LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Nonlinear Control and Servo Systems (FRTN05)

Exam - May 04, 2015 at 8.00 – 13.00

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for the exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

Results

Note!

In many cases the sub-problems can be solved independently of each other.

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
2015–05–04

1. Consider the control system

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1 \tag{1}$$

a. Write the system in first-order state-space form. (1 p)

b. Suppose $u(t) = 0$. Find and classify all equilibria and determine if they are stable or asymptotically stable if possible. Discuss if the stability results are global or local. (2 p)

c. Show that (1) is satisfied by the periodic solution $x(t) = \cos(t)$, $u(t) = \cos(2t)$. (1 p)

d. Design a state-feedback controller $u = u(x, \dot{x})$ for (1) , such that the origin of the closed loop system is globally asymptotically stable. (1 p)

Solution

a. Introduce $x_1 = x,\ x_2 = \dot{x}$ then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + 2x_2^2 + u - 1$$

b. Let $\dot{x}_1 = \dot{x}_2 = 0 \ \Rightarrow\ (x_1, x_2) = (-1, 0)$ is the only equilibrium. The linearization around this point is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 4x_2 \end{pmatrix}_{(x_1^0, x_2^0)=(-1,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The characteristic equation for the linearized system is $s^2 + 1 = 0 \ \Rightarrow\ s = i$ In general linearization only gives local behaviour of the nonlinear system, but as the linaerized sytem has a center point we can not conclude even local stability of the nonlinear system from this.

c. $x = \cos(t) \ \Rightarrow\ \dot{x} = -\sin(t) \ \Rightarrow\ \ddot{x} = -\cos(t)$ By inserting this in the system dynamics and using e.g., $u = cos(2t) = cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$ we get

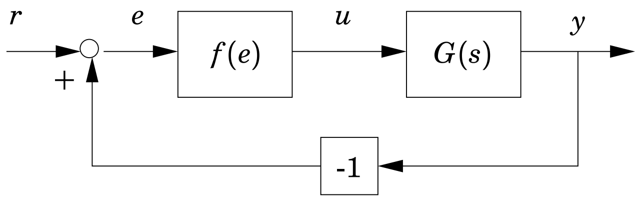
$$\ddot{x} - 2(\dot{x})2 + x = -\cos(t) - 2\sin^2(t) + \cos(t) = 2 + \cos^2(t) - 2 = u - 1$$

which shows that the trajectory is a solution.

d. The simplest way is to cancel the constant term and the nonlinearity with the control signal and introduce some linear feedback.

$$u = 1 - 2(\dot{x}^2)2 - a\dot{x},\ a > 0 \ \Rightarrow\ \ddot{x} + a\dot{x} + x = 0$$

As the resulting system is linear and time invariant with poles in the left half plane for all a $>$ 0 it is GAS.



Figur 1 Block diagram for Problem 3.

2. Use Lyapunov theory to prove that the system

$$\dot{x} = -x^3 + 4y^2x$$
$$\dot{y} = -x^2y - y^3$$

has a globally asymptotically stable equilibrium. (2 p)

Solution

One choice is the Lyapunov function

$$V(x, y) = x^2 + \alpha y^2.$$

Then

$$\frac{d}{dt}V = 2(x\dot{x} + \alpha y\dot{y}) = 2(-x^4 + 4y^2x^2 - \alpha y^2x^2 - \alpha y^4) < 0, \quad (x, y) \neq 0.$$

Hence, a possible choice for α is $\alpha = 4$. As V is positive and radially unbounded this proves global asymptotic stability of the system.

