Lecture 9 — Nonlinear Control Design

- Exact-linearization
- Lyapunov-based design
  - Lab 2
  - Adaptive control
  - Backstepping
- Hybrid / Piece-wise linear control
  - NOTE: Only overview!

Course Outline

Lecture 1-3	Modelling and basic phenomena (linearization, phase plane, limit cycles)
Lecture 2-6	Analysis methods (Lyapunov, circle criterion, describing functions)
Lecture 7-8	Common nonlinearities (Saturation, friction, backlash, quantization)
Lecture 9-13	Design methods (Lyapunov methods, Backstepping, Optimal control)
Lecture 14	Summary

Exact Feedback Linearization

Idea:

Find state feedback $u = u(x, v)$ so that the nonlinear system

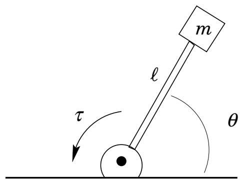
$$\dot{x} = f(x) + g(x)u$$

turns into the linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

Exact linearization: example [one-link robot]



$$m\ell^2\ddot{\theta} + d\dot{\theta} + m\ell g\cos\theta = u$$

where d is the viscous damping.

The control $u = \tau$ is the applied torque

Design state feedback controller $u = u(x)$ with $x = (\theta, \dot{\theta})^T$

Introduce new control variable v and let

$$u = m\ell^2 v + d\dot{\theta} + m\ell g\cos\theta$$

Then

$$\ddot{\theta} = v$$

Choose e.g. a PD-controller

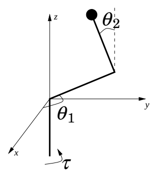
$$v = v(\theta, \dot{\theta}) = k_p(\theta_{\text{ref}} - \theta) - k_d\dot{\theta}$$

This gives the closed-loop system:

$$\ddot{\theta} + k_d\dot{\theta} + k_p\theta = k_p\theta_{\text{ref}}$$

Hence, $u = m\ell^2[k_p(\theta - \theta_{\text{ref}}) - k_d\dot{\theta}] + d\dot{\theta} + m\ell g\cos\theta$

Multi-link robot (n-joints)



General form

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

M	$n \times n$ inertia matrix, $M = M^T > 0$
$C\dot{\theta}$	$n \times 1$ vector of centrifugal and Coriolis forces
G	$n \times 1$ vector of gravitation terms

Computed torque

The computed torque
(also known as "Exact linearization", "dynamic inversion" , etc.)

$$\begin{aligned} u &= M(\theta)v + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) \\ v &= K_p(\theta_{ref} - \theta) - K_d\dot{\theta}, \end{aligned} \qquad (1)$$

gives closed-loop system

$$\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_{Ref}$$

The matrices K_d and K_p can be chosen diagonal (no cross-terms) and then this decouples into n independent second-order equations.

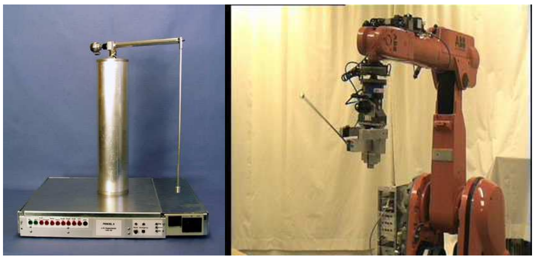
Lyapunov-Based Control Design Methods

$$\dot{x} = f(x, u)$$

- Select Lyapunov function $V(x)$for stability verification
- Find state feedback $u = u(x)$ that makes V decreasing
- Method depends on structure of f

Examples are energy shaping as in Lab 2 and e.g. **Back-stepping control design**, which require certain f discussed later.

Lab 2 : Energy shaping for swing-up control



[movie]

Use Lyapunov-based design for swing-up control.

Lab 2 : Energy shaping for swing-up control

Rough outline of method to get the pendulum to the upright position

- Find expression for total energy E of the pendulum (potential energy + kinetic energy)
- Let E_n be energy in upright position.
- Look at deviation $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find "swing strategy" of control torque u such that $\frac{dV}{dt} \leq 0$

Example of Lyapunov-based design

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2, \end{aligned} \tag{2}$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

We try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \ \forall (x_1,\, x_2) \neq (0,\, 0)$.

