

Lecture 13 — Nonlinear Control Synthesis Cont'd

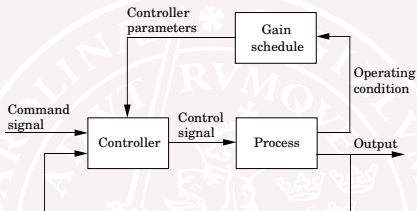
Today's Goal: *To understand the meaning of the concepts*

- *Gain scheduling*
- *Internal model control*
- *Model predictive control*
- *Nonlinear observers*
- *Lie brackets*

Material:

- Lecture notes
- Internal model, more info in e.g.,
 - Section 8.4 in [Glad&Ljung]
 - Ch 12.1 in [Khalil]

Gain Scheduling

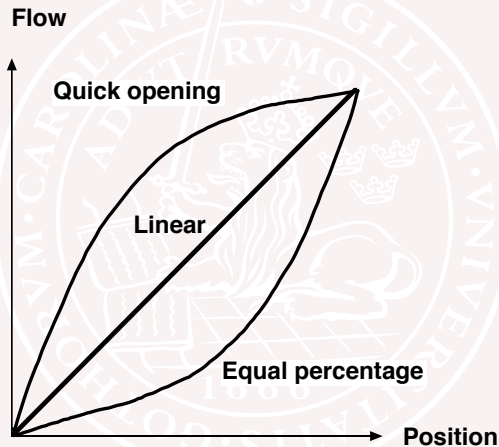


Example of scheduling variables

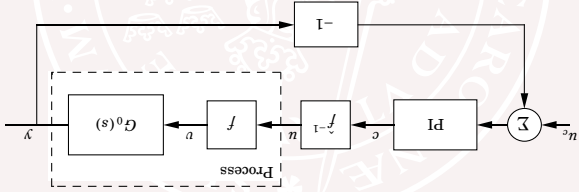
- Production rate
- Machine speed
- Mach number and dynamic pressure

Compare structure with adaptive control!

