Course Overview

L1-L6	Analysis Methods (Linearization, Lyapunov, Circle criterion, Describing functions)
L7,L9	Special nonlinearities
L10-L11	Optimal control
L8,L12,L13	Synthesis of nonlinear controllers
L14	Course summary

Lecture 12 — High-gain control design methods

- History: The Feedback Amplifier
- Linearization and inversion by high gain feedback
- Sliding mode control: example
- Sliding mode control: in general

Lecture 12 — High-gain control design methods

Today's Goal *You should be able to analyze and design*

- *high-gain control system*
- *sliding mode controller*

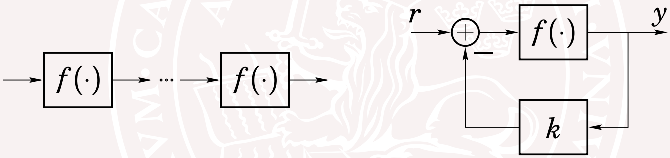
Material

- Lecture notes
  - Khalil: Sec 14.1.1 (pp.552–563)
  - Slotine and Li: Section 7.1, 8.5
  - Chapter 10 in *Adaptive Control* by Åström & Wittenmark

History of the Feedback Amplifier

New York–San Francisco communication link 1914.

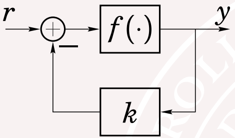
High signal amplification with low distortion was needed.

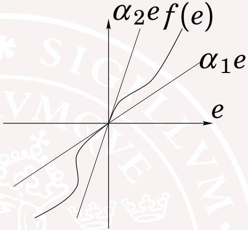


Feedback amplifiers was the solution!

Black, Bode, and Nyquist at Bell Labs 1920–1950.

Linearization Through High Gain





$\alpha_1 e \le f(e) \le \alpha_2 e$ gives $\frac{\alpha_1}{1+\alpha_1 k} r \le y \le \frac{\alpha_2}{1+\alpha_2 k} r$

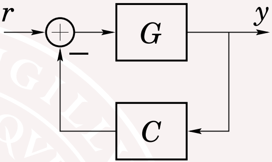
so if $\alpha_1 k, \alpha_2 k \gg 1$ then

$$y \approx \frac{1}{k} r$$

regardless of the nonlinearity. (Easier to design $k < 1$ with high accuracy.)

The Sensitivity Function $S = (1 + GC)^{-1}$

The closed-loop system is



$$G_{\text{cl}} = \frac{G}{1 + GC}$$

Small perturbations dG in G gives

$$dG_{\text{cl}} = \frac{dG}{1 + GC} - \frac{GCdG}{(1 + GC)^2} = \frac{dG}{(1 + GC)^2}$$

so that

$$\frac{dG_{\text{cl}}}{G_{\text{cl}}} = \underbrace{\frac{1}{1 + GC}}_{S} \frac{dG}{G}$$

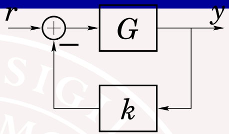
S is the closed-loop **sensitivity** to open-loop perturbations.

Distortion Reduction via Feedback Amplifiers

The feedback reduces distortion in each link.

Several links give distortion-free high gain.

Example—Distortion Reduction



Let $G = 1000$ with distortion $dG/G = 0.1$.

Choose $k = 0.1$ so that $S = (1 + Gk)^{-1} \approx 0.01$. Then

$$\frac{dG_{\text{cl}}}{G_{\text{cl}}} = S\frac{dG}{G} \approx 0.001$$

One hundred feedback amplifiers give total amplification

$$G_{\text{tot}} = (G_{\text{cl}})^{100} \approx 10^{100}$$

and total distortion

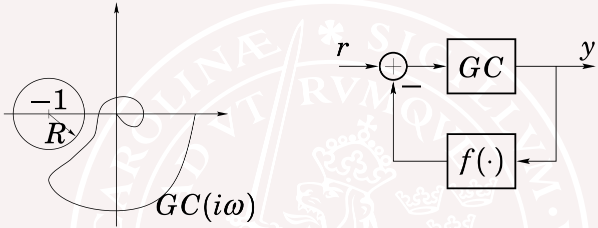
$$\frac{dG_{\text{tot}}}{G_{\text{tot}}} = (1 + 10^{-3})^{100} - 1 \approx 0.1$$

Transcontinental Communication Revolution

The feedback amplifier was patented by Black 1937.

