LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - January 14, 2015, 2 pm – 7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/“Collection of Formulae”. Pocket memoryless calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

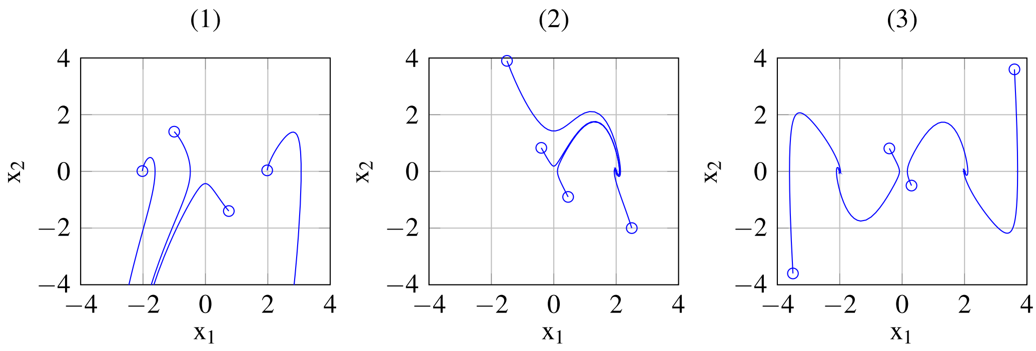

Figure 1 The simulated systems in problem 1. The initial value for each simulated trajectory is marked with a circle.

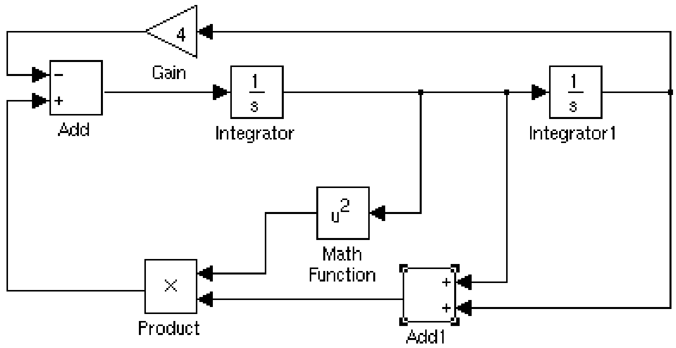

Figure 2 An (incorrect) attempt to describe the system in problem 1 in Simulink.

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^2 x_2 - x_1^3 + 4x_1 \end{aligned} \tag{1}$$

a. Determine all equilibria of the system. (1 p)

b. Classify all such equilibria according to the local behavior of the system in their proximity. (2 p)

c. Each of the three plots in Figure 1 depicts the trajectories of some dynamical system. The initial value of each simulated trajectory is marked with a small circle. Which one(s) of the three plots is compatible with the dynamical system (1)? Motivate your answer. (1 p)

d. An incorrect attempt to simulate system (1) in Simulink is displayed in Figure 2. Please sketch a correct implementation of (1). (1 p)

Solution

a. The equilibria are found by setting all derivatives to zero:

$$\left\{ \begin{aligned} 0 &= x_2 \\ 0 &= -x_1^2 x_2 - x_1^3 + 4x_1 \end{aligned} \right. \Leftrightarrow \left\{ \begin{aligned} x_2 &= 0 \\ 0 &= x_1(x_1^2 - 4) \end{aligned} \right.$$

and the solutions to this equation system are the three equilibrium points (-2,0), (0,0), and (2,0).

b. The local behavior in the proximity of the equilibria is found by investigating the linearizations around these points. The A-matrix will be given as

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \left.\begin{bmatrix} 0 & 1 \\ -2x_1x_2 - 3x_1^2 + 4 & -x_1^2 \end{bmatrix}\right|_{x=x_0}$$

The equilibrium in (0,0) gives

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

which has eigenvalues in 2 and -2 and the equilibrium is hence a saddle point.

