LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - March 13, 2013, 8 am – 1 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!
In many cases the sub-problems can be solved independently of each other.

Good Luck!

1. Consider the second order dynamical system

$$\ddot{x} + 3\dot{x} = 2\sin(2x)$$

a. Write it in state-space form. (1 p)

b. Verify that 0 is an equilibrium, and classify it. (1 p)

Solution

a. Choosing $x_1 = x$ and $x_2 = \dot{x}$, the state-space form can be written as

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -3x_2 + 2\sin(2x_1) \tag{2}$$

b. With $\dot{x}_1 = \dot{x}_2 = 0$, $x_2 = 0$ and $x_1 = k\pi$, where the origin is clearly an equilibria if $k = 0$. The Jacobian of the system in the origin is given by

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \tag{3}$$

The eigenvalues of the Jacobian are -1 and 4, implying that the equilibria is a saddle point.

2. Consider the controlled dynamical system

$$\dot{x}_1 = -x_1 - x_1^2 x_2 + a x_1^3 + u$$
$$\dot{x}_2 = x_1^3 - x_2$$

where $a \geq 0$ is a constant.

a. Assume $u \equiv 0$ and $a = 2$. Find all equilibria and determine their local stability properties using the linearization method. (3 p)

b. Assume that $u \equiv 0$ and $a = 0$. Use the candidate Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to prove stability of the origin $(0, 0)$. Specify what kind stability (i.e., local, local asymptotic, or global asymptotic) can be proved in this way, and justify your answer. (2 p)

c. Assume $a = 1$. Use the candidate Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ to design a feedback control law $u(x_1, x_2)$ such that the system is globally asymptotically stable. (1 p)

d. In practical applications it is uncommon to know the exact value of the parameter a, while it is more common to know that a belongs to a certain interval. How can the control signal designed in point **c.** be modified to guarantee global asymptotic stability for all $a \in [0.5, 1.5]$? (1 p)

Solution

a. Setting $\dot{x}_1 = \dot{x}_2 = 0$,three equilibria are obtained, $[x_1^e, x_2^e] = [0, 0], [1, 1], [-1, -1]$. The Jacobian of the system is given by

$$J(x_1^e, x_2^e) = \begin{bmatrix} -1 - 2x_1x_2 + 3ax_1^2 & -x_1^2 \\ 3x_1^2 & -1 \end{bmatrix} \tag{4}$$

Inserting the equilibria points and looking at the eigenvalues of the Jacobian, the points are concluded to be a stable focus ($\lambda_{1,2} = [-1, -1]$), unstable focus ($\lambda_{1,2} = [0, 2]$) and another unstable focus ($\lambda_{1,2} = [0, 2]$). The local stability properties are therefore asymptotically stable $x_{1,2}^e = [0, 0]$ and unstable $x_{1,2}^e = [1, 1], [-1, -1]$.

b. Using the classic Lyapunov-function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, its time-derivative, with the system dynamics inserted, becomes

$$\dot{V}(x) = -x_1^2 - x_2^2 \tag{5}$$

which is smaller than zero for all values of the states, thus globally asymptotically stable.

c. The time derivative of the Lyapunov-function is given by

$$\dot{V}(x) = -x_1^2 + x_1^4 - x_2^2 + x_1 u, \tag{6}$$

which should be smaller than zero. One possible controller choice that fulfills this is $u = -x_1^3$.

d. If a fluctuates to a value larger than its expected value, the system will become unstable. Given an upper bound on the value of a, the control signal can be modified to $u = -a_{max} x_1^3$, guaranteeing global asymptotic stability for all a within the given bounds. In this case, choosing $a_{max} \geq 1.5$ is sufficient.

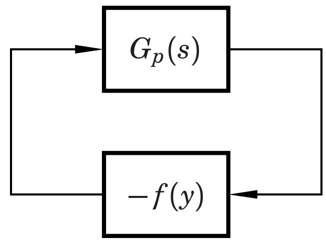

Figure 1 Feedback connection in Problem 3.