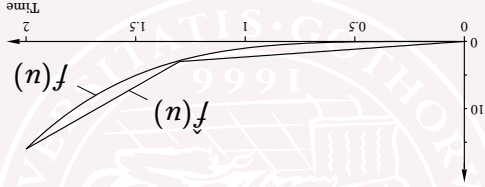
Valve Characteristics



Nonlinear Valve

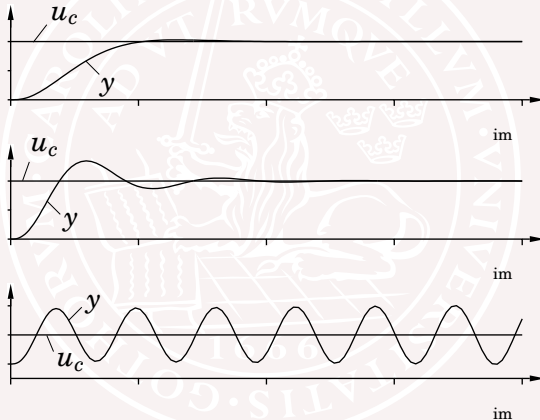


Valve characteristics



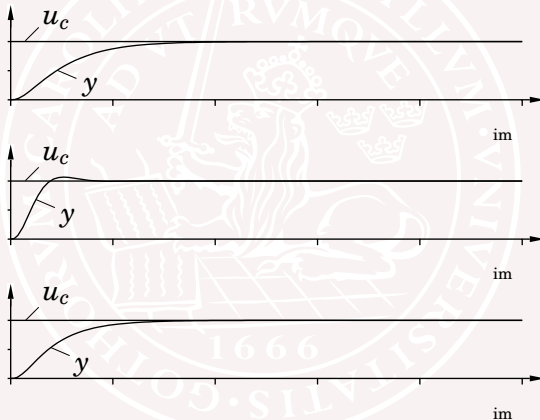
Results

Without gain scheduling



Results

With gain scheduling



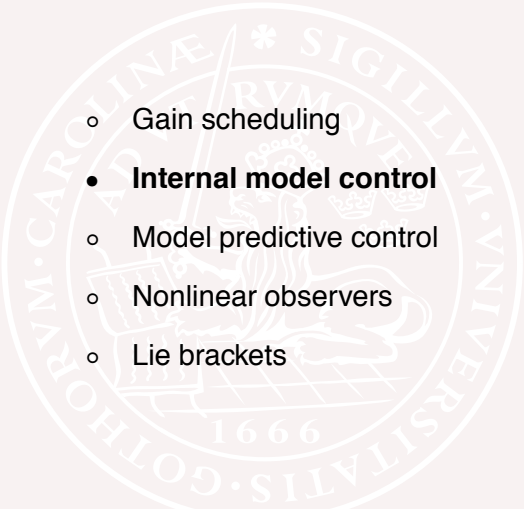
Gain Scheduling

- state dependent controller parameters.
 - $K = K(q)$
- design controllers for a number of operating points.
 - use the closest controller.

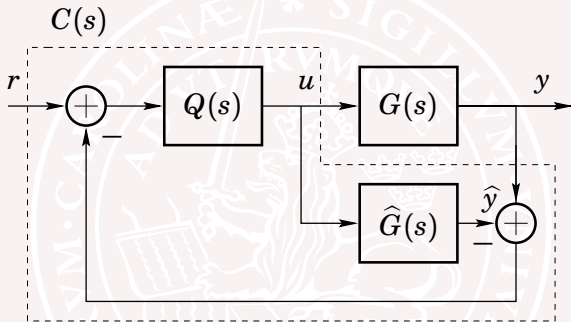
Problems:

- How should you switch between different controllers?
 - Bumpless transfer
- Switching between stabilizing controllers can cause instability.

Outline

- 
- Gain scheduling
 - **Internal model control**
 - Model predictive control
 - Nonlinear observers
 - Lie brackets

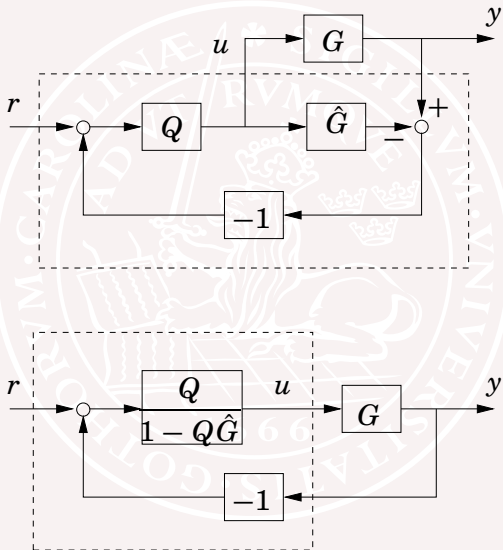
Internal Model Control



Feedback from model error $y - \hat{y}$.

Design: Choose $\hat{G} \approx G$ and Q stable with $Q \approx G^{-1}$.

Two equivalent diagrams



Example

$$G(s) = \frac{1}{1 + sT_1}$$

Choose

$$Q = \frac{1 + sT_1}{1 + \tau s}$$

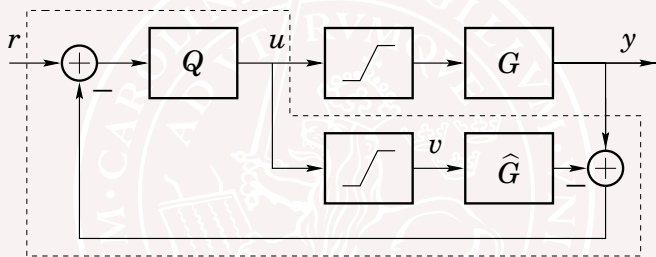
Gives the PI controller

$$C = \frac{1 + sT_1}{s\tau} = \frac{T_1}{\tau} \left(1 + \frac{1}{T_1 s} \right)$$

Internal Model Control Can Give Problems

- Unstable G
- $Q \neq G^{-1}$ due to RHP zeros
- Cancellation of process poles may show up in some signals

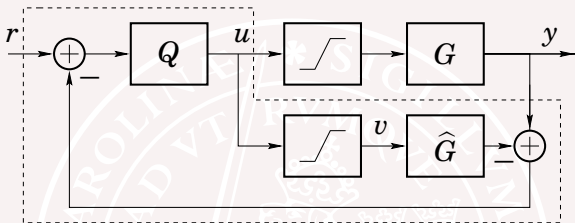
Internal Model Control with Static Nonlinearity



Include the nonlinearity in the model in the controller.