The equilibria in ($\pm$2,0) gives

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix}$$

which has eigenvalues in $-2 \pm 2i$ and the equilibria are hence stable foci.

c. For the given system, the equilibrium point in (0,0) is a saddle point, and the other two are stable foci. Subplot (1) is clearly not from the considered system, as initial values in the equilibrium points (-2,0) and (2,0) diverges.

In (2) there seems to be a saddle point in the origin and a stable focus in (2,0). This corresponds well with the properties of the given system (what happens around (-2,0) is not clear from the given simulated system trajectories).

In (3) there seems to be stable focuses in (-2,0) and (2,0), and a saddle point in the origin. This corresponds well with the given system.

To summarize, (2) and (3) may have been made with the given system in the problem.

d. Two errors are made in the implementation.

1. The addition block named ”Add” should switch the signs for the inputs
2. The input to the quadratic function block should come from the output of the second integrator (”Integrator1”) instead of the first integrator.

A correct implementation is displayed in Figure 3.

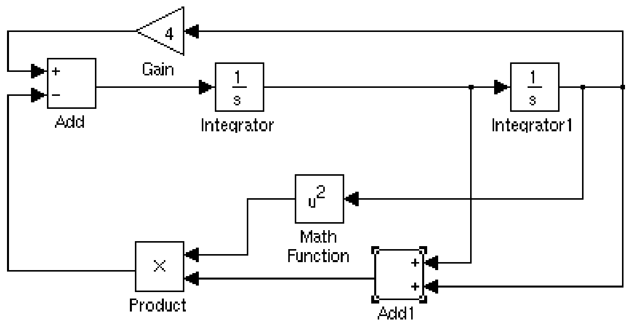

Figure 3 A correct implementation of the system in problem 1 in Simulink.

2. Consider the controlled dynamical system $\dot{x} = f(x,u)$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - x_2^3 + u(x) \end{bmatrix}.$$

a. Show that for $u(x) = 0$, the origin $x^* = (0,0)$ is an unstable equilibrium. (1 p)

b. Determine a feedback control $u(x)$ which makes the origin $x^* = (0,0)$ a globally asymptotically stable equilibrium for the system. (1 p)

c. Use the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left((x_1 - 1)^2 + x_2^2\right)$$

to prove that the feedback control

$$u(x) = 1 - x_1 - \sin(x_1)$$

makes the point $x' = (1,0)$ a globally asymptotically stable equilibrium of the system. (2 p)

Solution

a. The Jacobian in the origin equals

$$\nabla f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

hence it has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Then, $(0,0)$ is an unstable equilibrium for the system $\dot{x} = f(x)$.

b. By choosing $u(x) = x_2^3 - \sin(x_1) - x_1 - 2x_2$ one gets that $f(x,u(x)) = Ax$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

has a single eigenvalue -1. Then, the system $\dot{x} = f(x,u(x)) = Ax$ is linear and has a globally asymptotically stable equilibrium in the origin.

c. The function $V(x)$ satisfies $V(x') = 0$, $V(x) > 0$ for $x \neq x'$, and it is radially unbounded. Its time derivative is

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}(x)\dot{x}_1 + \frac{\partial V}{\partial x_2}(x)\dot{x}_2 = x_1\dot{x}_1 + x_2\dot{x}_2 = (x_1 - 1)x_2 + x_2(1 - x_1 - x_2^3) = -x_2^4.$$

This means that $\dot{V}(x)$ will be negative except for the set $E = \{x : x_2 = 0\}$ for which $\dot{V}(x) = 0$. Hence, one cannot simply use Lyapunov theorems to prove global asymptotic stability but has to revert to LaSalle's invariance principle (invariant set theorem). In order to apply that, observe that for $x_2 = 0$ we have

$$\dot{x}_2 = 1 - x_1$$

which is only zero for $x_1 = 1$. Hence, the largest subset of E which is invariant for the system is $\{x'\}$, which allows us to conclude that x' is a globally asymptotically stable equilibrium.