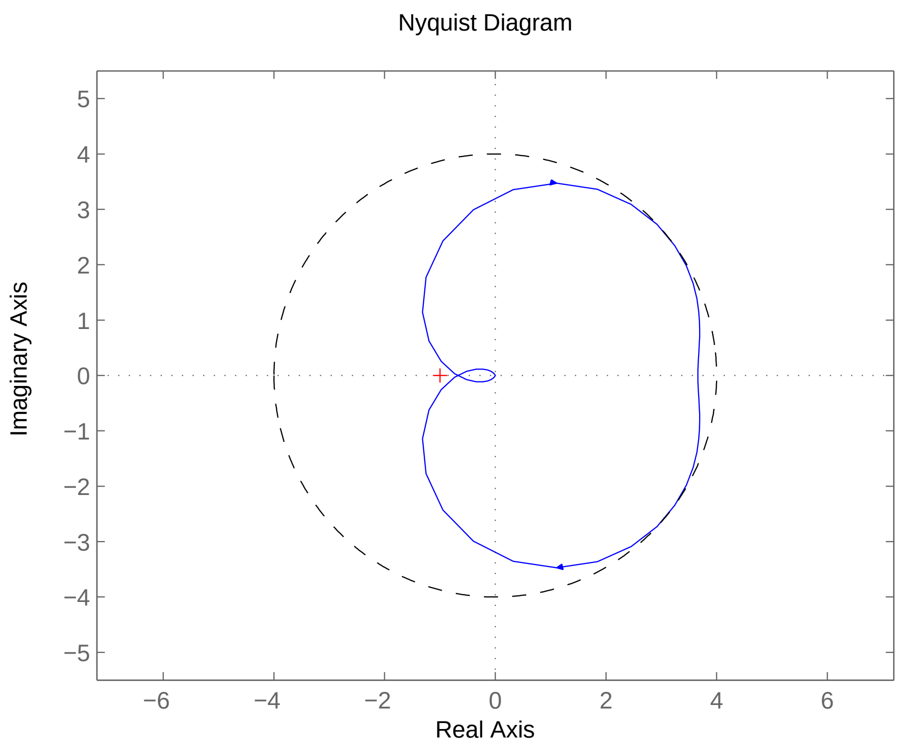

Figure 2 Nyquist-curve for the system $G_p(s)$ in Problem 3.

3. The Nyquist-curve of a stable linear system $G_p(s)$ is depicted in Figure 2. The system is fed back with a static nonlinearity of the form $u = f(y)$, according to the block diagram in Figure 1. For which of the three nonlinearities shown in Figures 3–5 can one determine stability of the closed-loop system using

a. the Small Gain Theorem? Please, specify the estimated gains. (2 p)

b. the Circle Criterion? Please, specify the estimated sector conditions for the non-linearities. (2 p)

Solution

a. The estimated maximum gains of the nonlinearities from the figures are given by 2, 0.45 and 0.2 respectively, and the process gain is seen to be