Choose $Q \approx G^{-1}$.

Example (cont'd)



Assume $r = 0$ and $\hat{G} = G$:

$$u = -Q(y - \hat{G}v) = -\frac{1 + sT_1}{1 + \tau_S}y + \frac{1}{1 + \tau_S}v$$

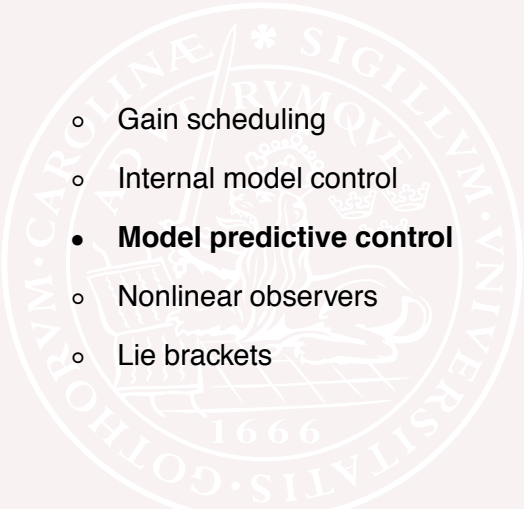
Same as before if $|u| \leq u_{\max}$: Integrating controller.

If $|u| > u_{\max}$ then

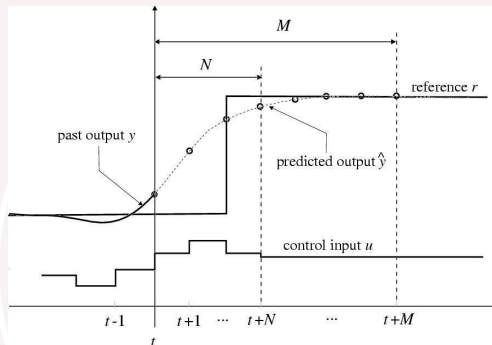
$$u = -\frac{1 + sT_1}{1 + \tau s}y \pm \frac{u_{\max}}{1 + \tau s}$$

No integration. (A way to implement anti-windup.)

Outline

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Model Predictive Control – MPC



- 1 Derive the future controls $u(t+j)$, $j = 0, 1, \dots, N-1$ that give an optimal predicted response.
- 2 Apply the first control $u(t)$.
- 3 Start over from 1 at next sample.

What is Optimal?

Minimize a cost function, V , of inputs and predicted outputs.

$$V = V(U_t, Y_t), \quad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \quad Y_t = \begin{bmatrix} \hat{y}(t+M|t) \\ \vdots \\ \hat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t \quad (1)$$

$\Rightarrow$ linear controller

$$u(t) = -L\hat{x}(t|t)$$

Model Predictive Control

- + Flexible method
 - * Many types of models for prediction:
 - state space, input–output, step response, FIR filters
 - * MIMO
 - * Time delays
- + Can include constraints on input signal and states
- + Can include future reference and disturbance information
- On-line optimization needed
- Stability (and performance) analysis can be complicated

Typical application:

Chemical processes with slow sampling (minutes)

A predictor for Linear Systems

Discrete-time model

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) + B_v v_1(t) \\ y(t) &= Cx(t) + v_2(t)\end{aligned} \quad t = 0, 1, \dots$$

Predictor (v unknown)

$$\begin{aligned}\hat{x}(t+k+1|t) &= A\hat{x}(t+k|t) + Bu(t+k) \\ \hat{y}(t+k|t) &= C\hat{x}(t+k|t)\end{aligned}$$

The M -step predictor for Linear Systems

$\hat{x}(t|t)$ is predicted by a standard Kalman filter, using outputs up to time t , and inputs up to time $t - 1$.