3. Consider the feedback loop in Figure 1. The linear system

$$G(s) = \frac{1}{(s+1)^4}$$

is connected with the static nonlinearity $u(t) = f(e(t))$, where $f(e) = 4e^3 + \frac{3\pi}{8}|e|e$

a. Find the describing function for the nonlinearity $f(e)$. (2 p)

b. What is the amplitude and frequency of a possible limit cycle? Will it be a stable limit cycle according to the describing function method? (3 p)

Solution

a. The expression $e|e|$ is an odd static nonlinearity

$$N_{e|e|}(A) = \frac{b_1 + ia_1}{A}, \text{ where}$$
$$b_1 = \frac{1}{\pi}\int_0^{2\pi} A\sin(\phi)|A\sin(\phi)|\sin(\phi)d\phi = \frac{8A^2}{3\pi}$$
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} A\sin(\phi)|A\sin(\phi)|\cos(\phi)d\phi = 0$$

The expression e^3 is an odd static nonlinearity, too

$$
\begin{aligned}
N_{e^3}(A) &= \frac{b_1 + i a_1}{A}, \text{ where} \\
b_1 &= \frac{1}{\pi}\int_0^{2\pi} A^3 \sin(\phi)^4 d\phi = \frac{3A^3}{4} \\
a_1 &= \frac{1}{\pi}\int_0^{2\pi} A^3 \sin(\phi)^3 \cos(\phi) d\phi = 0
\end{aligned}
$$

Hence, $N_{e|e|}(A) = \dfrac{8A}{3\pi}$ and $N_{e^3}(A) = \dfrac{3}{4}A^2$. The total describing function will be $N(A) = A + 3A^2$.

b. Possible crossing points between $G(i\omega)$ and $1/N(A)$ must be on the negative real axis. Thus we need the frequency ω' where

$$\arg G(i\omega') = -\pi.$$

Now $\arg G(i\omega) = -4 \arctan \omega$ which leads to the solution $\omega' = 1$. Also $|G(i\omega')| = 1/4$. This gives us the equation

$$1/N(A') = 1/4$$

with the solution $A' = 1$.

The limit cycle will be unstable.

4. Consider the linear system

$$\dot{x} = x + u, \quad x(0) = x_0.$$

Derive a control signal $u(t)$ which takes the system to

$$x(1) = 1$$

while minimizing the control energy

$$\int_0^1 u^2(t)dt.$$

(3 p)

Solution

Define the Hamiltonian

$$H(x, u, \lambda, n_0) = n_0 u^2 + \lambda x + \lambda u.$$

The optimal control $u^*(t)$ minimizes H:

$$H_u = 2n_0 u^* + \lambda = 0 \Rightarrow u^*(t) = -\frac{\lambda(t)}{2n_0}.$$

n_0 must be greater than zero for the optimum to exist, so choose $n_0 = 1$. $\lambda(t)$ satisfies

$$\dot{\lambda} = -H_x^T = -\lambda$$
$$\lambda(1) = \mu.$$

Thus $\lambda(t) = \mu e^{-t+1}$. To find μ we must use the boundary condition on x:

$$x(t) = x_0 e^t + \int_0^t e^{t-\tau} u^*(\tau) d\tau$$

and

$$x(1) = 1.$$

This gives

$$1 = x_0 e + \int_0^1 e^{1-\tau}(-\frac{\mu e^{-\tau+1}}{2}) d\tau = x_0 e + \frac{\mu e^2}{4}(e^{-2} - 1)$$

$$\Rightarrow \mu = \frac{4(1 - x_0 e)}{1 - e^2}.$$

This gives the control

$$u^*(t) = -\frac{2(1 - x_0 e)e^{-t+1}}{1 - e^2}.$$

5. An exponentially stable linear system $G(s)$ is negative feedback interconnected with a nonlinearity Ψ. The Nyquist diagram of the linear system is shown in Figure 2. (Note: For your answer it is more important to clearly mark in a figure where you get your data from than to have all digits correct.)

a. What is the largest sector $[-k, k]$, such that if $-kx \leq \Psi(x) \leq kx$, *the small gain theorem* guarantees stability for the closed loop? (1 p)