Example - cont'd

$$\begin{aligned} \dot{V} &= \dot{x}_1x_1 + \dot{x}_2x_2 = (-3x_1 + 2x_1x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4 \end{aligned}$$

We would like to have

$$\dot{V} < 0 \qquad \forall (x_1,\, x_2) \neq (0,\, 0)$$

Inserting the control law, $u = -2x_1x_2^2$, we get

$$\dot{V} = -3x_1^2 - x_2^2 \underbrace{-2x_1^2x_2^2 + 2x_1^2x_2^2}_{=0} - x_2^4 = -3x_1^2 - x_2^2 - x_2^4 < 0, \quad \forall x \neq 0$$

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u \end{aligned} \tag{3}$$

Find a globally asymptotically stabilizing control law $u = u(x)$.

Attempt 1: Try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \ \forall (x_1,\, x_2) \neq (0,\, 0)$.

$$\dot{V} = \dot{x}_1x_1 + \dot{x}_2x_2 = x_2^3 \cdot x_1 + u \cdot x_2 = x_2\underbrace{(x_2^2x_1 + u)}_{-x_2} = -x_2^2 \leq 0$$

where we chose

$$u = -x_2 - x_2^2x_1$$

However $\dot{V} = 0$ as soon as $x_2 = 0$ (Note: x_1 could be anything).

According to LaSalle's theorem the set $E = \{x | \dot{V} = 0\} = \{(x_1, 0)\} \ \forall x_1$

What is the largest invariant set M?

Plugging in the control law $u = -x_2 - x_2^2x_1$, we get

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_2 - x_2^2x_1 \end{aligned} \tag{4}$$

and we see that if we start anywhere on the line $\{(x_1, 0)\}$ we will stay in the same point as both $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, thus M=E and we will not converge to the origin, but get stuck on the line $x_2 = 0$.

Draw phase-plot with e.g., `pplane and study the behaviour.`

Attempt 2:

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u \end{aligned} \tag{5}$$

Try the Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4,$$

which satisfies

- $V(0,0) = 0$
- $V(x_1, x_2) > 0, \quad \forall (x_1,\, x_2) \neq (0,\, 0).$
- radially unbounded,

$$\frac{dV}{dt} = \dot{x}_1x_1 + \dot{x}_2x_2^3 = x_2^3(x_1 + u) = -x_2^4 \leq 0$$

$u = -x_1 - x_2$if we use $u = -x_1 - x_2$

With

$$u = -x_1 - x_2$$

we get the dynamics

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned} \tag{6}$$

$\dot{V} = 0$ if $x_2 = 0$, thus

$$E = \{x | \dot{V} = 0\} = \{(x_1, 0)\} \ \forall x_1$$

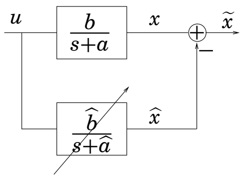
However, now the only possibility to stay on $x_2 = 0$ is if $x_1 = 0$, (else $\dot{x}_2 \neq 0$ and we will leave the line $x_2 = 0$).
Thus, the largest invariant set

$$M = (0, 0)$$

According to the Invariant Set Theorem (LaSalle) all solutions will end up in M and so the origin is GAS.

Draw phase-plot with e.g., `pplane and study the behaviour.`

Adaptive Noise Cancellation Revisited



$$\dot{x} + ax = bu$$
$$\dot{\widehat{x}} + \widehat{a}\widehat{x} = \widehat{b}u$$

Introduce $\widetilde{x} = x - \widehat{x}$, $\widetilde{a} = a - \widehat{a}$, $\widetilde{b} = b - \widehat{b}$.

