LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - April 25, 2014, 1 pm – 6 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16 points

4: 16.5 – 20.5 points

5: 21 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!
In many cases the sub-problems can be solved independently of each other.

Good Luck!

1.

a. Prove that the origin is globally asymptotically stable for the two systems below

$$\text{I:}\quad \begin{aligned}\dot{x}_1 &= -4x_1^5 - 2x_1^3x_2^4\\ \dot{x}_2 &= -5x_2^7\end{aligned} \qquad \text{II:}\quad \begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1 - x_2^3\end{aligned}$$

(2 p)

b. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 - x_2x_3\\ \dot{x}_2 &= -x_2^5 + x_3x_1^2\\ \dot{x}_3 &= x_1x_2^2 + u\end{aligned}$$

Design a feedback controller $u(x)$ that renders the origin globally asymptotically stable. (2 p)

Solution

a. Along solutions of the first system, the candidate Lyapunov function

$$V_I(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

has derivative

$$\dot{V}_I(x) = x_1(-4x_1^5 - 2x_1^3x_2^4) + x_2(-5x_2^7) = -4x_1^6 - 2x_1^4x_2^4 - 5x_2^7\,.$$

Since $V_I(0) = 0$, $V_I(x) \geq 0$ for all x, $\dot{V}_I(x) < 0$ for all $x \neq 0$, and $V_I(x)$ is radially unbounded, thus Lyapunov's stability theorem implies that the origin is a globally asymptotically stable equilibrium.

For the second system, consider the candidate Lyapunov function

$$V_{II}(x) = x_1^2 + \frac{1}{2}x_2^2\,.$$

Along the solutions, one has

$$\dot{V}_{II}(x) = 2x_1x_2 + x_2(-2x_1 - x_2^3) = -x_2^4\,.$$

One has $V_{II}(0) = 0$, $V_{II}(x) \geq 0$ for all x, $\dot{V}_{II}(x) \leq 0$, and $V_{II}(x)$ is radially unbounded. Since $\dot{V}_{II}(x) = 0$ for all $x = (x_1, 0)$, and not only for $x = 0$, one cannot simply use Lyapunov's stability theorem, but should apply LaSalle's theorem. For that, observe that $\dot{x}_2 = -2x_1$ when $x_2 = 0$, so that the largest invariant subset of $E = \{x : \dot{V}_{II}(x) = 0\} = \{(x_1, 0)\}$ is $\{0\}$. Then, the origin is a globally asymptotically stable equilibrium.

b. Let us choose the candidate Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\,,$$

which is radially unbounded and has a minimum $V(0) = 0$ in the origin. Along solutions of the system, its derivative is given by

$$\begin{aligned}\dot{V}(x) &= x_1(-x_1^3 - x_2x_3) + x_2(-x_2^5 + x_3x_1^2) + x_3(x_1x_2^2 + u) \\ &= -x_1^4 - x_2^6 + x_3u + x_3g(x)\end{aligned}$$

where $g(x) := -x_1x_2 + x_1^2x_2 + x_1x_2^2$. Then, choosing

$$u(x) = -x_3 - g(x)$$

gives

$$\dot{V}(x) = h(x)\,, \qquad h(x) = -x_1^4 - x_2^6 - x_3^2\,.$$

Since $h(x) < 0$ for all $x \neq 0$, Lyapunov's stability theorem allows one to prove global asymptotic stability of the origin with this choice of the control $u(x)$.

2. Consider the sliding mode controlled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \eta & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(x_1, x_2)\,,$$

where η is a scalar parameter and

$$u(x_1, x_2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \text{if} \ \ x_2 > 0\,, \qquad\qquad u(x_1, x_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ \text{if} \ \ x_2 < 0\,.$$

a. Determine the sliding set. (1.5 p)

b. Find the sliding dynamics. (1.5 p)

c. For which values of the parameter η is $x^* = [0, 0]^T$ a stable equilibrium for the sliding dynamics? (1 p)

Solution

Let us rewrite the system as

$$\dot{x} = \begin{cases} f^+(x) & \text{if} \quad x_2 > 0 \\ f^-(x) & \text{if} \quad x_2 < 0 \end{cases},$$

with

$$f^+(x) = \begin{bmatrix} \eta & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad f^-(x) := \begin{bmatrix} \eta & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

a. The sliding surface is $\{x_2 = 0\}$, with normal vector $[0, 1]^T$. In order to determine the sliding set, we need to find the subset of the sliding surface where

$$[0, 1]f^+(x) < 0\,, \qquad [0, 1]f^-(x) > 0\,.$$

The above gives $4x_1 - 2 < 0$, and $4x_1 + 2 > 0$, i.e., $-1/2 < x_1 < 1/2$. Hence, the sliding set is $\{(x_1, 0) : \ |x_1| < 1/2\}$.