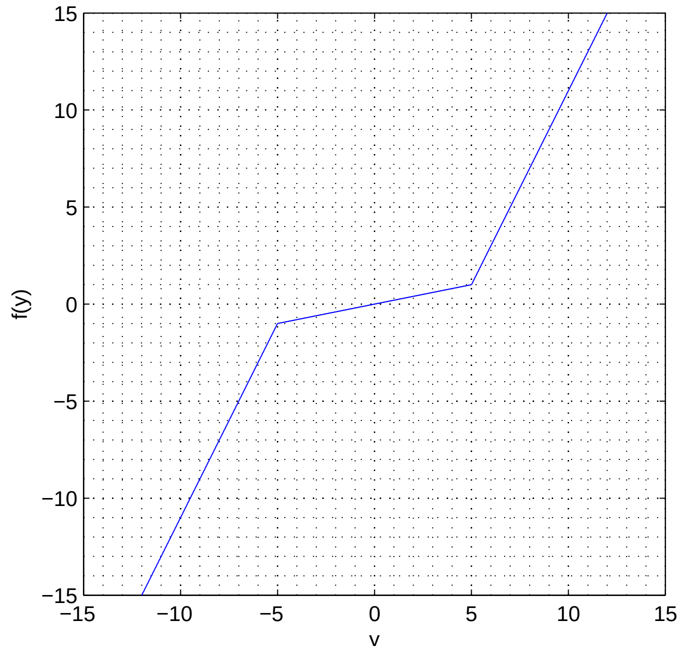

Figure 3 Nonlinearity 1 in Problem 3.

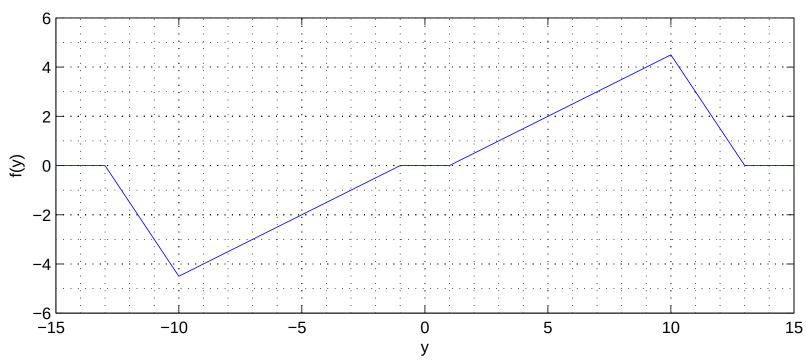

Figure 4 Nonlinearity 2 in Problem 3.

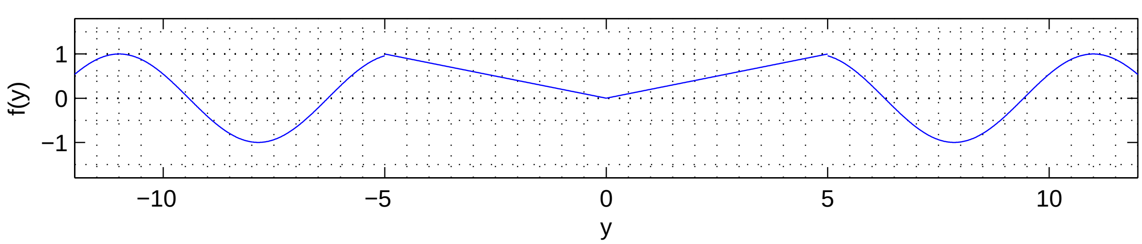

Figure 5 Nonlinearity 3 in Problem 3.

