Lecture 10 — Optimal Control

- Introduction
- Static Optimization with Constraints
- Optimization with Dynamic Constraints
- The Maximum Principle
- Examples

Material

- Lecture slides
- References to Glad & Ljung, part of Chapter 18
- D. Liberzon, Calculus of Variations and Optimal Control Theory: A concise Introduction, Princeton University Press, 2010 (linked from course webpage)

Goal

To be able to

- solve simple optimal control problems by hand
- design controllers

using the maximum principle

Optimal Control Problems

Idea: Formulate the design problem as optimization problem

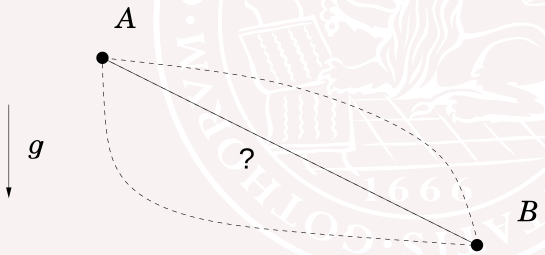
+ Gives systematic design procedure
+ Can use on nonlinear models
+ Can capture limitations etc as constraints
- Hard to find suitable criterium?!
- Can be hard to find the optimal controller

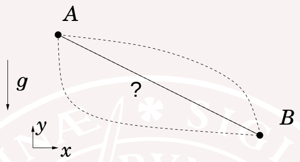
Solutions will often be of “bang-bang” character if control signal is bounded, compare lecture on sliding mode controllers.

The beginning

- John Bernoulli: The brachistochrone problem 1696

  Let a particle slide along a frictionless curve. Find the curve that takes the particle from A to B in shortest time





$$\frac{1}{2}v^2 = gy, \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = v\sin\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -v\cos\theta$$

Find $y(x)$, with $y(0)$ and $y(1)$ given, that minimizes

$$J(y) = \int_0^1 \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gy(x)}}\,dx$$

- Solved by John and James Bernoulli, Newton, l'Hospital
- Euler: Isoperimetric problems
  - Example: The largest area covered by a curve of given length is a circle [see also Dido/cow-skin/Carthage].

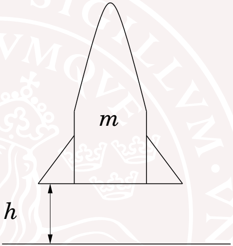
Optimal Control

- The space race (Sputnik 1957)
- Putting satellites in orbit
- Trajectory planning for interplanetary travel
- Reentry into atmosphere
- Minimum time problems
- Pontryagin's maximum principle, 1956
- Dynamic programming, Bellman 1957
- Vitalization of a classical field

An example: Goddard's Rocket Problem (1910)

How to send a rocket as high up in the air as possible?

$$\frac{d}{dt}\begin{pmatrix} v \\ h \\ m \end{pmatrix} = \begin{pmatrix} \frac{u-D}{m} - g \\ v \\ -\gamma u \end{pmatrix}$$



where $u =$ motor force, $D(v,h) =$ air resistance, $m =$ mass.

Constraints

$$0 \le u \le u_{max}, \quad m(t_f) \ge m_1$$

Criterium

$$\text{Maximize } h(t_f), \qquad t_f \text{ given}$$

Goddard's Problem

Can you guess the solution when $D(v, h) = 0$?

Much harder when $D(v, h) \neq 0$

Can be optimal to have low v when air resistance is high. Burn fuel at higher level.

Took about 50 years before a complete solution was found.

Read more about Goddard at `http://www.nasa.gov/centers/goddard/`

Optimal Control Problem. Constituents

Control signal $u(t), 0 \leq t \leq t_f$

Criterium $h(t_f)$.

Differential equations relating $h(t_f)$ and u

Constraints on u

Constraints on $x(0)$ and $x(t_f)$

t_f can be fixed or a free variable

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Preliminary: Static Optimization

Minimize $g_1(x,u)$
over $x \in R^n$ and $u \in R^m$ s.t. $g_2(x,u) = 0$
(Assume $g_2(x,u) = 0 \Rightarrow \partial g_2(x,u)/\partial x$ non-singular)

Lagrangian: $$\mathcal{L}(x,u,\lambda) = g_1(x,u) + \lambda^T g_2(x,u)$$

Local minima of $g_1(x,u)$ constrained on $g_2(x,u) = 0$
can be mapped into critical points of $\mathcal{L}(x,u,\lambda)$

Necessary conditions for local minimum

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial u} = 0 \qquad \left(\frac{\partial \mathcal{L}}{\partial \lambda} = g_2(x,u) = 0 \right)$$

Note: Difference if constrained control!