Year	*Channels*	*Loss (dB)*	*No amp's*
1914	1	60	3–6
1923	1–4	150–400	6–20
1938	16	1000	40
1941	480	30000	600

Sensitivity and the Circle Criterion



$|S(i\omega)| = 1/R$ corresponds in the Nyquist diagram to a circle with center in -1 and radius R.

Circle criterion gives stability if

$$\frac{1}{1+R} \leq \frac{f(y)}{y} \leq \frac{1}{1-R}$$

$|S(i\omega)|$ small implies low sensitivity to nonlinearities.

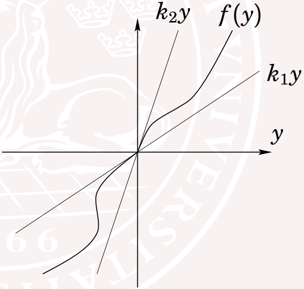
Small Sensitivity Allows Large Uncertainty

If $|S(i\omega)|$ is small, we can choose R large (close to one).

This corresponds to a large sector for $f(\cdot)$.

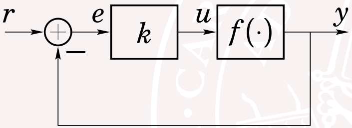
Hence, $|S(i\omega)|$ small implies low sensitivity to nonlinearities.

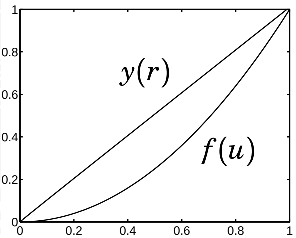
$$k_1 = \frac{1}{1+R}$$
$$k_2 = \frac{1}{1-R}$$



High Gain Linearization of Static Nonlinearity

Same idea often applicable





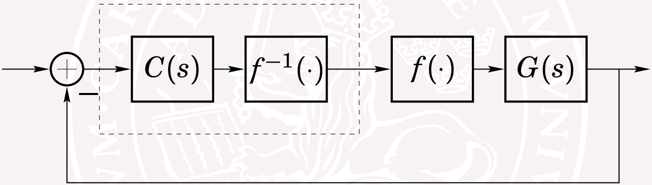
Linearization of $f(u) = u^2$ through feedback.

The case $k = 100$ is shown in the plot: $y(r) \approx r$.

Warning: High gain can give a noise sensitive system.

Inverting Nonlinearities

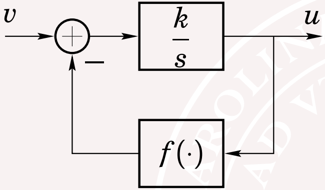
Compensation of static nonlinearity through inversion:

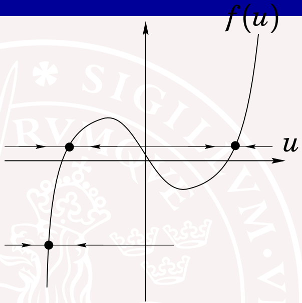


Should be combined with feedback as in the figure!

What if (a) $f(x) = x^3$? (b) $f(x) = x^2$?

How to Obtain f^{-1} from f





$$\dot{u} = k\big(v - f(u)\big)$$

If $k > 0$ large and $df/du > 0$, then $\dot{u} \to 0$ and

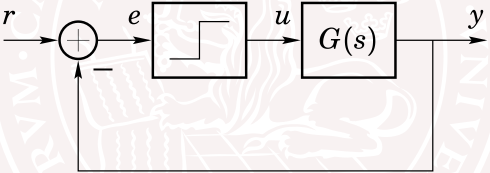
$$0 = k\big(v - f(u)\big) \qquad \text{that is} \qquad u = f^{-1}(v)$$

Note that the function f above does not have a well-defined inverse! What “inverse” can you get with the scheme above?

On–Off Control

On–off control is the simplest control strategy.

Common in temperature control, level control etc.



The relay feedback corresponds to extreme high-gain control.

A Control Design Idea, and a Problem

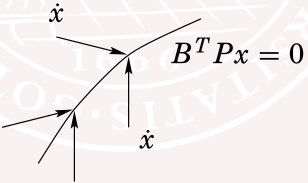
Assume $V(x) = x^T P x$, $P > 0$, represents the energy of

$$\dot{x} = Ax + Bu, \qquad u \in [-1, 1]$$

Idea: Choose u such that V decays as fast as possible

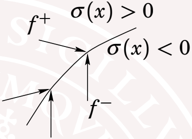
$$\dot{V} = x^T(A^T P + AP)x + 2B^T Px \cdot u$$
$$u = -\operatorname{sgn}(B^T Px)$$

The following situation might then occur ("system is not Lipschitz")



Sliding Modes

$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



The **switching set/ sliding set** is where $\sigma(x) = 0$ and f^+ and f^- point towards $\sigma(x) = 0$.