b. The sliding dynamics are given by the convex combination

$$\dot{x} = \alpha f^+(x) + (1 - \alpha)f^-(x)\,,$$

where $\alpha = \alpha(x)$ is determined by the condition

$$[0, 1]\left(\alpha f^+(x) + (1 - \alpha)f^-(x)\right) = 0\,.$$

The above gives

$$4x_1 - 2\alpha + 2(1 - \alpha) = 0\,,$$

that is $\alpha = x_1 + 1/2$. Substituting back, one finds that the sliding dynamics is given by

$$\dot{x} = \alpha f^+(x) + (1 - \alpha)f^-(x) = \begin{bmatrix} \eta + 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c. On the sliding set the dynamics are given by $\dot{x}_1 = (\eta + 2)x_1$. The origin is a locally stable equilibrium if and only if $\eta \leq -2$, and is an asymptotically stable equilibrium if and only if $\eta < -2$.

3. Consider the following controlled dynamical system

$$\dot{v}(t) = -2v(t) + u(t)\,, \qquad v(0) = 0$$

describing the velocity of a particle affected by viscous friction and driven by a controlled force $u(t)$. One is interested in choosing $u(t)$ so as to meet the constraint

$$v(1) = 3\,,$$

while minimizing the acceleration cost

$$\int_0^1 u^2(t)\ \mathrm{d}t\,.$$

a. Write down the Hamiltonian and the co-state equations. (1 p)

b. Solve the co-state equations. (1 p)

c. Find the optimal control $u^*(t)$ (2 p)

Solution

a. The state is $x = v$, the running cost $L(x, u) = u^2$, the time horizon $t_f = 1$, the final cost $\phi(x) = 0$, and the dynamics $\dot{x} = f(x, u) = -2x + u$. Note that we have a final time constraint $\psi(v(t_f)) = 0$, and $\psi(v) = v - 3$. Then, we need the more general version of the Pontryagin maximum principle, with the Hamiltonian given by

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda f(x, u) = n_0 u^2 + \lambda(-2x + u)$$

where the multiplier n_0 can take the values $n_0 = 0$ or $n_0 = 1$, and the co-state equation reads

$$\dot{\lambda} = -\frac{\partial}{\partial x} H = 2\lambda$$

with final time condition

$$\lambda(1) = n_0 \frac{\partial}{\partial x}\phi(x(1)) + \mu \frac{\partial}{\partial x}\psi(x(1)) = \mu\,,$$

where $\mu \geq 0$.

b. The solution of the co-state equation

$$\dot{\lambda} = 2\lambda\,, \qquad \lambda(1) = \mu\,,$$

is given by

$$\lambda(t) = \mu e^{2(t-1)}\,.$$

c. We should consider both the abnormal case $n_0 = 0$ and the normal one $n_0 = 1$, separately.

For the abnormal case, observe that, since $[n_0, \mu] \neq [0, 0]$, one has either $\mu < 0$, or $\mu > 0$. For $\mu < 0$, one gets $\lambda(t) < 0$, so that the Hamiltonian $H(x, u, \lambda(t), n_0 = 0) = \lambda(t)(-2x + u)$ does not admit a minimum in u. (The infimum $-\infty$ is achieved by taking arbitrarily negative u.) For $\mu > 0$, one gets $\lambda(t) > 0$, so that the Hamiltonian $H(x, u, \lambda(t), n_0 = 0) = \lambda(t)(-2x + u)$ is minimized by taking $u(t) = 0$, for all $t \in [0, 1]$. However, choosing $u(t) = 0$ for all $t \in [0, 1]$ gives state equation

$$\dot{x} = 0\,, \qquad x(0) = 0$$

whose solution $x(t) = 0$ for all $t \in [0, 1]$ violates the constraint $x(1) = 3$.

Then, we are left with considering the normal case $n_0 = 1$. Here the Hamiltonian at time t reads

$$H(x, u, \lambda(t), n_0 = 1) = u^2 + \lambda(t)(-2x + u)\,.$$

The minimum of $u^2 + \lambda(t)(-2x + u)$ with respect to u is found by solving $\frac{\partial}{\partial u}(u^2 + \lambda(-2x + u)) = 2u + \lambda = 0$, which gives us

$$u^*(t) = -\frac{1}{2}\lambda(t) = -\frac{\mu}{2}e^{2(t-1)}\,, \qquad t \in [0, 1]\,.$$

It remains to determine the value of μ. To get it, one needs to solve the state equation

$$\dot{x} = -2x + u = -2x - \frac{\mu}{2}e^{2(t-1)}\,,$$

and impose the constraint $x(1) = 3$. The solution is

$$x(t) = -\frac{\mu}{8}e^{2(t-1)}$$

so that $x(1) = \mu/8 = 3$ gives $\mu = 24$.