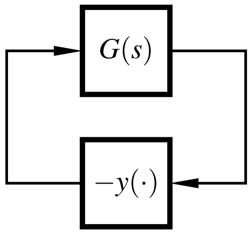

Figure 4 The setup for the simulation in problem 3.b, where $y(\cdot)$ is the relay described in (2).

3. The following asymmetric relay

$$y(u) = \begin{cases} a & \text{if } u \geq 0 \\ b & \text{if } u < 0 \end{cases} \tag{2}$$

–where $a > b$ are two constants– is useful in applications to autotuning of PID controllers.

a. Compute the describing function for an asymmetric relay as in (2). (2 p)

b. A simulation of the linear time-invariant system with transfer function

$$G(s) = \frac{1}{s(s+1)^2}$$

was performed in negative feedback with an asymmetric relay as in (2) with $a = 1$ and $b = -2$, see Figure 4. Which of the three subplots in Figure 5 is consistent with the prediction given by the describing function method for the negative feedback interconnection of $G(s)$ and the asymmetric relay (2)? (2 p)

Note: In case you have not been able to solve subproblem 3.a, you can answer using the describing function for a symmetric relay ($a = -b = 1$) given in the lecture slides.

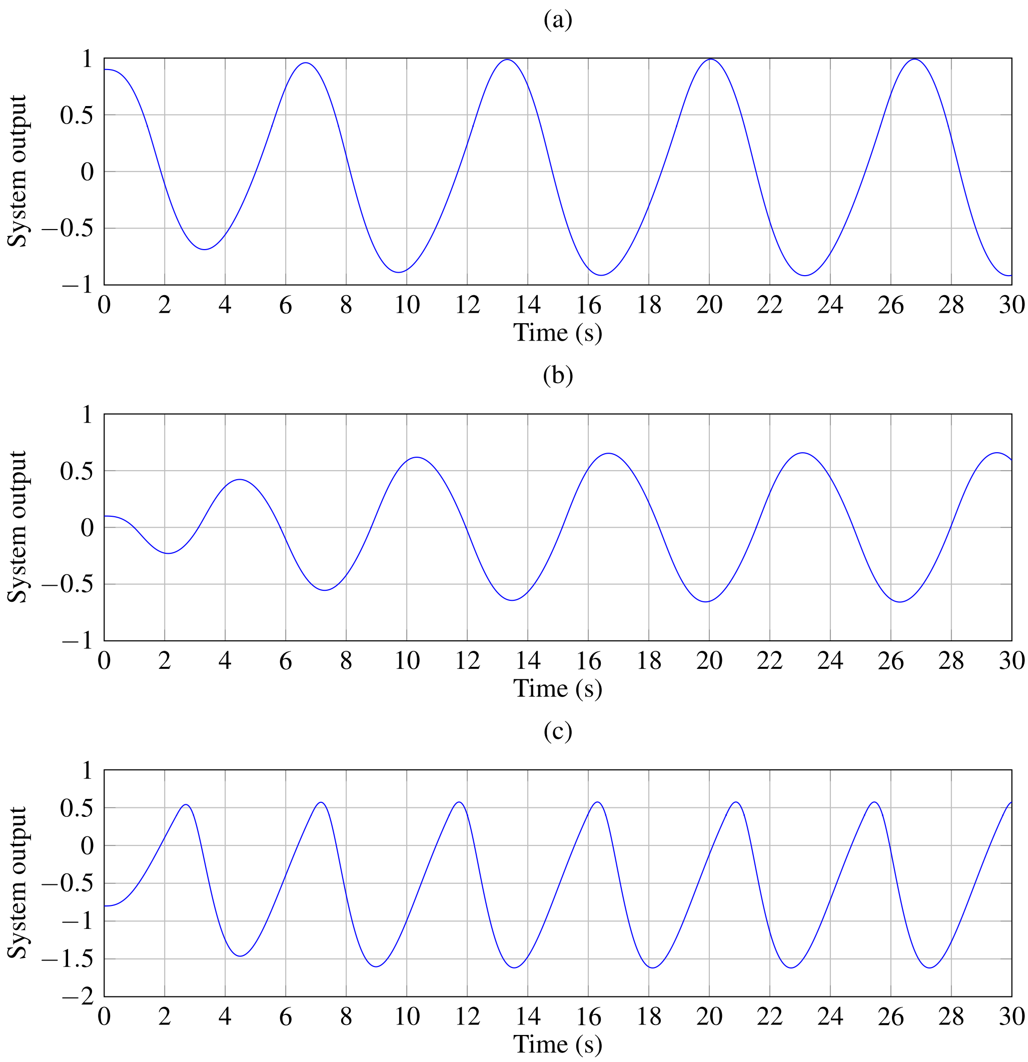

Figure 5 The simulated systems in problem 3.

Solution

a. Sending the signal $A\sin\varphi$ through the asymmetric relay gives the following output

$$\Psi(\varphi) = \begin{cases} a & ,0 < \varphi < \pi \\ b & ,\pi < \varphi < 2\pi \end{cases}$$

The coefficients needed for the describing function are now calculated as