Want to design adaptation law so that $\widetilde{x} \to 0$

Let us try the Lyapunov function

$$V = \frac{1}{2}(\widetilde{x}^2 + \gamma_a \widetilde{a}^2 + \gamma_b \widetilde{b}^2)$$
$$\dot{V} = \widetilde{x}\dot{\widetilde{x}} + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} =$$
$$= \widetilde{x}(-a\widetilde{x} - \widetilde{a}\widehat{x} + \widetilde{b}u) + \gamma_a \widetilde{a}\dot{\widetilde{a}} + \gamma_b \widetilde{b}\dot{\widetilde{b}} = -a\widetilde{x}^2$$

where the last equality follows if we choose

$$\dot{\widetilde{a}} = -\dot{\widehat{a}} = \frac{1}{\gamma_a}\widetilde{x}\widehat{x} \qquad \dot{\widetilde{b}} = -\dot{\widehat{b}} = -\frac{1}{\gamma_b}\widetilde{x}u$$

Invariant set: $\widetilde{x} = 0$.

This proves that $\widetilde{x} \to 0$.

(The parameters $\widetilde{a}$ and $\widetilde{b}$ do not necessarily converge: $u \equiv 0$.)

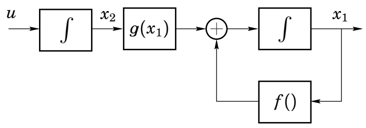
Back-Stepping Control Design

We want to design a state feedback $u = u(x)$ that stabilizes

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \qquad (7)$$

at $x = 0$ with $f(0) = 0$.

Idea: See the system as a cascade connection. Design controller first for the inner loop and then for the outer.



Suppose the partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\bar{v}$$

can be stabilized by $\bar{v} = \phi(x_1)$ and there exists Lyapunov fcn $V_1 = V_1(x_1)$ such that

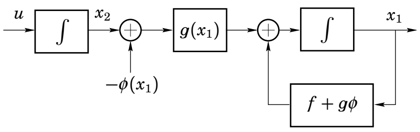
$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) \le -W(x_1)$$

for some positive definite function W.

The Trick

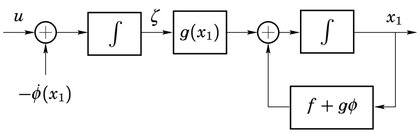
Equation (7) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)\phi(x_1) + g(x_1)[x_2 - \phi(x_1)] \\ \dot{x}_2 &= u \end{aligned}$$



Introduce new state $\zeta = x_2 - \phi(x_1)$ and control $v = u - \dot{\phi}$:

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)\phi(x_1) + g(x_1)\zeta \\ \dot{\zeta} &= v \end{aligned}$$



Consider $V_2(x_1, x_2) = V_1(x_1) + \zeta^2/2$. Then,

$$\begin{aligned} \dot{V}_2(x_1, x_2) &= \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v \\ &\le -W(x_1) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v \end{aligned}$$

Choosing

$$v = -\frac{dV_1}{dx_1}g(x_1) - k\zeta, \qquad k > 0$$

gives

$$\dot{V}_2(x_1, x_2) \le -W(x_1) - k\zeta^2$$

Hence, $x = 0$ is asymptotically stable for (7) with control law $u(x) = \dot{\phi}(x) + v(x)$.

If V_1 radially unbounded, then global stability.

Back-Stepping Lemma

Lemma: Let $z = (x_1, \ldots, x_{k-1})^T$ and

$$\begin{aligned} \dot{z} &= f(z) + g(z)x_k \\ \dot{x}_k &= u \end{aligned}$$

Assume $\phi(0) = 0$, $f(0) = 0$,

$$\dot{z} = f(z) + g(z)\phi(z)$$

stable, and $V(z)$ a Lyapunov fcn (with $\dot{V} \le -W$). Then,

$$u = \frac{d\phi}{dz}\Big(f(z) + g(z)x_k\Big) - \frac{dV}{dz}g(z) - (x_k - \phi(z))$$

stabilizes $x = 0$ with $V(z) + (x_k - \phi(z))^2/2$ being a Lyapunov fcn.

Strict Feedback Systems

Back-stepping Lemma can be applied to stabilize systems on strict feedback form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\
\dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)u
\end{aligned}$$

where $g_k \neq 0$

Note: $x_1,\dots,x_k$ do not depend on $x_{k+2},\dots,x_n$.