Future predicted outputs are given by

$$\begin{bmatrix} \hat{y}(t+M|t) \\ \vdots \\ \hat{y}(t+1|t) \end{bmatrix} = \begin{bmatrix} CA^M \\ \vdots \\ CA \end{bmatrix} \hat{x}(t|t) + \begin{bmatrix} CB & CAB & CA^2B & \dots \\ 0 & CB & CAB & \dots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u(t+M-1) \\ \vdots \\ u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}$$

$$Y_t = D_x \hat{x}(t|t) + D_u U_t$$

Limitations

Limitations on control signals, states and outputs,

$$|u(t)| \leq C_u \quad |x_i(t)| \leq C_{x_i} \quad |y(t)| \leq C_y,$$

leads to linear programming or quadratic optimization.

Efficient optimization software exists.

Design Parameters

- Model
- M (look on settling time)
- N as long as computational time allows
- If $N < M - 1$ assumption on $u(t + N), \dots, u(t + M - 1)$ needed (e.g., $= 0, = u(t + N - 1)$.)
- Q_y, Q_u (trade-offs between control effort etc)
- C_y, C_u limitations often given
- Sampling time

Product: ABB Advant

Example—Motor

$$A = \begin{pmatrix} 1 & 0.139 \\ 0 & 0.861 \end{pmatrix}, \quad B = \begin{pmatrix} 0.214 \\ 2.786 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Minimize $V(U_t) = \|Y_t - R\|$ where $R = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$, r =reference,

$$M = 8, N = 2, u(t+2) = u(t+3) = u(t+7) = \dots = 0$$

Example—Motor

$$\begin{aligned} Y_t &= \begin{pmatrix} CA^8 \\ \vdots \\ CA \end{pmatrix} x(t) + \begin{pmatrix} CA^6 B & CA^7 B \\ \vdots & \vdots \\ 0 & CB \end{pmatrix} \begin{pmatrix} u(t+1) \\ u(t) \end{pmatrix} \\ &= D_x x(t) + D_u U_t \end{aligned}$$

Solution without control constraints

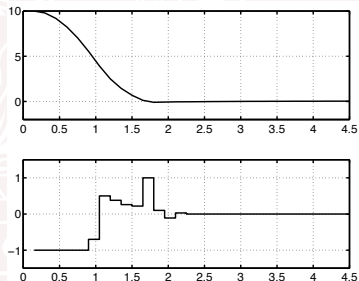
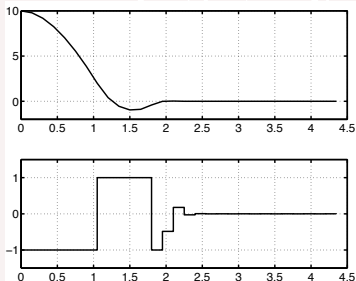
$$\begin{aligned} U_t &= -(D_u^T D_u)^{-1} D_u^T D_x x + (D_u^T D_u)^{-1} D_u^T R = \\ &= - \begin{pmatrix} -2.50 & -0.18 \\ 2.77 & 0.51 \end{pmatrix} \begin{pmatrix} x_1(t) - r \\ x_2(t) \end{pmatrix} \end{aligned}$$

Use

$$u(t) = -2.77(x_1(t) - r) - 0.51x_2(t)$$

Example–Motor–Results

No control constraints in optimization (but in simulation) Control constraints $|u(t)| \leq 1$ in optimization.