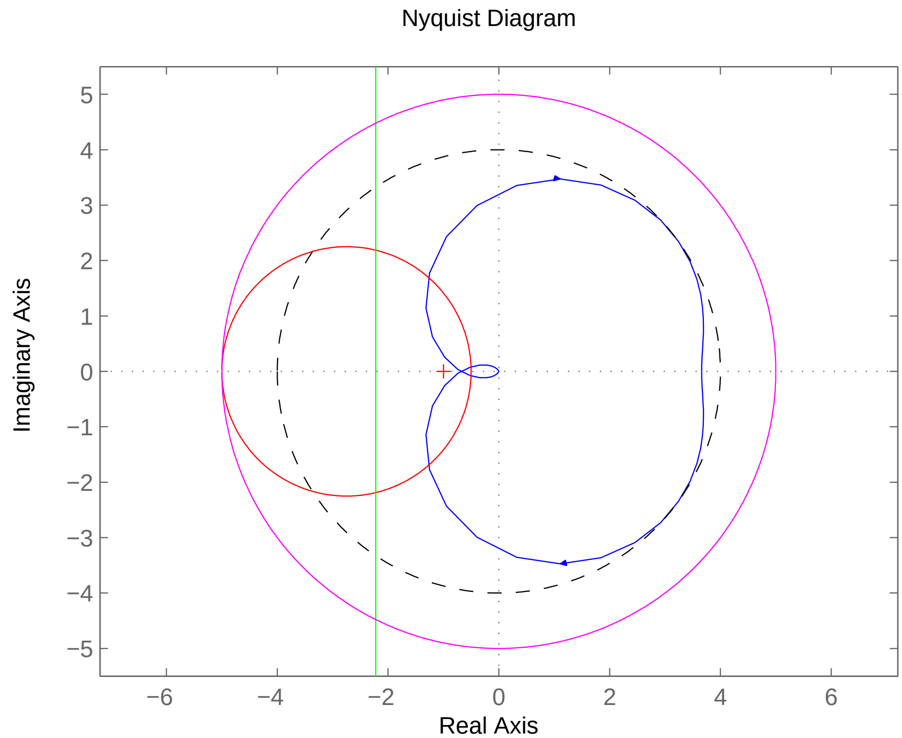

Figure 6 Circle criterion solution. Nonlinearity 1,2 and 3 in red, green and magenta, respectively.

$\gamma_G = ||G_p(s)||_\infty = 4$ from the Nyquist plot. The Small Gain Theorem states that a sufficient condition for closed-loop stability is

$$4\gamma_f = \gamma_G \gamma_f < 1\,, \tag{7}$$

where γ_f is the gain of the static non linearity $f(y)$, which is given by the smallest $K \geq 0$ such that $|f(y)| \leq K|y|$ for every y. From the plots, one finds that $\gamma_{f_1} = 2$, $\gamma_{f_2} = 1/2$, and $\gamma_{f_3} = 1/5$. Hence, the small gain theorem makes it it possible to prove stability only for nonlinearity f_3 (since $\gamma_G\gamma_{f_3} = 4/5 < 1$), while nothing can be said about the nonlinearities f_1 (since $\gamma_G\gamma_{f_1} = 4 \cdot 2 \geq 1$) and f_2 (since $\gamma_G\gamma_{f_2} = 4/2 \geq 1$).

b. Containing the nonlinearities, the sectors are given by $[0.2, 2]$, $[0, 0.45]$ and $[-0.2, 0.2]$ respectively. Drawing these sectors into the Nyquist diagram (see Fig. 6), it can be concluded that the feedback loops with nonlinearities 2 and 3 are asymptotically stable, whereas nothing can be said about the one with nonlinearity 1.

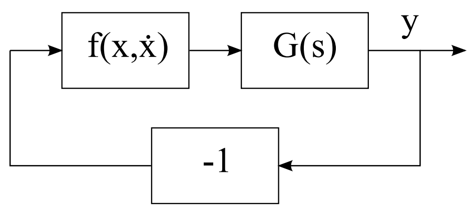

Figure 7 The Van Der Pol Oscillator

4. The Van Der Pol oscillator was introduced in computer exercise 5. The equation for the oscillator is

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0,$$

where $\alpha > 0$ is a constant. This can be written as a linear system with transfer function

$$G(s) = \frac{\alpha}{s^2 - \alpha s + 1}$$

in feedback with a nonlinear system given by

$$z = f(x, \dot{x}), \qquad f(x, \dot{x}) = x^2 \dot{x}$$

according to the block diagram of Figure 7.

a. Derive the describing function of the non-linear block with input $x(t)$ and output $z(t) = f(x(t), \dot{x}(t))$. (2 p)

Hint: You might find the following relationship useful:

$$\sin(\omega t)^2 \cos(\omega t) = \frac{1}{4}(\cos(\omega t) - \cos(3\omega t)).$$

b. Use the describing function method to predict the frequency and amplitude of possible limit cycles for the output signal $y(t)$.