The **switching set/ sliding set** is given by x such that

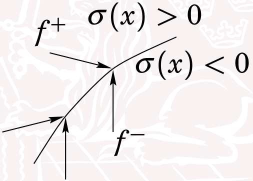
$$\sigma(x) = 0$$
$$\frac{\partial \sigma}{\partial x} f^+ = (\nabla \sigma) f^+ < 0$$
$$\frac{\partial \sigma}{\partial x} f^- = (\nabla \sigma) f^- > 0$$

Note: If f^+ and f^- point “in the same direction” on both sides of the set $\sigma(x) = 0$ then the solution curves will just pass through and this region will not belong to the sliding set.

Sliding Mode

If f^+ and f^- both points towards $\sigma(x) = 0$, what will happen then?

The sliding dynamics are $\dot{x} = \alpha f^+ + (1-\alpha) f^-$, where α is obtained from $\dfrac{d\sigma}{dt} = \dfrac{\partial\sigma}{\partial x} \cdot \dot{x} = 0$ on $\{\sigma(x) = 0\}$.



More precisely, find α such that the components of f^+ and f^- perpendicular to the switching surface cancel: $\alpha f^+_\perp + (1-\alpha) f^-_\perp = 0$
The resulting dynamics is then the sum of the corresponding components along the surface.

4 minute exercise

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u = Ax + Bu$$

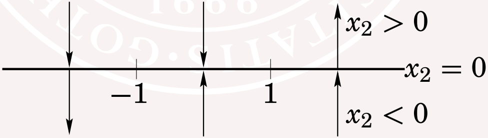
$$u = -\operatorname{sgn}\sigma(x) = -\operatorname{sgn} x_2 = -\operatorname{sgn}(Cx)$$

which means that

$$\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}$$

Determine the *switching set* and the *sliding dynamics*.

Approximate figure:



4 minute exercise — Solution

$$\dot{x}_1 = -x_2 + u = -x_2 - \text{sgn}(x_2)$$
$$\dot{x}_2 = x_1 - x_2 + u = x_1 - x_2 - \text{sgn}(x_2)$$

$$f^+ = \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} \qquad f^- = \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix}$$

$$\sigma(x) = x_2 = 0 \;\Rightarrow \underline{x_2 = 0}$$
$$\frac{\partial \sigma}{\partial x} f^+ = \begin{bmatrix} 0 & 1 \end{bmatrix} f^+ = x_1 - x_2 - 1 < 0 \;\Rightarrow \underline{x_1 < 1}$$
$$\frac{\partial \sigma}{\partial x} f^- = \begin{bmatrix} 0 & 1 \end{bmatrix} f^- = x_1 - x_2 + 1 > 0 \;\Rightarrow \underline{x_1 > -1}$$

We will thus have a sliding set for $\{-1 < x_1 < 1,\ x_2 = 0\}$

The normal projections of $(f^+,\ f^-)$ to $\sigma(x) = x_2 = 0$ are

$$f_\perp^+ = \begin{bmatrix} 0 \\ x_1 - x_2 - 1 \end{bmatrix} \qquad f_\perp^- = \begin{bmatrix} 0 \\ x_1 - x_2 + 1 \end{bmatrix}$$

Find $\alpha \in [0,1]$ such that $\alpha f_\perp^+ + (1-\alpha) f_\perp^- = 0$ on $\{x_2 = 0\}$

$$\alpha(x_1 - x_2 - 1) + (1-\alpha)(x_1 - x_2 + 1) = 0$$
$$\Rightarrow$$
$$\alpha = \frac{x_1 + 1}{2} \text{ as } x_2 = 0$$