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Nonlinear Observers

What if x is not measurable?

$$\dot{x} = f(x, u), \quad y = h(x)$$

Simplest observer (open loop – only works for as. stable systems).

$$\dot{\hat{x}} = f(\hat{x}, u)$$

Correction, as in linear case,

$$\dot{\hat{x}} = f(\hat{x}, u) + K(y - h(\hat{x}))$$

Choices of K

- Linearize f at x_0 , find K for the linearization
- Linearize f at $\hat{x}(t)$, find $K(t)$ for the linearization

Second case is called *Extended Kalman Filter*

A Nonlinear Observer for the Pendulum



Control tasks:

- 1 Swing up
- 2 Catch
- 3 Stabilize in upward position

The observer must to be valid for a complete revolution

A Nonlinear Observer for the Pendulum

$$\frac{d^2\theta}{dt^2} = \sin \theta + u \cos \theta$$

$$x_1 = \theta, x_2 = \frac{d\theta}{dt} \Rightarrow$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \sin x_1 + u \cos x_1$$

Observer structure:

$$\frac{d\hat{x}_1}{dt} = \hat{x}_2 + k_1(x_1 - \hat{x}_1)$$

$$\frac{d\hat{x}_2}{dt} = \sin \hat{x}_1 + u \cos \hat{x}_1 + k_2(x_1 - \hat{x}_1)$$

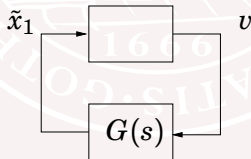
A Nonlinear Observer for the Pendulum

Introduce the error $\tilde{x} = \hat{x} - x$

$$\begin{cases} \frac{d\tilde{x}_1}{dt} = -k_1\tilde{x}_1 + \tilde{x}_2 \\ \frac{d\tilde{x}_2}{dt} = \sin \hat{x}_1 - \sin x_1 + u(\cos \hat{x}_1 - \cos x_1) - k_2\tilde{x}_1 \end{cases}$$

$$\frac{d}{dt} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$v = 2 \sin \frac{\tilde{x}_1}{2} \left(\cos \left(x_1 + \frac{\tilde{x}_1}{2} \right) - u \sin \left(x_1 + \frac{\tilde{x}_1}{2} \right) \right)$$



Stability with Small Gain Theorem

The linear block:

$$\begin{aligned} G(s) &= \frac{1}{s^2 + k_1 s + k_2} \\ \left| \frac{1}{G(i\omega)} \right|^2 &= \omega^4 + (k_1^2 - 2k_2)\omega^2 + k_2^2 \\ &= (\omega^2 - k_2 + k_1^2/2)^2 - k_1^4/4 + k_1^2 k_2 \\ \gamma_G = \max |G(i\omega)| &= \begin{cases} \frac{1}{\sqrt{k_1^2 k_2 - k_1^4/4}}, & \text{if } k_1^2 < 2k_2 \\ \frac{1}{k_2}, & \text{if } k_1^2 \geq 2k_2 \end{cases} \end{aligned}$$

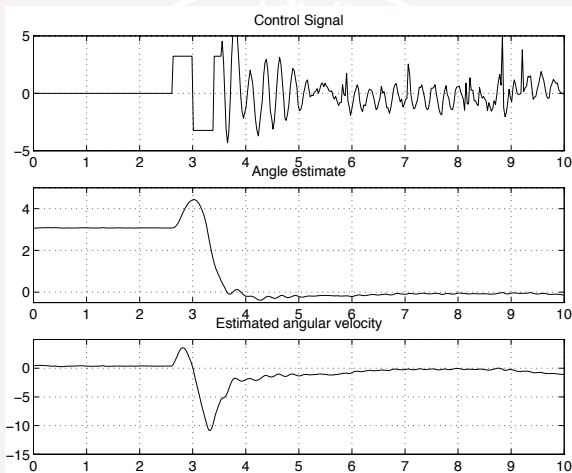
Stability with Small Gain Theorem

$$v = 2 \sin \frac{\tilde{x}_1}{2} \left(\cos \left(x_1 + \frac{\tilde{x}_1}{2} \right) - u \sin \left(x_1 + \frac{\tilde{x}_1}{2} \right) \right)$$
$$|v| \leq |\tilde{x}_1| \sqrt{1 + u_{max}^2} = \beta |\tilde{x}_1|$$