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} \Psi(\varphi)\cos\varphi d\varphi = \frac{1}{\pi}\left(a\int_0^{\pi}\cos\varphi d\varphi + b\int_{\pi}^{2\pi}\cos\varphi d\ \varphi\right) = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} \Psi(\varphi)\sin\varphi d\varphi = \frac{1}{\pi}\left(a\int_0^{\pi}\sin\varphi d\varphi + b\int_{\pi}^{2\pi}\sin\varphi d\varphi\right)$$

$$= \frac{a}{\pi}[-\cos\varphi]_0^{\pi} + \frac{b}{\pi}[-\cos\varphi]_{\pi}^{2\pi} = \frac{2(a-b)}{\pi}$$

and the describing function is given by $N(A) = \frac{2(a-b)}{\pi A}$ (inserting $b = -a$ gives $N(A) = \frac{4a}{\pi A}$, which is the describing function for a symmetric relay).

b. The describing function of the asymmetric relay is positive real, and $-1/N(A)$ will therefore only exist on the negative real axis. We should thus look for when the Nyquist curve of the system intersects with the negative real axis. The frequency for this intersection is found by solving

$$-\pi = \arg\, G(i\omega) = -\frac{\pi}{2} - 2\arctan\,\omega$$

$$\Leftrightarrow \arctan\,\omega = \frac{\pi}{4} \Leftrightarrow \omega = 1 \text{ rad/s}$$

and the magnitude of the Nyquist curve when the negative real axis is crossed is found by calculating

$$|G(1i)| = \left|\frac{1}{i(i+1)^2}\right| = \frac{1}{|i+1|^2} = \frac{1}{2}$$

The describing function method now predicts a limit cycle with angular frequency 1 rad/s and an amplitude A which is the solution to ($N(A) = \frac{2(a-b)}{\pi A} = \frac{6}{\pi A}$)

$$-\frac{1}{2} = -\frac{1}{N(A)} = -\frac{\pi A}{6} \Leftrightarrow A = \frac{3}{\pi} \approx 0.95$$

The angular frequency 1 rad/s corresponds to the frequency $1/(2\pi)$ Hz, and the period time 2π s ≈ 6.28 s. The only subplot that is close to these numbers are (a).