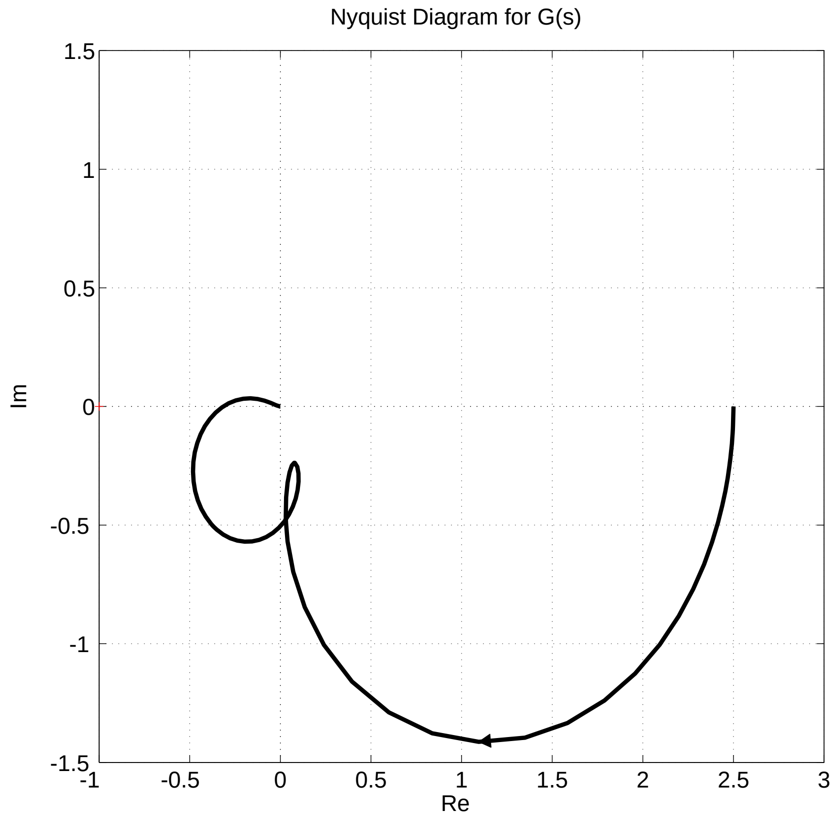
b. What is the largest sector $[0, \beta]$, such that if $0 \leq \Psi(x) \leq \beta x$, *the circle criterion* guarantees stability for the closed loop? (1 p)

c. Find the largest sector $[\alpha, 0]$, where $\alpha < 0$ such that if $\alpha x \leq \Psi(x) \leq 0$, *the circle criterion* guarantees stability for the closed loop. (1 p)

Solution

a. In this case we first want to find the maximum gain of the linear system which equals the largest magnitude ('radius') of the Nyquist curve. From the Nyquist curve we see that this is about 2.5. The small gain theorem then allows the sector to have $k < \frac{1}{2.5} = \frac{2}{5}$.

b. According to the circle criterion, in this case the closed loop will be stable for the nonlinearity in the sector $[0, \beta]$ if the Nyquist curve stays to the right of the vertical line $-1/\beta$. From the Nyquist curve we see that we can take $\beta \approx 1/0.5 = 2$.



Figur 2 Nyquist diagram for linear system $G(s)$ in Problem 5.

c. Multiply the nonlinearity and the system by -1 and apply the "ordinary" circle criterion. This means that the mirrored Nyquist curve must stay to the right of $-1/(-\alpha)$. The mirrored Nyquist curve is to the right of the vertical line -2.5, which means we can choose $\alpha = -1/2.5$.

6. In this problem we are going to examine how to stabilize a system using the control signal $u = \text{sat}_k(v)$, i.e.,

$$u(v) = \begin{cases} k, & v \geq k; \\ v, & -k \leq v \leq k; \\ -k, & v \leq -k; \end{cases}$$

where $\infty \geq k > 0$ and $k = \infty$ means $u(v) = v$ for all v.