The observer is stable if $\gamma_G \beta < 1$

$$\Rightarrow k_2 > \begin{cases} \beta^2 k_1^{-2} + k_1^2/4, & \text{if } k_1 < \sqrt{2\beta}, \\ \beta, & \text{if } k_1 \geq \sqrt{2\beta} \end{cases}$$

A Nonlinear Observer for the Pendulum



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Controllability

Linear case

$$\dot{x} = Ax + Bu$$

All controllability definitions coincide

$$0 \rightarrow x(T),$$

$$x(0) \rightarrow 0,$$

$$x(0) \rightarrow x(T)$$

T either fixed or free

Rank condition System is controllable iff

$$W_n = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \text{ full rank}$$

Is there a corresponding result for nonlinear systems?

Lie Brackets

Lie bracket between $f(x)$ and $g(x)$ is defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Example:

$$\begin{aligned} f &= \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix}, & g &= \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \\ [f, g] &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & -\sin x_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos x_2 + \sin x_2 \\ -x_1 \end{pmatrix} \end{aligned}$$

Why interesting?

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

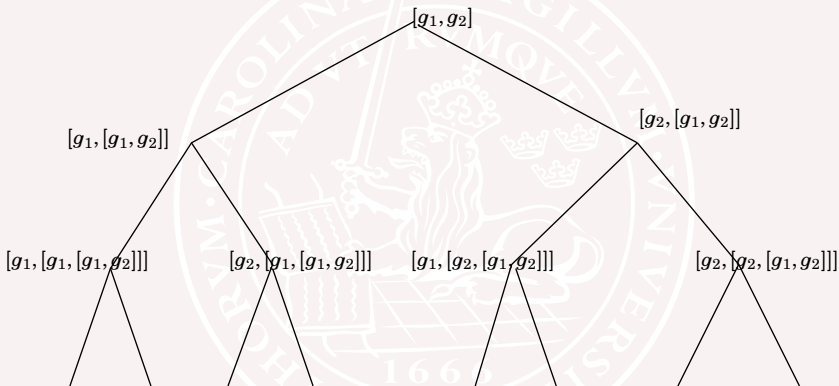
- The motion $(u_1, u_2) = \begin{cases} (1, 0), & t \in [0, \epsilon] \\ (0, 1), & t \in [\epsilon, 2\epsilon] \\ (-1, 0), & t \in [2\epsilon, 3\epsilon] \\ (0, -1), & t \in [3\epsilon, 4\epsilon] \end{cases}$

gives motion $x(4\epsilon) = x(0) + \epsilon^2[g_1, g_2] + O(\epsilon^3)$

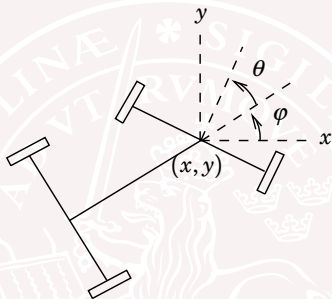
- $\Phi_{[g_1, g_2]}^t = \lim_{n \rightarrow \infty} (\Phi_{-g_2}^{\sqrt{\frac{t}{n}}} \Phi_{-g_1}^{\sqrt{\frac{t}{n}}} \Phi_{g_2}^{\sqrt{\frac{t}{n}}} \Phi_{g_1}^{\sqrt{\frac{t}{n}}})^n$

- The system is controllable if the **Lie bracket tree** has full rank (controllable=the states you can reach from $x = 0$ at fixed time T contains a ball around $x = 0$)

The Lie Bracket Tree



Parking Your Car Using Lie-Brackets



$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \\ \sin(\theta) \\ 0 \end{pmatrix} u_2$$

Parking the Car

Can the car be moved sideways?