Example

Design back-stepping controller for

$$\dot{x}_1 = x_1^2 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

Step 0 Verify strict feedback form
Step 1 Consider first subsystem

$$\dot{x}_1 = x_1^2 + \phi_1(x_1), \quad \dot{x}_2 = u_1$$

where $\phi_1(x_1) = -x_1^2 - x_1$ stabilizes the first equation. With $V_1(x_1) = x_1^2/2$, Back-Stepping Lemma gives

$$\begin{aligned}
u_1 &= (-2x_1 - 1)(x_1^2 + x_2) - x_1 - (x_2 + x_1^2 + x_1) = \phi_2(x_1,x_2) \\
V_2 &= x_1^2/2 + (x_2 + x_1^2 + x_1)^2/2
\end{aligned}$$

Hybrid Control

Control problems where there is a mixture between continuous states and discrete state variables.

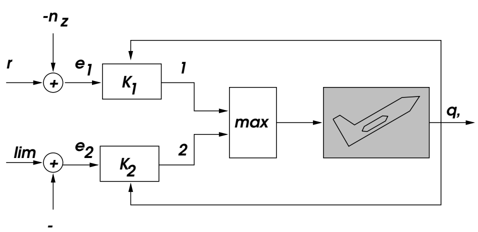
Continuous states: position, velocity, temperature, pressure

Discrete states: on/off variables, controller modes, loss of actuators, loss of sensors, relays, etc

Discontinuous differential equations

Much active field, much left to understand

Aircraft Example



(Branicky, 1993)

Back-Stepping

Back-Stepping Lemma can be applied **recursively** to a system

$$\dot{x} = f(x) + g(x)u$$

on strict feedback form.

Back-stepping generates stabilizing feedbacks $\phi_k(x_1,\dots,x_k)$ (equal to u in Back-Stepping Lemma) and Lyapunov functions

$$V_k(x_1,\dots,x_k) = V_{k-1}(x_1,\dots,x_{k-1}) + [x_k - \phi_{k-1}]^2/2$$

by "stepping back" from x_1 to u

Back-stepping results in the final state feedback

$$u = \phi_n(x_1,\dots,x_n)$$

Step 2 Applying Back-Stepping Lemma on

$$\begin{aligned}
\dot{x}_1 &= x_1^2 + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{aligned}$$

gives

$$\begin{aligned}
u = u_2 &= \frac{d\phi_2}{dz}\Big(f(z) + g(z)x_n\Big) - \frac{dV_2}{dz}g(z) - (x_n - \phi_2(z)) \\
&= \frac{\partial\phi_2}{\partial x_1}(x_1^2 + x_2) + \frac{\partial\phi_2}{\partial x_2}x_3 - \frac{\partial V_2}{\partial x_2} - (x_3 - \phi_2(x_1,x_2))
\end{aligned}$$

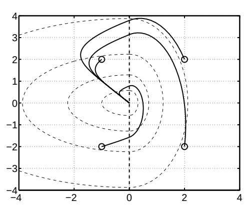
which globally stabilizes the system.

Example of hybrid control

Control law that switches between different modes, e.g. between

- Time optimal control – during large set point changes
- Linear control – close to set point

Phase Plane



No common *quadratic* Lyapunov function exists.

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

Piecewise quadratic Lyapunov functions

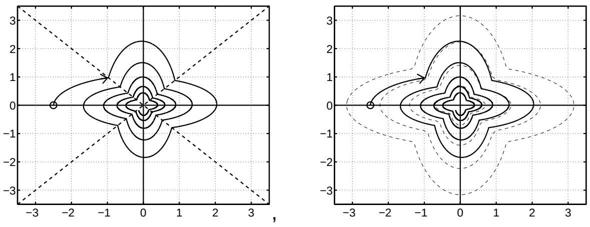
$$V(x) = \begin{cases} x^*Px & \text{if } x_1 < 0 \\ x^*Px + \eta x_1^2 & \text{if } x_1 \geq 0 \end{cases}$$

The matrix inequalities

$$\begin{aligned} A_1^*P + PA_1 &< 0 \\ P &> 0 \\ A_2^*(P + \eta E^*E) + (P + \eta E^*E)A_2 &< 0 \\ P + \eta E^*E &> 0 \end{aligned}$$

with $E = [1\ 0]$, have the solution $P = \text{diag}\{1, 3\}$, $\eta = 7$.

Flower Example



Next Lecture

- Optimization.

Read chapter 18 in [Glad & Ljung] for preparation.