} S(\alpha u) &= \alpha S(u), && \text{scaling} \\ S(u_1 + u_2) &= S(u_1) + S(u_2), && \text{superposition} \end{aligned}$$

A system is *time-invariant* if delaying the input results in a delayed output:

$$y(t - \tau) = S(u(t - \tau))$$

# Linear time-invariant systems are easy to analyze

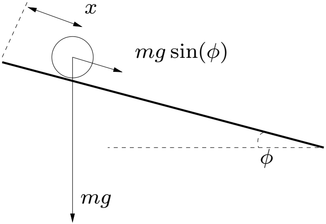
Different representations of same system/behavior

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0 \\ y(t) &= g(t) \star u(t) = \int g(r)u(t-r)dr \\ Y(s) &= G(s)U(s) \end{aligned}$$

Local stability = global stability:
Eigenvalues of $A$ (= poles of $G(s)$) in left half plane
Superposition:
Enough to know step (or impulse) response
Frequency analysis possible:
Sinusoidal inputs give sinusoidal outputs

# Linear models are not always enough

**Example:** Ball and beam




Linear model (acceleration along beam) :
Combine $F = m \cdot a = m\frac{d^2x}{dt^2}$ with $F = mg\sin(\phi)$:

$$\ddot{x}(t) = g\sin(\phi(t))$$

# Linear models are not enough

$x =$ position (m) $\qquad \phi =$ angle (rad) $\qquad g = 9.81$ (m/s$^2$)
Can the ball move 0.1 meter in 0.1 seconds with constant $\phi$?
Linearization: $\sin\phi \sim \phi$ for $\phi \sim 0$

$$\begin{cases} \ddot{x}(t) = g\phi \\ x(0) = 0 \end{cases}$$

Solving the above gives $x(t) = \frac{t^2}{2} g\phi$

For $x(0.1) = 0.1$, one needs $\phi = \frac{2*0.1}{0.1^2*g} \geq 2$ rad

Clearly outside linear region!
Contact problem, friction, centripetal force, saturation

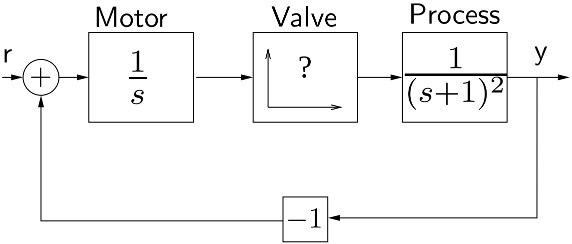
How fast can it be done? (Optimal control)

# Warm-Up Exercise: 1-D Nonlinear Control System

$$\dot{x} = x^2 - x + u$$

- stability for $u = 0$?
- stability for constant $u = b$?
- stability with linear feedback $u = ax + b$?
- stability with non-linear feedback $u(x) =$?