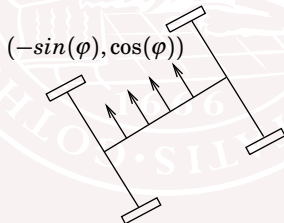
Sideways: in the $(-\sin(\varphi), \cos(\varphi), 0, 0)^T$ -direction?

$$\begin{aligned} [g_1, g_2] &= \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 \\ &= \begin{pmatrix} 0 & 0 & -\sin(\varphi + \theta) & -\sin(\varphi + \theta) \\ 0 & 0 & \cos(\varphi + \theta) & \cos(\varphi + \theta) \\ 0 & 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 0 \\ &= \begin{pmatrix} -\sin(\varphi + \theta) \\ \cos(\varphi + \theta) \\ \cos(\theta) \\ 0 \end{pmatrix} =: g_3 = \text{"wriggle"} \end{aligned}$$

Once More

$$\begin{aligned}[g_3, g_2] &= \frac{\partial g_2}{\partial x} g_3 - \frac{\partial g_3}{\partial x} g_2 = \dots \\ &= \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \\ 0 \end{pmatrix} = \text{"sideways"}\end{aligned}$$

The motion $[g_3, g_2]$ takes the car sideways.



The Parking Theorem

You can get out of any parking lot that is bigger than your car.
Use the following control sequence:

Wriggle, Drive, –Wriggle(this requires a cool head), –Drive
(repeat).

Outline

- Gain scheduling
- Internal model control
- Model predictive control
- Nonlinear observers
- Lie brackets
- **Extra: Integral quadratic constraints**

Integral Quadratic Constraint

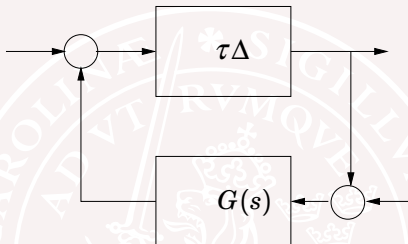


The (possibly nonlinear) operator Δ on $\mathbf{L}_2^m[0, \infty)$ is said to *satisfy the IQC defined by Π* if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \geq 0$$

for all $v \in \mathbf{L}_2[0, \infty)$.

IQC Stability Theorem



Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0, 1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0, \infty]$$

then the feedback system is input/output stable.

Δ structure

$\Pi(i\omega)$

Condition

Δ passive

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$\|\Delta(i\omega)\| \leq 1$$

$$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$$

$$x(i\omega) \geq 0$$

$$\delta \in [-1, 1]$$

$$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$$

$$\begin{aligned} X &= X^* \geq 0 \\ Y &= -Y^* \end{aligned}$$

$$\delta(t) \in [-1, 1]$$

$$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$$

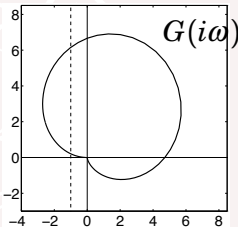
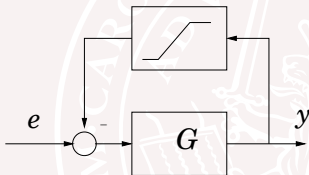
$$\Delta(s) = e^{-\theta s} - 1$$

$$\begin{bmatrix} x(i\omega)\rho(\omega)^2 & 0 \\ 0 & -x(i\omega) \end{bmatrix}$$