(1 p)

Solution

a. Assume the input to the nonlinearity to be

$$x(t) = A\sin(\omega t). \tag{8}$$

The corresponding output is given by

$$z(t) = x^2\dot{x} = A^3\omega\sin^2(\omega t)\cos(\omega t) = \frac{A^3\omega}{4}(\cos(\omega t) - \cos(3\omega t)). \tag{9}$$

Assume $G(s)$ has low pass characteristics so the higher order harmonics is attenuated. The describing function is given by

$$N(A, \omega) := \frac{b(A, \omega) + ja(A, \omega)}{A} \tag{10}$$

with

$$a(A,\omega) := \frac{\omega}{\pi} \int_0^{2\pi/\omega} z(t)\cos(\omega t)dt = \cdots = \frac{A^3\omega}{4} \tag{11}$$

$$b(A,\omega) := \frac{\omega}{\pi} \int_0^{2\pi/\omega} z(t)\sin(\omega t)dt = \cdots = 0 \tag{12}$$

and hence

$$N(A,\omega) = \frac{A^2}{4} j\omega \tag{13}$$

b. According to the describing function method, the system will oscillate when

$$N(A,\omega)G(j\omega) = -1 \tag{14}$$

and after insertion

$$1 + \frac{A^2 j\omega}{4} \frac{\alpha}{(j\omega)^2 - \alpha j\omega + 1} = 0 \tag{15}$$

it is easy to see that this is only achievable for $A = 2$ and $\omega = 1$.

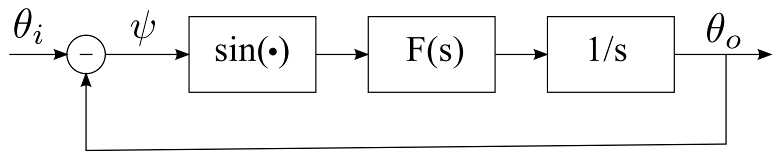

Figure 8 PLL block schematics.

5. A Phase Locked Loop (PLL) is one of the building block in a radio receiver. A phase domain block diagram is shown in Figure 8. The role of the PLL is to synchronize the phase of the output signal θ_o to that of the input signal θ_i so that the phase error ψ is constant.

a. A first order PLL has $F(s) = K$, for some constant $K > 0$. Introduce the phase error ψ as a state and derive a differential equation for ψ. (1 p)

b. When receiving a signal with the radio, a ramp signal is applied on the input:

$$\theta_i(t) = \gamma t\,, \qquad t \geq 0\,.$$

What is the largest value of $\gamma > 0$ such that there exist equilibria for ψ? Prove that, if such equilibria exist, $\psi(t)$ always converges to one of them. (2 p)

c. By changing the filter $F(s)$ to

$$F(s) = \frac{K}{s+a}, \quad K, a > 0$$

a second order PLL is formed. Introduce two states, $x_1 = \psi$ and $x_2 = \dot{\psi}$, derive the state equations and use the Lyapunov candidate function

$$V(x) = 1 - \cos(x_1) + \frac{1}{2}px_2^2, \quad p > 0$$

to determine the stability of the PLL when the input is $\theta_i \equiv 0$. (2 p)

Solution

a. The equation becomes

$$\dot{\psi} = \dot{\theta}_i - K\sin(\psi) \tag{16}$$

b. A ramp with slope γ gives

$$\dot{\theta}_i = \gamma \tag{17}$$

and the equilibria of Eq. (16) is found by putting $\dot{\psi} = 0$

$$\gamma = K\sin(\psi) \tag{18}$$

This equation has no solution if $\gamma > K$ which gives the requirement of $\gamma \leq K$. With $\gamma := \gamma_0$ Eq. (16) will have equilibriums at

$$\begin{aligned}
\psi_0 &= \arcsin(\frac{\gamma_0}{K}) + 2\pi k, \quad k = [0, 1, 2, \ldots] \\
\psi_1 &= \pi - \arcsin(\frac{\gamma_0}{K}) + 2\pi k, \quad k = [0, 1, 2, \ldots]
\end{aligned} \tag{19}$$