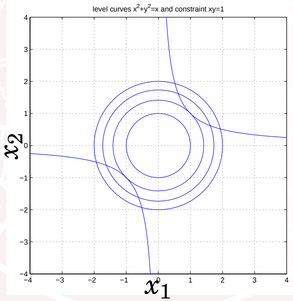
Example - static optimization

Minimize

$$g_1(x_1, x_2) = x_1^2 + x_2^2$$

with the constraint that

$$g_2(x_1, x_2) = x_1 \cdot x_2 - 1 = 0$$



Level curves for constant g_1 and the constraint $g_2 = 0$, repectively.

Static Optimization cont'd

Solving the equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial g_1}{\partial x} + \lambda^T \frac{\partial g_2}{\partial x} = 0 \Rightarrow \lambda^T = -\frac{\partial g_1}{\partial x}\left(\frac{\partial g_2}{\partial x}\right)^{-1}$$

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{\partial g_1}{\partial u} + \lambda^T \frac{\partial g_2}{\partial u} = 0 \Rightarrow \frac{\partial g_1}{\partial u} - \frac{\partial g_1}{\partial x}\left(\frac{\partial g_2}{\partial x}\right)^{-1}\frac{\partial g_2}{\partial u} = 0$$

This gives m equations to solve for u.

Sufficient condition for local minimum

$$\frac{\partial^2 \mathcal{L}}{\partial u^2} > 0$$

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Optimization with Dynamic Constraint

Optimal Control Problem

$$\min_u J = \min_u \left\{ \phi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)\, dt \right\}$$

subject to

$$\dot{x} = f(x,u), \quad x(t_0) = x_0$$

Introduce *Hamiltonian*: $H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$

$$J = \phi(x(t_f)) + \int_{t_0}^{t_f} \Big(L(x,u) + \lambda^T (f - \dot{x}) \Big)\, dt$$

$$= \phi(x(t_f)) - \Big[\lambda^T x \Big]_{t_0}^{t_f} + \int_{t_0}^{t_f} \Big(H + \dot{\lambda}^T x \Big)\, dt$$

second equality obtained from "integration by parts".

Optimization with Dynamic Constraint cont'd

Variation of J:

$$\delta J = \left[\left(\frac{\partial \phi}{\partial x} - \lambda^T \right) \delta x \right]_{t=t_f} + \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \frac{\partial H}{\partial u} \delta u \right] dt$$

Necessary conditions for local minimum ($\delta J = 0$)

$$\lambda(t_f)^T = \left. \frac{\partial \phi}{\partial x} \right|_{t=t_f} \qquad \dot{\lambda}^T = -\frac{\partial H}{\partial x} \qquad \frac{\partial H}{\partial u} = 0$$

- Adjoined, or co-state, variables, $\lambda(t)$
- λ specified at $t = t_f$ and x at $t = t_0$
- Two Point Boundary Value Problem (TPBV)
- For sufficiency $\frac{\partial^2 H}{\partial u^2} \geq 0$

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Problem Formulation (1)

Standard form (1):

$$\text{Minimize} \int_0^{t_f} \overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\,dt + \overbrace{\phi(x(t_f))}^{\text{Final cost}}$$
$$\dot{x}(t) = f(x(t),u(t))$$
$$u(t) \in U, \quad 0 \le t \le t_f, \qquad t_f \text{ given}$$
$$x(0) = x_0$$

$x(t) \in R^n$, $u(t) \in R^m$

$U \subseteq R^m$ control constraints

Here we have a fixed end-time t_f. This will be relaxed later on.

The Maximum Principle (18.2)

Introduce the **Hamiltonian**

$$H(x,u,\lambda) = L(x,u) + \lambda^T(t) f(x,u).$$

Assume optimization (1) has a solution $\{u^*(t), x^*(t)\}$. Then

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \leq t \leq t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$\frac{d\lambda}{dt} = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \text{with} \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

Notation

$$H_x = \frac{\partial H}{\partial x} = \begin{pmatrix} \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} & \dots \end{pmatrix}$$

Remarks

Proof: If you are theoretically interested look in [Glad & Ljung].

Idea: note that every change of $u(t)$ from the suggested optimal $u^*(t)$ must lead to larger value of the criterium.

Should be called “minimum principle”

$\lambda(t)$ are called the **Lagrange multipliers** or the **adjoint variables**

Remarks

The Maximum Principle gives **necessary** conditions

A pair $(u^*(\cdot), x^*(\cdot))$ is called **extremal** the conditions of the Maximum Principle are satisfied. Many extremals can exist.

The maximum principle gives all possible candidates.

However, there might not exist a minimum!

Example

Minimize $x(1)$ when $\dot{x}(t) = u(t)$, $x(0) = 0$ and $u(t)$ is free

Why doesn't there exist a minimum?