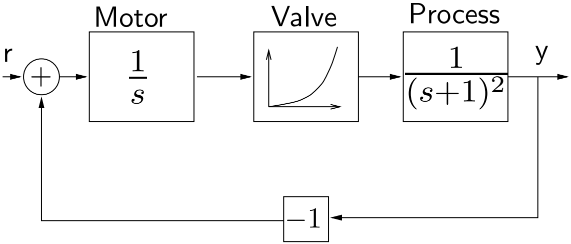
# Stability Can Depend on Amplitude




Valve characteristic $f(x) =???$

Step changes of amplitude, $r = 0.2$, $r = 1.68$, and $r = 1.72$

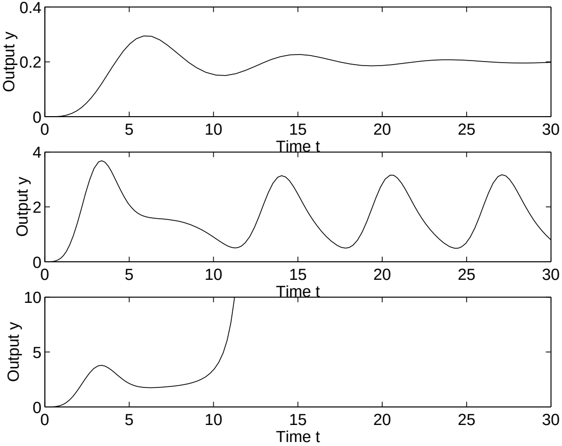
# Stability Can Depend on Amplitude




Valve characteristic $f(x) = x^2$

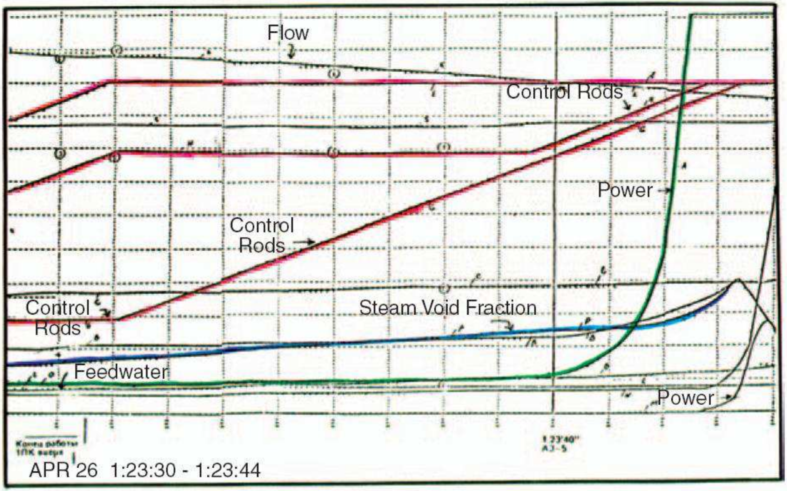
Step changes of amplitude, $r = 0.2$, $r = 1.68$, and $r = 1.72$

# Step Responses




$r = 0.2$

$r = 1.68$


Flow
Control Rods
Power
Control
Rods
Control
Rods
Steam Void Fraction
Feedwater
Power
APR 26 1:23:30 - 1:23:44

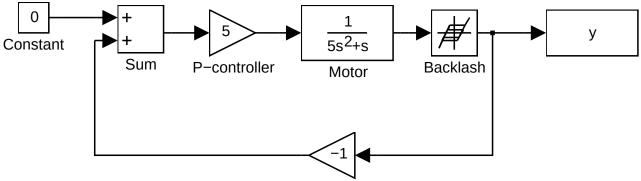
$\sim e^{2t}$



**Figure 2.** *Chernobyl nuclear power plant shortly after the accident on 26 April 1986.*

# Stable Periodic Solutions

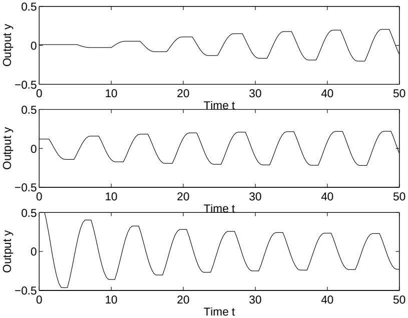
**Example:** Motor with back-lash




Motor: $G(s) = \frac{1}{s(1+5s)}$
Controller: $K = 5$

# Stable Periodic Solutions

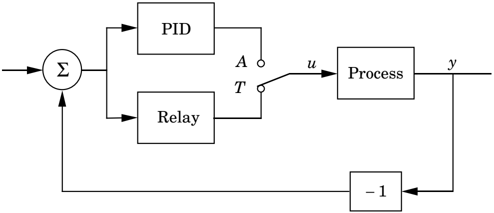
Output for different initial conditions:

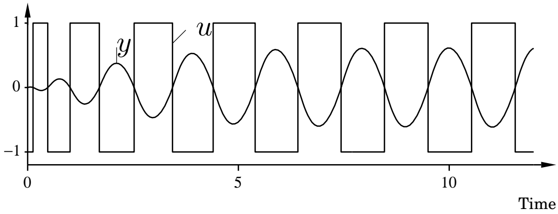



Frequency and amplitude independent of initial conditions!
Several systems use the existence of such a phenomenon

# Relay Feedback Example

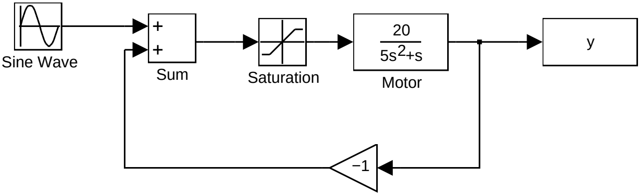
Period and amplitude of limit cycle are used for autotuning




[ patent: T Hgglund and K J strm]

# Jump Resonances

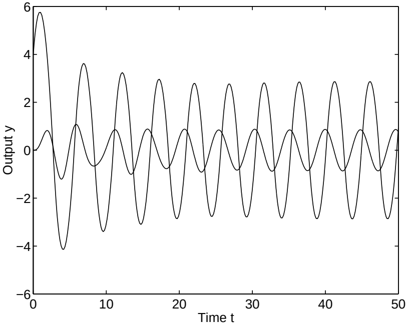



Response for sinusoidal depends on initial condition
Problem when doing frequency response measurement

# Jump Resonances

$$u = 0.5 \sin(1.3t), \quad \text{saturation level} = 1.0$$

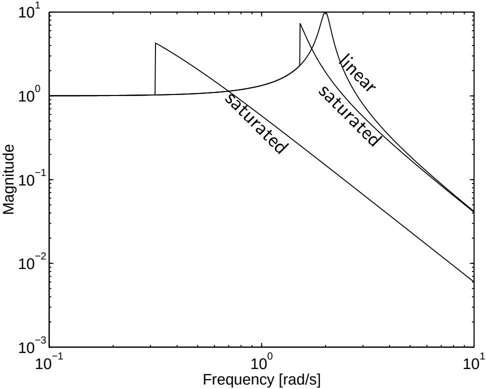
Two different initial conditions




give two different amplifications for same sinusoid!

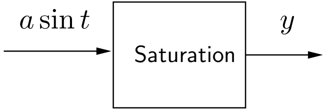
# Jump Resonances

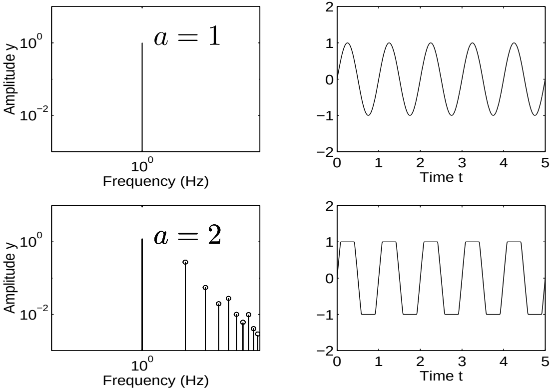
Measured frequency response (many-valued)



# New Frequencies

**Example:** Sinusoidal input, saturation level 1



# New Frequencies

**Example:** Electrical power distribution

THD = Total Harmonic Distortion = $\frac{\sum_{k=2}^{\infty} \text{energy in tone } k}{\text{energy in tone 1}}$