4. Consider the sliding mode control system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= -x_1 + \left(\tfrac{29}{18} - x_2^2\right) u + \tfrac{11}{18} \end{aligned} \qquad u(x) = \begin{cases} -1 & \text{if} \quad x_1 + 3x_2 > 0 \\ +1 & \text{if} \quad x_1 + 3x_2 < 0 \,. \end{cases}$$

a. Determine the sliding set. (2 p)

b. Is the origin a locally asymptotically stable equilibrium for the sliding dynamics? Motivate your answer. (1 p)

Solution

a. We can rewrite the system as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} f^+(x_1,x_2) & \text{if} \quad \sigma(x_1,x_2) > 0 \\ f^-(x_1,x_2) & \text{if} \quad \sigma(x_1,x_2) < 0 \,. \end{cases}$$

where

$$f^+(x_1,x_2) = \begin{bmatrix} -x_2 \\ x_2^2 - x_1 - 1 \end{bmatrix}, \qquad f^-(x_1,x_2) = \begin{bmatrix} -x_2 \\ \frac{20}{9} - x_1 - x_2^2 \end{bmatrix}, \qquad \sigma(x) = x_1 + 3x_2 \,.$$

The sliding set is the subset of the sliding surface $\{x : \sigma(x) = 0\}$ where

$$\nabla\sigma(x_1,x_2) \cdot f^+(x_1,x_2) < 0 \,, \qquad \nabla\sigma(x_1,x_2) \cdot f^-(x_1,x_2) > 0 \,.$$

The equations above read

$$-x_2 + 3(x_2^2 - x_1 - 1) < 0 \,, \qquad -x_2 + 3\left(\frac{20}{9} - x_1 - x_2^2\right) > 0$$

so that, using the identity $x_1 + 3x_2 = 0$, one gets

$$3x_2^2 + 8x_2 - 3 < 0 \,, \qquad 3x_2^2 - 8x_2 - \frac{20}{3} < 0 \,.$$

Solving the inequalities above one gets

$$-9 < x_2 < 1 \,, \qquad -2 < x_2 < 10 \,.$$

Hence the sliding set is

$$\{(-3x_2, x_2) : -2 < x_2 < 1\} \,.$$

b. On the sliding set, one has $x_1 = -3x_2$, so that the sliding dynamics is

$$\dot{x}_1 = -x_2 = x_1/3 \,, \qquad \dot{x}_2 = -\dot{x}_1/3 = x_2/3 \,.$$

Hence, the origin is an unstable equilibrium of the sliding dynamics.

5. A stable linear time-invariant process with transfer function $P(s)$ and Nyquist curve shown in Figure 6 is to be controlled using a proportional controller $C(s) = K$, see Figure 7.

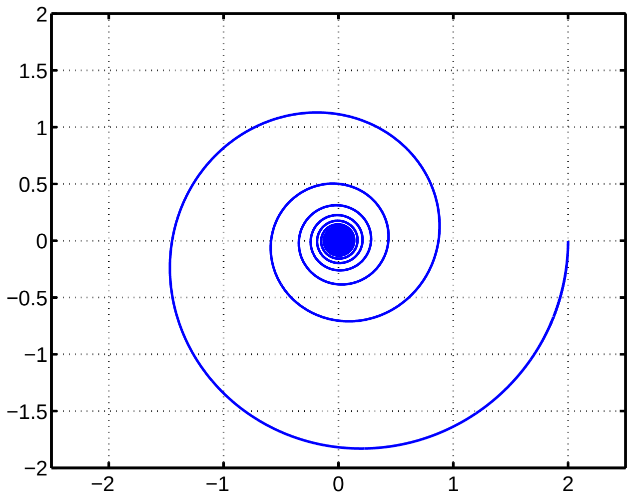

Figure 6 The Nyquist curve of the process in Problem 5

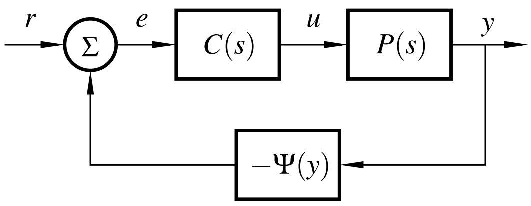

Figure 7 The feedback system in Problem 5

a. For a regular feedback connection, i.e. $\Psi(y) = y$, apply the Nyquist theorem to determine whether the closed-loop system is stable for $K = 1$ and for $K = 0.6$. (1 p)

b. Consider the presence of a dead zone ψ in the feedback connection given by

$$\psi(y) = \begin{cases} 0, & \text{if } |y| < D, \\ y - \text{sign}(y) \cdot D, & \text{if } |y| \geq D. \end{cases}$$