The equilibria in ψ_0 will be stable and ψ_1 will be unstable. This can be checked with linearization around the equilibria.

c. The dynamic equations are given by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -ax_2 - K\sin(x_1) \end{aligned} \tag{20}$$

Start with finding the equilibrium points with no input as

$$\begin{aligned} x_1 &= k\pi\,, \qquad k \in \mathbb{Z} \\ x_2 &= 0 \end{aligned} \tag{21}$$

and the Lyapunov function given is a candidate function.
The derivative of the Lyapunov function becomes

$$\dot{V} = \sin(x_1)x_2 - px_2(K\sin(x_1) + ax_2) \tag{22}$$

Choose

$$p = \frac{1}{K} \tag{23}$$

which gives

$$\dot{V} = -\frac{a}{K}x_2^2\,. \tag{24}$$

Let $E := \{\dot{V}(x) = 0\} = \{x_2 = 0\}$. Then, where the only solutions that can stay in are $x_1 = k\pi$, $x_2 = 0$, $k \in \mathbb{Z}$, so that the largest invariant subset of E is $M := \{(k\pi, 0) : k \in \mathbb{Z}\}$. Then, LaSalle's theorem implies that $x(t)$ converges to M for every initial condition $x(0)$.

6. A velocity-controlled particle moving under influence of viscous friction is described by the dynamics

$$\dot{v}(t) = -v(t) + u(t), \quad v(0) = 1,$$

where $v(t)$ is the velocity and $u(t)$ is the applied force. We are interested in designing an open-loop control $u(t)$, $t \in [0, 1]$, such that the particle reaches a complete stop at $t = 1$. Assume we want to do so while minimizing the the following quadratic cost

$$\int_0^1 u^2(t) \; \mathrm{d}t.$$

a. State the above problem as an optimal control problem and determine its Hamiltonian. (1 p)

b. Apply Pontryagin's maximum principle to determine the optimal control signal $u^*(t)$, $t \in [0, 1]$. (Assuming that the optimal control exists) (2 p)

c. Determine the optimal control $u^*(t)$, $t \in [0, 1]$ for the same problem and cost function as in point **a.** with the additional constraint

$$-\frac{3}{4} \leq u(t) \leq 0\,,$$

for all $t \in [0, 1]$. (1 p)

Solution

a. Introduce the Hamiltonian

$$\mathcal{H}(v(t), u(t), \lambda(t)) = u(t)^2 + \lambda(t)(-v(t) + u(t)),$$

where the adjoint state $\lambda(t)$ satisfies the differential equation

$$\dot{\lambda}(t) = -\mathcal{H}_v = \lambda(t).$$

Since we have no explicit bounds on the control signal $u(t)$, the optimal control signal $u^*(t)$ is directly given by the optimality condition

$$\mathcal{H}_u = 2u^*(t) + \lambda(t) = 0, \;\; \forall t \in [0, 1],$$

which means that

$$u^*(t) = -\frac{1}{2}\lambda(t).$$

Solving the differential equation for the adjoint state, we get

$$\lambda(t) = Ce^t,$$

where C is a constant. Inserting the calculated control signal $u^*(t)$ in the system dynamics results in the differential equation

$$\dot{v}(t) = -v(t) - \frac{C}{2}e^t, \; v(0) = 1.$$

Integrating this equation, we get that the state dynamics with this particular control signal is given by

$$v(t) = \left(1 + \frac{C}{4}\right) e^{-t} - \frac{C}{4} e^{t}.$$

Finally, using the boundary condition $v(1) = 0$, the constant C can be determined to