Nonlinear loads: Rectifiers, switched electronics, transformers

Important, increasing problem

Guarantee electrical quality

Standards, such as $THD < 5\%$

# New Frequencies

**Example:** Mobile telephone

Effective amplifiers work in nonlinear region

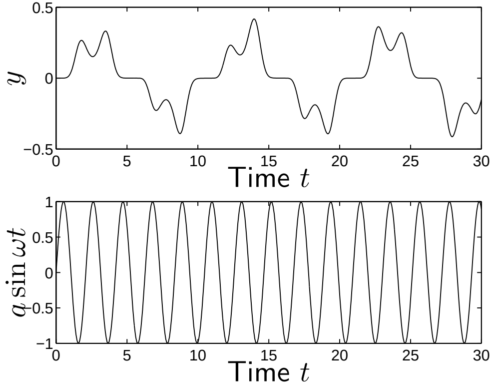
Introduces spectrum leakage

Channels close to each other

Trade-off between effectivity and linearity

# Subresonances

**Example:** Duffing's equation $\ddot{y} + \dot{y} + y - y^3 = a\sin(\omega t)$



# When is Nonlinear Theory Needed?

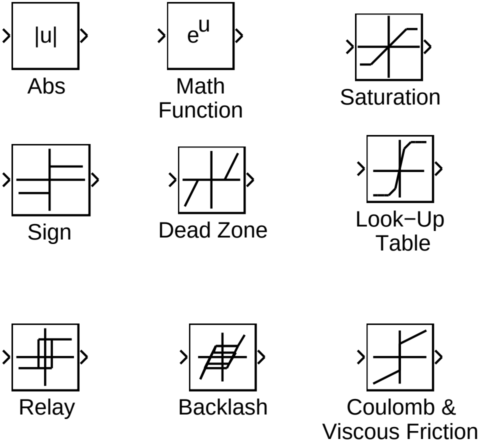
- Hard to know when - Try simple things first!
- Regulator problem versus servo problem
- Change of working conditions (production on demand, short batches, many startups)
- Mode switches
- Nonlinear components

How to detect? Make step responses, Bode plots

- Step up/step down
- Vary amplitude
- Sweep frequency up/frequency down

# Some Nonlinearities

Static – dynamic



# Nonlinear Differential Equations

Problems

- No analytic solutions
- Existence?
- Uniqueness?
- etc

# Finite escape time

**Example:** The differential equation

$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0$$

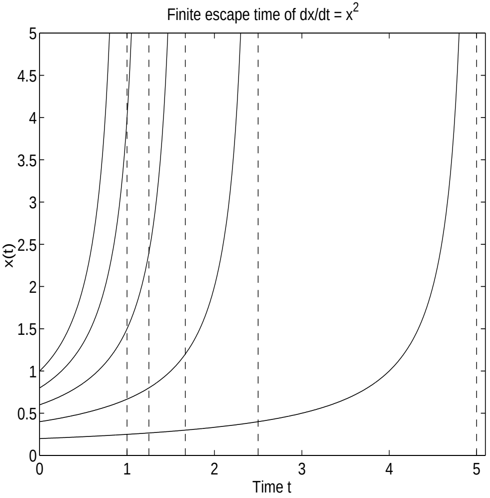
has solution

$$x(t) = \frac{x_0}{1 - x_0 t}, \qquad 0 \leq t < \frac{1}{x_0}$$

Finite escape time

$$t_f = \frac{1}{x_0}$$

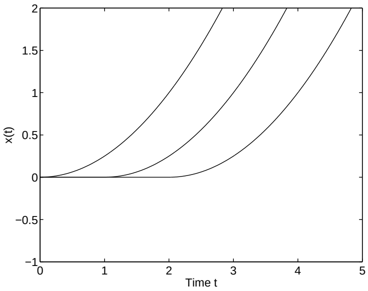
# Finite Escape Time



# Uniqueness Problems

**Example:** The equation $\dot{x} = \sqrt{x},\ \ x(0) = 0$ has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$




Compare with water tank:

$$dh/dt = -a\sqrt{h}, \qquad h : \text{height (water level)}$$

Change to backward-time: "If I see it empty, when was it full?")

# Local Existence and Uniqueness

For $R > 0$, let $\Omega_R$ denote the ball $\Omega_R = \{z : \|z - a\| \leq R\}$.