$$\rho(\omega) = 2 \max_{|\theta| \leq \theta_0} \sin(\theta\omega/2)$$

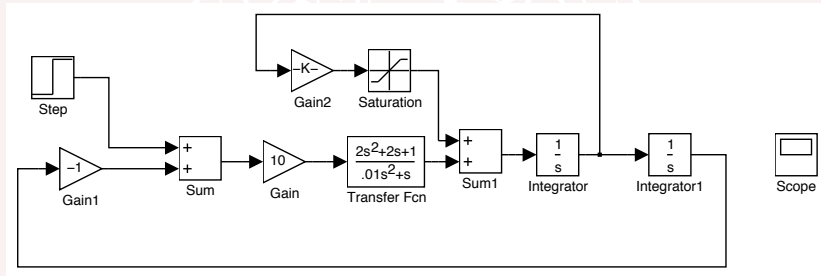
A Matlab toolbox for system analysis

<http://www.ee.mu.oz.au/staff/cykao/>

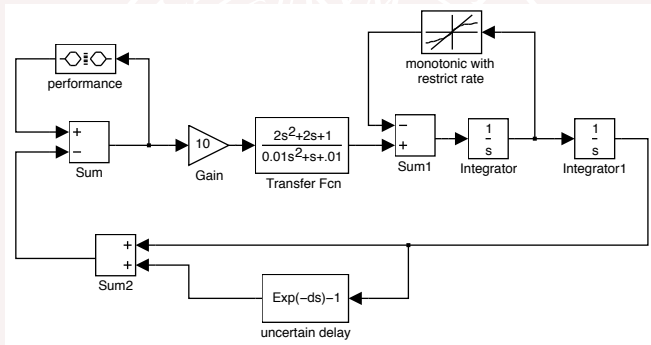


```
>> abst_init_iqc;  
>> G = tf([10 0 0],[1 2 2 1]);  
>> e = signal  
>> w = signal  
>> y = -G*(e+w)  
>> w==iqc_monotonic(y)  
>> iqc_gain_tbx(e,y)
```

A servo with friction



An analysis model defined graphically



```
z iqc_gui('fricSYSTEM')
```

```
extracting information from fricSYSTEM ...
```

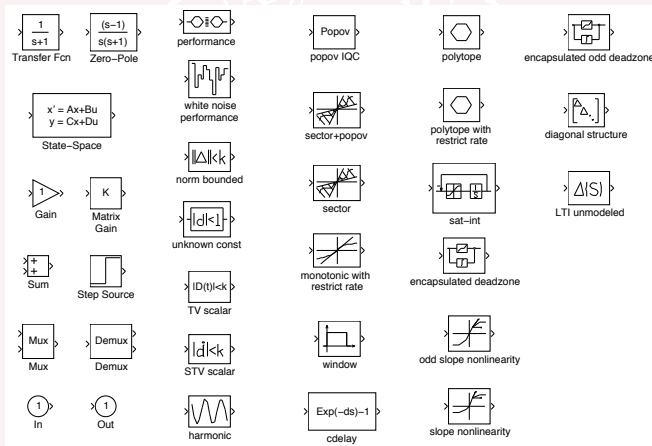
```
scalar inputs: 5  
states:        10  
simple q-forms: 7
```

```
LMI #1    size = 1    states: 0  
LMI #2    size = 1    states: 0  
LMI #3    size = 1    states: 0  
LMI #4    size = 1    states: 0  
LMI #5    size = 1    states: 0
```

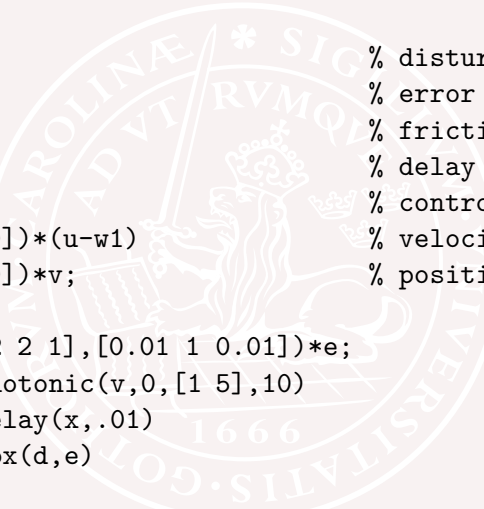
```
Solving with 62 decision variables ...
```

```
ans =      4.7139
```

A library of analysis objects



The friction example in text format



```
d=signal; % disturbance signal
e=signal; % error signal
w1=signal; % friction force
w2=signal; % delay perturbation
u=signal; % control force
v=tf(1,[1 0])*(u-w1) % velocity
x=tf(1,[1 0])*v; % position
e==d-x-w2;
u==10*tf([2 2 1],[0.01 1 0.01])*e;
w1==iqc_monotonic(v,0,[1 5],10)
w2==iqc_cdelay(x,.01)
iqc_gain_tbx(d,e)
```

Summary

- Gain scheduling
- Internal model control
- Model predictive control
- Nonlinear observers
- Lie brackets
- Extra: Integral quadratic constraints

Next: Lecture 14

- Course Summary