For what positive values of D do the small gain theorem and, respectively, the circle criterion allow one to prove stability of the closed-loop system with $K = 0.6$? Motivate your answer. (2 p)

Solution

a. For $K = 1$, the open-loop Nyquist curve is the same as the one for the process, which encircles -1. Thus the system is unstable.

For $K = 0.6$, the open-loop Nyquist curve is obtained by simply multiplying the given Nyquist curve by 0.6, in which case the curve does not encircle -1 (the intersection with the negative real line furthest away from the origin occurs at approximately -0.87). Thus the system is stable.

b. The nonlinearity is shown in Figure 8.

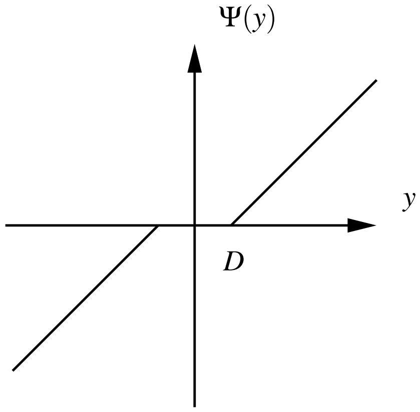

Figure 8 The deadzone in Problem 5

We see that the nonlinearity lies in the stability sector $[k_1, k_2] = [0, 1]$ for all values of D. The gain of the nonlinearity is thus 1. In the Nyquist curve see that the linear system has gain 2. Straightforward application of the small gain theorem thus yields nothing, since the product of the gains is 2, which is greater than 1. Thus, this approach with the small gain theorem does not yield stability for any D.

For the circle criterion, we see in the given Nyquist curve that

$$\min_{\omega} \text{Re } P(i\omega) > -1.5.$$

The Nyquist curve for the open-loop system with $K = 0.6$ thus stays to the right of the line

$$\text{Re } s = -0.6 \cdot 1.5 = -0.9 > -\frac{1}{k_2} = -1.$$

The circle criterion thus yields that the closed-loop system is stable for all positive (and even non-positive) values of D.

6. Consider the following open-loop optimal control problem

$$\text{minimize} \quad \int_0^2 \left(-\frac{1}{2}x^2(t) + u_1^2(t) + u_2^2(t) \right) \mathrm{d}t, \tag{3}$$

$$\text{subject to} \quad \dot{x}(t) = -x(t) + u_1(t) + u_2(t), \quad x(0) = 1. \tag{4}$$

a. Write down the Hamiltonian. (1 p)

b. Write down the adjoint equation and the final time condition on the adjoint variable $\lambda(t)$. (1 p)

c. Verify that the pair

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} 1 - (1+\lambda_0)t \\ \lambda_0 + (1+\lambda_0)t \end{bmatrix}$$

jointly satisfies (4) and the adjoint equation. Determine the value of the constant λ_0. (1 p)

d. Determine the optimal open loop control $u^*(t)$ solution of the problem (3)–(4). (1 p)

e. Find the open-loop optimal control for

$$\begin{aligned} \text{minimize} \quad & \int_0^2 \left(-0.5x^2(t) + u_1^2(t) + u_2^2(t) \right) \mathrm{d}t, \\ \text{subject to} \quad & \dot{x}(t) = -x(t) + u_1(t) + u_2(t), \\ & x(0) = 1, \quad x(2) = 2. \end{aligned}$$