Note: $\alpha \in [0,1] \Rightarrow x_1 \in [-1,1]$

The *sliding dynamics* are the given by

$$
\begin{aligned}
\dot{x} &= \alpha f^+ + (1-\alpha) f^- \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \alpha \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -2\alpha - x_2 - 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -x_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

where we inserted $x_2 = 0$ and $\alpha = \frac{x_1+1}{2}$

We see that on the sliding set $\{-1 < x < 1,\, x_2 = 0\}$ we have

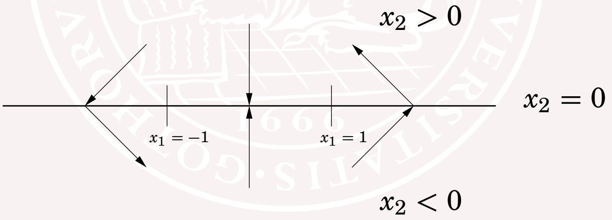
$$
\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= 0
\end{aligned}
$$

For any initial condition starting on the sliding set, there will be exponential convergence to $x_1 = x_2 = 0$.

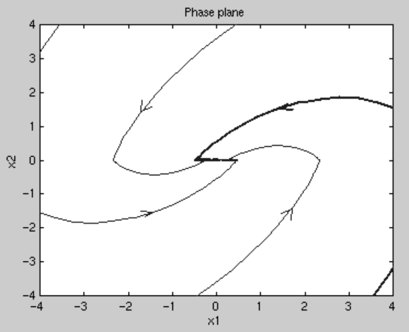
For small x_2 we have

$$\begin{cases} \dot{x}_2(t) \approx x_1 - 1, \quad \dfrac{dx_2}{dx_1} = \dfrac{dx_2/dt}{dx_1/dt} \approx 1 - x_1 & x_2 > 0 \\ \\ \dot{x}_2(t) \approx x_1 + 1, \quad \dfrac{dx_2}{dx_1} = \dfrac{dx_2/dt}{dx_1/dt} \approx 1 + x_1 & x_2 < 0 \end{cases}$$

This implies the following behavior



Sliding Mode Dynamics



The dynamics along the sliding set in $\sigma(x) = 0$ can also be obtained by finding $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.

Example (cont'd)

Finding $u = u_{\text{eq}}$ such that $\dot{\sigma}(x) = \dot{x}_2 = 0$ gives

$$0 = \dot{x}_2 = x_1 - \underbrace{x_2}_{=0} + u_{\text{eq}} = x_1 + u_{\text{eq}}$$

Insert $u_{\text{eq}} = -x_1$ in the equation for $\dot{x}_1$:

$$\dot{x}_1 = -\underbrace{x_2}_{=0} + u_{\text{eq}} = -x_1$$

gives the dynamics on the sliding set (where $x_2 = 0$)

Remember: $u_{eq} \in [-1, 1]$ so can only satisfy $u_{\text{eq}} = -x_1$ on the interval $x_1 \in [-1, 1]$!

Equivalent Control

Assume

$$\dot{x} = f(x) + g(x)u$$
$$u = -\operatorname{sgn}\sigma(x)$$

has a sliding set on $\sigma(x) = 0$. Then, for $x(t)$ staying on the sliding set we should have

$$0 = \dot{\sigma}(x) = \frac{\partial \sigma}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial \sigma}{\partial x}\Big(f(x) + g(x)u \Big)$$

The equivalent control is thus given by solving

$$u_{\text{eq}} = -\left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x} f(x)$$

for all those x such that $\sigma(x) = 0$ and $\frac{\partial \sigma}{\partial x} g(x) \neq 0$.

Equivalent Control for Linear System

$$\dot{x} = Ax + Bu$$
$$u = -\operatorname{sgn}\sigma(x) = -\operatorname{sgn}(Cx)$$

Assume $CB > 0$. The sliding set lies in $\sigma(x) = Cx = 0$.

$$0 = \dot{\sigma}(x) = \frac{d\sigma}{dx}\Big(f(x) + g(x)u\Big) = C\big(Ax + Bu_{\text{eq}}\big)$$

gives $CBu_{\text{eq}} = -CAx$.

Example (cont'd) For the previous system

$$u_{\text{eq}} = -CAx/CB = -(x_1 - x_2)/1 = -x_1,$$

because $\sigma(x) = x_2 = 0$. Same result as above.

More on the Sliding Dynamics

If $CB > 0$ then the dynamics along a sliding set in $Cx = 0$ is

$$\dot{x} = Ax + Bu_{\text{eq}} = \left(I - \frac{BC}{CB}\right)Ax,$$

One can show that the eigenvalues of $(I - BC/CB)A$ equals the zeros of $G(s) = C(sI - A)^{-1}B$. (exercise for PhD students)

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$.
Choose $\sigma(x)$ such that the sliding mode tends to the origin.