$$C = \frac{4}{e^2 - 1}.$$

Consequently, the optimal control signal $u^*(t)$ is given by

$$u^*(t) = -\frac{2}{e^2 - 1} e^t, \ t \in [0, 1].$$

b. First, we note that the optimal control signal calculated in exercise **a)** is in the interval

$$-0.85 \le u^*(t) \le -0.31.$$

Hence, it is clear that the bounds introduced on the control signal in this subproblem will influence the solution. The same Hamiltonian as in the previous subproblem is considered. To find the optimal control signal, we utilize the condition

$$\begin{aligned} u^*(t) &= \arg \min_{-3/4 \le w(t) \le 0} H(v^*(t), w(t), \lambda^*(t)) \\ &= \arg \min_{-3/4 \le w(t) \le 0} \left\{ w^2(t) + \lambda^*(t)(-v^*(t) + w(t)) \right\}. \end{aligned}$$

Similarly as in the previous subproblem, the adjoint state is given by

$$\lambda(t) = Ce^t,$$

where C is a constant. Hence, the optimal control signal can be written as follows, conditioned on the adjoint state value

$$u^*(t) = \begin{cases} 0 & \lambda < 0, \\ -3/4 & \lambda > 3/2, \\ -\dfrac{C}{2} e^t & \text{otherwise.} \end{cases}$$

It is clear that $C > 0$, otherwise $u^*(t) \equiv 0$ and the boundary condition $v(1) = 0$ would not be satisfied. Since $\lambda(t)$ is strictly increasing, the structure of the optimal control signal is

$$u^*(t) = \begin{cases} -\dfrac{C}{2} e^t & 0 \le t \le t_1, \\ -3/4 & t_1 < t \le 1. \end{cases}$$

To find the constant C and the time t_1, we solve the state dynamics in each time interval $[0, t_1]$ and $(t_1, 1]$.

1. $t \in [0, t_1]$: In this time interval, the state dynamics is given by

$$\dot{v}(t) = -v(t) - \frac{C}{2}e^t.$$

   Solving this differential equation and utilizing the initial value $v(0) = 1$, we get

$$v_{(a)}(t) = \left(1 + \frac{C}{4}\right)e^{-t} - \frac{C}{4}e^t.$$

2. $t \in (t_1, 1]$: In this time interval, the state dynamics is given by

$$\dot{v}(t) = -v(t) - \frac{3}{4}.$$

   Solving this differnetial equation and utilizing the boundary condition $v(1) = 0$, we get

$$v_{(b)}(t) = \frac{3}{4}(e^{1-t} - 1).$$

To determine the constant C we utilize that the Hamiltonian $\mathcal{H}$ is constant along the optimal trajectory. In the time interval $[0, t_1]$ we have that

$$\mathcal{H}_{(a)} = -C\left(1 + \frac{C}{4}\right),$$

and in the time interval $(t_1, 1]$ we have that

$$\mathcal{H}_{(b)} = \frac{9}{16} - \frac{3C}{4}e.$$

From the condition that $\mathcal{H}_{(a)} = \mathcal{H}_{(b)}$, two possible values of C are obtained

$$C_1 = 3.51 \quad , \quad C_2 = 0.640.$$

Finally, for finding the time t_1 we utilize that the state $v(t)$ must be continuous at $t = t_1$, *i.e.*, $v_{(a)}(t_1) = v_{(b)}(t_1)$. For the respective value of C, this equation can be solved for t_1. Then, it turns out that for C_1 the time t_1 is outside the interval $[0, 1]$. Hence, the desired time t_1 is obtained for C_2. A numerical or graphical solution of the equation

$$v_{(a)}(t_1) = v_{(b)}(t_1),$$

provides the solution as $t_1 = 0.837$. Hence, the optimal control signal is given by

$$u^*(t) = \begin{cases} -\dfrac{C}{2}e^t & 0 \le t \le t_1, \\ -3/4 & t_1 < t \le 1, \end{cases}$$

with $C = 0.64$ and $t_1 = 0.837$.