**Theorem**

If, $f$ is Lipschitz-continuous in $\Omega_R$, i.e.,

$$\|f(z) - f(y)\| \leq K\|z - y\|, \qquad \text{for all } z, y \in \Omega_R\,,$$

then

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$$

has a unique solution

$$x(t)\,, \qquad 0 \leq t < R/C_R,$$

where $C_R = \max_{x \in \Omega_R} \|f(x)\|$

# Global Existence and Uniqueness

**Theorem**

If $f$ is Lipschitz-continuous in $R^n$, i.e.,

$$\|f(z) - f(y)\| \leq K\|z - y\|, \qquad \text{for all } z, y \in R^n\,,$$

then

$$\dot{x}(t) = f(x(t)), x(0) = a$$

has a unique solution

$$x(t)\,, \qquad t \geq 0\,.$$

# State-Space Models

- State vector $x$
- Input vector $u$
- Output vector $y$

| | |
|---|---|
| general: | $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \ldots) = 0$ |
| explicit: | $\dot{x} = f(x, u), \quad y = h(x)$ |
| affine in $u$: | $\dot{x} = f(x) + g(x)u, \quad y = h(x)$ |
| linear time-invariant: | $\dot{x} = Ax + Bu, \quad y = Cx$ |

# Transformation to Autonomous System

Nonautonomous:

$$\dot{x} = f(x, t)$$

Always possible to transform to autonomous system
Introduce $x_{n+1}$ = time

$$\begin{aligned} \dot{x} &= f(x, x_{n+1}) \\ \dot{x}_{n+1} &= 1 \end{aligned}$$

# Transformation to First-Order System

Assume $\frac{d^k y}{dt^k}$ highest derivative of $y$

Introduce $x = \begin{bmatrix} y & \frac{dy}{dt} & \ldots & \frac{d^{k-1}y}{dt^{k-1}} \end{bmatrix}^T$

**Example**: Pendulum

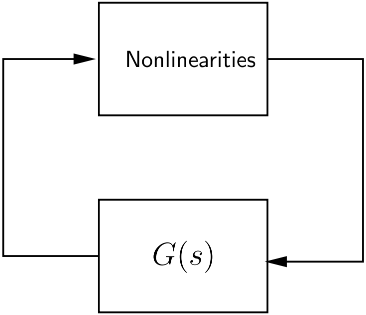
$$MR\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

$x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ gives

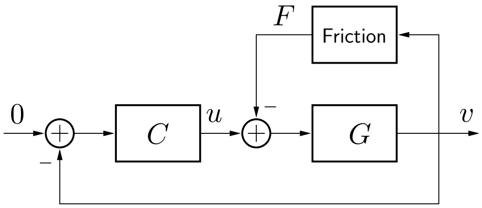
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1 \end{aligned}$$

# A Standard Form for Analysis

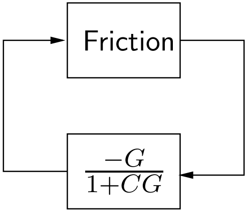
Transform to the following form



# Example, Closed Loop with Friction




$\Longleftrightarrow$



# Equilibria (=singular points)

*Put all derivatives to zero!*

General: $f(x_0, u_0, y_0, 0, 0, 0, \ldots) = 0$
Explicit: $f(x_0, u_0) = 0$
Linear: $Ax_0 + Bu_0 = 0$ (has analytical solution(s)!)

# Multiple Equilibria

Example: Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

Equilibria given by $\ddot{\theta} = \dot{\theta} = 0 \Longrightarrow \sin\theta = 0 \Longrightarrow \theta = n\pi$
Alternatively,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1 \end{aligned}$$

gives $x_2 = 0$, $\sin(x_1) = 0$, etc

# Next Lecture

- Linearization
- Stability definitions
- Simulation in Matlab