Assume system has form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = f(x) + g(x)u$$

Choose control law

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \operatorname{sgn} \sigma(x),$$

where $\mu > 0$ is a design parameter, $\sigma(x) = p^T x$, and
$p^T = \begin{pmatrix} p_1 & \dots & p_n \end{pmatrix}$ represents a stable polynomial.

Sliding Mode Control gives Closed-Loop Stability

Consider $\mathcal{V}(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$\dot{\mathcal{V}} = \sigma(x)\dot{\sigma}(x) = x^T p\big(p^T f(x) + p^T g(x)u\big)$$

With the chosen control law, we get

$$\dot{\mathcal{V}} = -\mu\sigma(x)\,\mathrm{sgn}\,\sigma(x) \le 0$$

so $\sigma(x) \to 0$ in finite time.

$$\begin{aligned} 0 = \sigma(x) &= p_1 x_1 + \cdots + p_{n-1}x_{n-1} + p_n x_n \\ &= p_1 x_n^{(n-1)} + \cdots + p_{n-1}x_n^{(1)} + p_n x_n^{(0)} \end{aligned}$$

where $x^{(k)}$ denote time derivative. P stable gives that $x(t) \to 0$.

Note: $\mathcal{V}$ is itself not a true Lyapunov function. It only guarantees convergence to the line $\{\sigma(x) = 0\} = \{p^T x = 0\}$.

Time to Switch

Consider an initial point x such that $\sigma_0 = \sigma(x) > 0$. Then

$$\sigma(x)\dot{\sigma}(x) = -\mu\sigma(x)\,\mathrm{sgn}\,\sigma(x)$$

so

$$\dot{\sigma}(x) = -\mu$$

Hence, the time to the first switch is

$$t_{\mathrm{s}} = \frac{\sigma_0}{\mu} < \infty$$

Note that $t_{\mathrm{s}} \to 0$ as $\mu \to \infty$.

Example—Sliding Mode Controller

Design state-feedback controller for

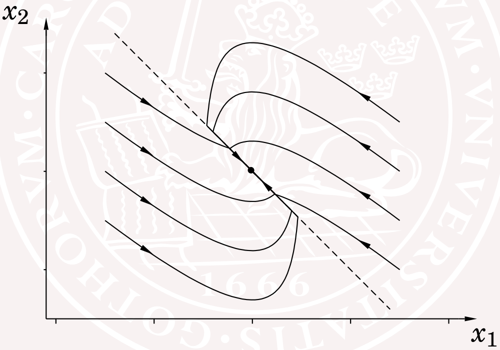
$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$\begin{aligned} u &= -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \operatorname{sgn} \sigma(x) \\ &= -2x_1 - \mu \operatorname{sgn}(x_1 + x_2) \end{aligned}$$

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding set is in $\sigma(x) = x_1 + x_2$.



Time Plots

Initial condition

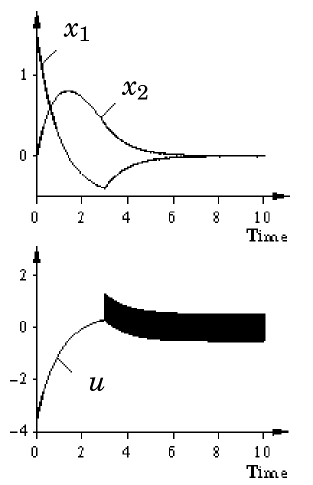
$$x(0) = \begin{pmatrix} 1.5 & 0 \end{pmatrix}^T.$$

Simulation agrees well with time to switch

$$t_{\mathsf{s}} = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \widehat{f}(x) + \widehat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{V} = \sigma(x)\left[\frac{p^T(f\widehat{g}^T - \widehat{f}g^T)p}{p^T\widehat{g}} - \mu\frac{p^Tg}{p^T\widehat{g}}\operatorname{sgn}\sigma(x)\right] < 0$$

if $\operatorname{sgn}(p^Tg) = \operatorname{sgn}(p^T\widehat{g})$ and $\mu > 0$ is sufficiently large.

Closed-loop system is quite robust against model errors!

(High gain control with stable open loop zeros)

Implementation

A relay with hysteresis or a smooth (e.g. linear) region is often used in practice.

Choice of hysteresis or smoothing parameter can be critical for performance

More complicated structures with several relays possible. Harder to design and analyze.

Next Lectures

- L13: Other synthesis methods
- L14: Course summary