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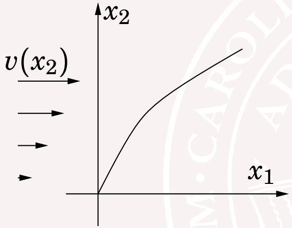
Example–Boat in Stream



$$
\begin{aligned}
&\text{min } -x_1(T)\\
&\dot{x}_1 = v(x_2) + u_1\\
&\dot{x}_2 = u_2\\
&x_1(0) = 0\\
&x_2(0) = 0\\
&u_1^2 + u_2^2 = 1
\end{aligned}
$$

Speed of water $v(x_2)$ in x_1 direction. Move maximum distance in x_1-direction in fixed time T

Assume v linear so that $v'(x_2) = 1$

Solution

Hamiltonian:

$$H = 0 + \lambda^T f = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \lambda_1 (v(x_2) + u_1) + \lambda_2 u_2$$

Adjoint equation:

$$\begin{bmatrix} \dot\lambda_1 \\ \dot\lambda_2 \end{bmatrix} = \begin{bmatrix} -\partial H / \partial x_1 \\ -\partial H / \partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -v'(x_2)\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

with boundary conditions

$$\begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} \partial\phi / \partial x_1|_{x=x^*(t_f)} \\ \partial\phi / \partial x_2|_{x=x^*(t_f)} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

This gives $\lambda_1(t) = -1, \quad \lambda_2(t) = t - T$

Solution

Optimality: Control signal should solve

$$\min_{u_1^2+u_2^2=1} \lambda_1(v(x_2)+u_1)+\lambda_2 u_2$$

Minimize $\lambda_1 u_1 + \lambda_2 u_2$ so that (u_1, u_2) has length 1

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}$$

$$u_1(t) = \frac{1}{\sqrt{1+(t-T)^2}}, \quad u_2(t) = \frac{T-t}{\sqrt{1+(t-T)^2}}$$

See fig 18.1 for plots

Remark: It can be shown that this optimal control problem has a minimum. Hence it must be the one we found, since this was the only solution to MP

5 min exercise

Solve the optimal control problem

$$\min \int_0^1 u^4 dt + x(1)$$
$$\dot{x} = -x + u$$
$$x(0) = 0$$

5 min exercise - solution

Compare with standard formulation:

$$t_f = 1 \qquad L = u^4 \qquad \phi = x \qquad f(x) = -x + u$$

Need to introduce one adjoint state

Hamiltonian:

$$H = L + \lambda^T \cdot f = u^4 + \lambda(-x + u)$$

Adjoint equation:

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -(-\lambda) \qquad \Longrightarrow \qquad \lambda(t) = Ce^t$$

$$\lambda(t_f) = \frac{\partial \phi}{\partial x} = 1 \qquad \Longrightarrow \qquad \lambda(t) = e^{t-1}$$

At optimality:

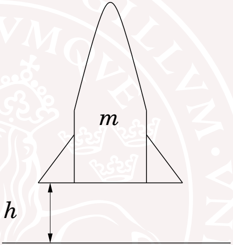
$$0 = \frac{\partial H}{\partial u} = 4u^3 + \lambda$$

$$\Longrightarrow \quad u(t) = \sqrt[3]{-\lambda(t)/4} = \sqrt[3]{-e^{(t-1)}/4}$$

Goddard's Rocket Problem revisited

How to send a rocket as high up in the air as possible?

$$\frac{d}{dt}\begin{pmatrix} v \\ h \\ m \end{pmatrix} = \begin{pmatrix} \frac{u-D}{m} - g \\ v \\ -\gamma u \end{pmatrix}$$



$(v(0), h(0), m(0)) = (0, 0, m_0)$, $g, \gamma > 0$
u motor force, $D = D(v, h)$ air resistance

Constraints: $0 \leq u \leq u_{max}$ and $m(t_f) = m_1$ (empty)

Optimization criterion: $\max_{t_f, u} h(t_f)$

Problem Formulation (2)

$$\min_{\substack{t_f \geq 0 \\ u:[0,t_f]\to U}} \int_0^{t_f} L(x(t),u(t))\,dt + \phi(t_f, x(t_f))$$

$$\dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0$$

$$\psi(t_f, x(t_f)) = 0$$

Note the differences compared to standard form:

- t_f free variable (i.e., not specified *a priori*)
- r end constraints

$$\Psi(t_f, x(t_f)) = \begin{pmatrix} \Psi_1(t_f, x(t_f)) \\ \vdots \\ \Psi_r(t_f, x(t_f)) \end{pmatrix} = 0$$

- time varying final penalty, $\phi(t_f, x(t_f))$

The Maximum Principle will be generalized in the next lecture!

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