Your task is to choose the control signal $v = v(x_1, x_2)$, such that the system (2)

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= u \\ u &= \text{sat}_k(v) \end{aligned} \tag{2}$$

is globally asymptotically stabilized.

a. What is the smallest possible choice of k if one is using the Lyapunov candidate

$$V_a = x_1^2 + x_2^2$$

to design a globally stabilized system. (2 p)

b. Now try with the Lyapunov function candidate

$$V_b = \log(1 + x_1^2) + x_2^2$$

and choose $v(x_1, x_2)$ so that the system is globally asymptotically stabilized. What is the smallest possible k now? (2 p)

Solution

Both V_a and V_b are positive definite with respect to (x_1, x_2) and radially unbounded.

a.

$$\frac{d}{dt}V_a = 2\left(x_1\dot{x}_1 + x_2\dot{x}_2\right) = 2(x_1^2 + u)x_2$$

u would need to be $-x_1^2 - f_{odd}(x_2)$ to get the derivative $\frac{dV_a}{dt}$ negative (semi-)definite. This can only be done with an unbounded u, which means k has to be infinitely large.

b.

$$\frac{d}{dt}V_b = 2\left(\frac{x_1\dot{x}_1}{1+x_1^2} + x_2\dot{x}_2\right) = 2\left(\frac{x_1^2}{1+x_1^2} + u\right)x_2$$

As $0 \leq \dfrac{x_1^2}{1+x_1^2} \leq 1$ we can always compensate for this term with u and by choosing "the rest of our available control signal" u as for instance $-4sat(x_2)$

$$v = -\frac{x_1^2}{1+x_1^2} - (k-1)sat(x_2) \Rightarrow u = -\frac{x_1^2}{1+x_1^2} - (k-1)sat(x_2)$$

However, this will leave us with $\frac{d}{dt}V_b = -(k-1)sat(x_2)x_2 \leq 0$. If $x_2 = 0 \Rightarrow \dot{x}_1 = 0$, but with the chosen control law $\dot{x}_2 = 0$ only if $x_1 = 0$, so the origin will be the only equilibrium if $k > 1$.

7. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1^2 - \operatorname{sign}(x_1 - x_2) + u \\ \dot{x}_2 &= -x_2\end{aligned}$$

a. Set u=0 and calculate the sliding surface. Also determine if the dynamics on the sliding surface is stable. (2 p)

b. Design a **continuous** controller, u, that brings all solution to the switching line $x_1 - x_2 = 0$. Does this control change the behaviour on the switching line? (1 p)

Solution

a. Set $\sigma(x) = x_1 - x_2$ and use equivalent control to calculate the sliding surface. Use $u_{eq} \in [-1\ 1]$

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1^2 - u_{eq}\\ \dot{x}_2 &= -x_2\end{aligned}$$

Set $\dot{\sigma}(x) = 0$

$$\dot{\sigma}(x) = \dot{x}_1 - \dot{x}_2 = -x_2 + x_1^2 - u_{eq} + x_2 = 0 \tag{3}$$

Thus $u_{eq} = x_1^2$. Since $u_{eq} \in [-1\ 1]$ the sliding surface is between $x_1 = -1$ and $x_1 = 1$. The dynamics on the sliding surface are $\dot{\sigma}(x) = \dot{x}_1 - \dot{x}_2 = 0 \Rightarrow \dot{x}_1 = \dot{x}_2 = -x_2 = -x_1$ which means that the dynamics are $\dot{x}_1 = -x_1$, $\dot{x}_2 = -x_2$ which is asymptotically stable.

b. Choose Lyapunov function $V(x) = \sigma^2/2$ which gives

$$\frac{dV}{dt} = (x_1 - x_2)(x_1^2 - \text{sign}(x_1 - x_2) + u) \tag{4}$$

Choose $u = -x_1^2$. This gives

$$\frac{dV}{dt} = -2(x_1 - x_2)\text{sign}(x_1 - x_2) = -2|x_1 - x_2| \leq 0 \tag{5}$$

This means that we will reach the surface $\sigma(x)$ in finite time and we will stay there. The dynamics **on the sliding line** with the chosen control is

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= -x_2\end{aligned}$$

which in this case is the same as without the continous feedback.