(2 p)

Solution

a. The Hamiltonian is

$$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u) = -0.5x^2 + u_1^2 + u_2^2 + \lambda(-x + u_1 + u_2).$$

b. The adjoint equation is

$$\dot{\lambda} = -\nabla_x H(x,u,\lambda) = x + \lambda, \quad \lambda(2) = \nabla\phi^T(x(2)) = 0.$$

c. We solve points c and d jointly.

d. Since H is quadratic in u and u is unbounded, the optimal u is given by the unique stationary point satisfying

$$\begin{aligned} 0 = \nabla_u H(x,u,\lambda) &= [2u_1 + \lambda \quad 2u_2 + \lambda] \\ \iff u_1 &= u_2 = -0.5\lambda. \end{aligned}$$

Combining the adjoint equation with the state equation gives us the linear ordinary differential equation system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -x + u_1 + u_2 \\ x + \lambda \end{bmatrix} = \begin{bmatrix} -x - \lambda \\ x + \lambda \end{bmatrix}, \quad \begin{bmatrix} x(0) = 1 \\ \lambda(2) = 0 \end{bmatrix}.$$

Insertion of the provided pair yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} -1+(1+\lambda_0)t-\lambda_0-(1+\lambda_0)t \\ 1-(1+\lambda_0)t+\lambda_0+(1+\lambda_0)t \end{bmatrix} = \begin{bmatrix} -1-\lambda_0 \\ 1+\lambda_0 \end{bmatrix}. \tag{5}$$

Differentiation of the provided state and adjoint solution yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} -1-\lambda_0 \\ 1+\lambda_0 \end{bmatrix},$$

which clearly coincides with (5).

We use the adjoint boundary condition and the provided adjoint solution to solve for $\lambda_0 = \lambda(0)$, which gives us

$$\begin{aligned} 0 = \lambda(2) &= \lambda_0 + 2(1+\lambda_0) \\ \iff \lambda_0 &= -\frac{2}{3}. \end{aligned}$$

Thus, the optimal open-loop control is

$$u_1(t) = u_2(t) = -0.5\lambda(t) = -0.5(\lambda_0 + (1+\lambda_0)t) = -0.5\left(-\frac{2}{3}+\frac{t}{3}\right) = \frac{2-t}{6}.$$

e. We start by checking the abnormal case, which gives us the Hamiltonian

$$H(x,u,\lambda,n_0) = \lambda^T f(x,u) = \lambda(-x+u_1+u_2).$$

Since H is a linear function in u and u is unbounded, there is no minimizing u and so the abnormal case does not have any extremals. Thus, the problem is normal.

Since the problem is normal, we have the same Hamiltonian and adjoint equation as before, except that we now have a different terminal condition for the adjoint state due to the introduced state terminal constraints. We thus get

$$\lambda(t_f) = n_0 \nabla\phi^T(x(t_f)) + \nabla\Psi^T(x(t_f))\mu = \mu.$$

The provided state and adjoint solution still hold, but the new adjoint terminal constraint yields

$$\mu = \lambda(t_f) = \lambda_0 + 2(1+\lambda_0) \iff \lambda_0 = \frac{\mu-2}{3}.$$

We find μ by enforcing the terminal constraint in the provided state solution, yielding

$$2 = x(2) = 1-(1+\lambda_0)\cdot 2 = 1-\frac{2}{3}(\mu+1) \iff \mu = -2.5.$$

Thus, the optimal open-loop control is

$$\begin{aligned} u_1(t) = u_2(t) = -0.5(\lambda_0+(1+\lambda_0)t) &= -0.5\left(\frac{\mu-2}{3}+\left(1+\frac{\mu-2}{3}\right)t\right) \\ &= -0.5\left(\frac{-4.5}{3}+\left(1-\frac{4.5}{3}\right)t\right) = \frac{3+t}{4}. \end{aligned}$$