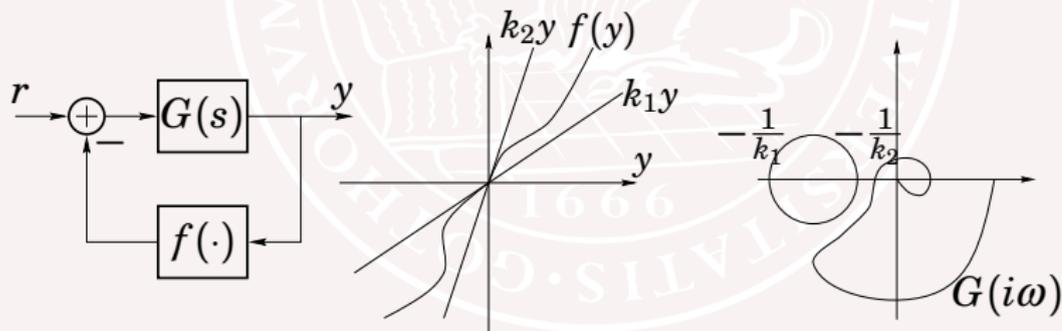


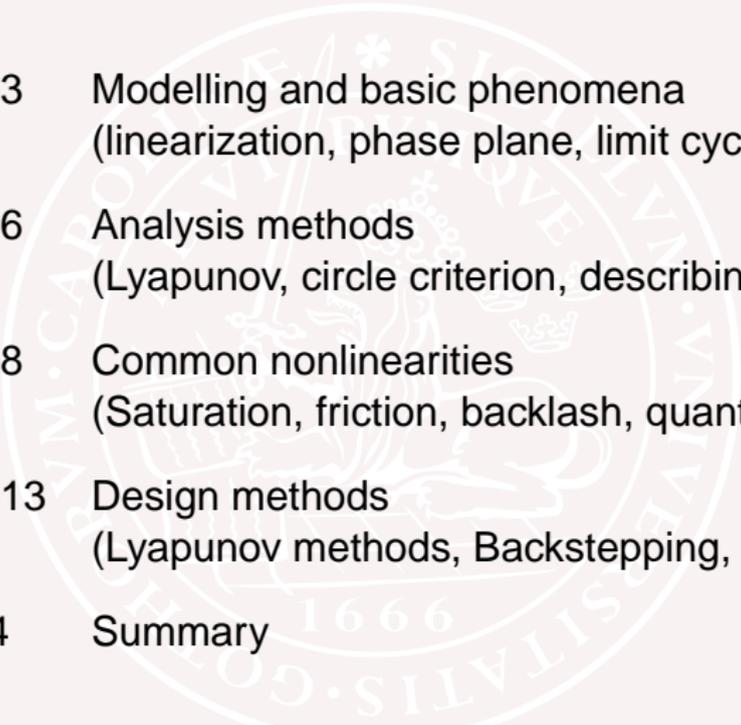
# Lecture 5 — Input–output stability

or

“How to make a circle out of the point  $-1 + 0i$ , and different ways to stay away from it ...”



# Course Outline

- 
- Lecture 1-3 Modelling and basic phenomena  
(linearization, phase plane, limit cycles)
- Lecture 2-6 Analysis methods  
(Lyapunov, circle criterion, describing functions)
- Lecture 7-8 Common nonlinearities  
(Saturation, friction, backlash, quantization)
- Lecture 9-13 Design methods  
(Lyapunov methods, Backstepping, Optimal control)
- Lecture 14 Summary

# Today's Goal

*To understand*

- *signal norms*
- *system gain*
- *bounded input bounded output (BIBO) stability*

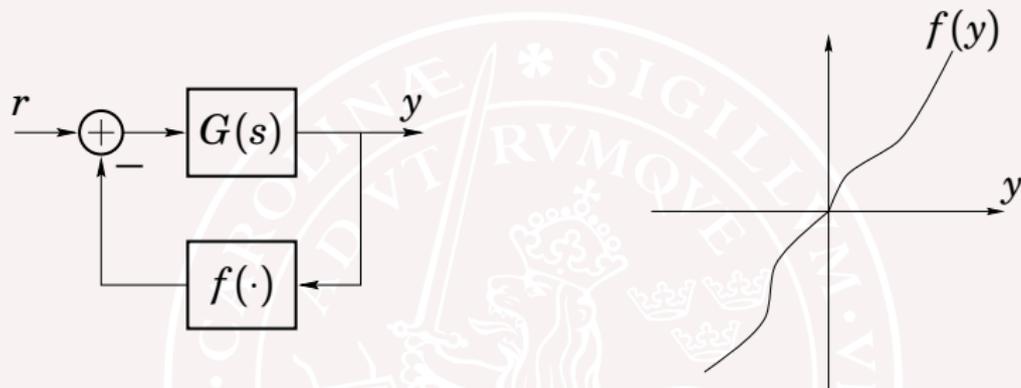
*To be able to analyze stability using*

- *the Small Gain Theorem,*
- *the Circle Criterion,*
- *Passivity*

Material

- *[Glad & Ljung]: Ch 1.5-1.6, 12.3*  
*[Khalil]: Ch 5–7.1; [Slotine & Li]: Ch.4.7–4.8*
- *lecture slides*

# History



For what  $G(s)$  and  $f(\cdot)$  is the closed-loop system stable?

- Lur'e and Postnikov's problem (1944)
- Aizerman's conjecture (1949) (False!)
- Kalman's conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)

# Gain

**Idea:** Generalize static gain to nonlinear dynamical systems



The gain  $\gamma$  of  $S$  should tell what is the largest amplification from  $u$  to  $y$

Here  $S$  can be a constant, a matrix, a linear time-invariant system, etc

**Question:** How should we measure the size of  $u$  and  $y$ ?

# Norms

A norm  $\|\cdot\|$  measures size.

A **norm** is a function from a space  $\Omega$  to  $\mathbf{R}^+$ , such that for all  $x, y \in \Omega$

- $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| \cdot \|x\|$ , for all  $\alpha \in \mathbf{R}$

## Examples

Euclidean norm:  $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$

Max norm:  $\|x\| = \max\{|x_1|, \dots, |x_n|\}$

# Signal Norms

A signal  $x(t)$  is a function from  $\mathbf{R}^+$  to  $\mathbf{R}$ .

A signal norm is a way to measure the size of  $x(t)$ .

## Examples

2-norm (energy norm):  $\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}$

sup-norm:  $\|x\|_\infty = \sup_{t \in \mathbf{R}^+} |x(t)|$

The space of signals with  $\|x\|_2 < \infty$  is denoted  $\mathcal{L}_2$ .

# Parseval's Theorem

**Theorem** If  $x, y \in \mathcal{L}_2$  have the Fourier transforms

$$X(i\omega) = \int_0^{\infty} e^{-i\omega t} x(t) dt, \quad Y(i\omega) = \int_0^{\infty} e^{-i\omega t} y(t) dt,$$

then

$$\int_0^{\infty} y^T(t)x(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(i\omega)X(i\omega) d\omega.$$

In particular,

$$\|x\|_2^2 = \int_0^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(i\omega)|^2 d\omega.$$

$\|x\|_2 < \infty$  corresponds to bounded energy.

# System Gain

A system  $S$  is a map between two signal spaces:  $y = S(u)$ .



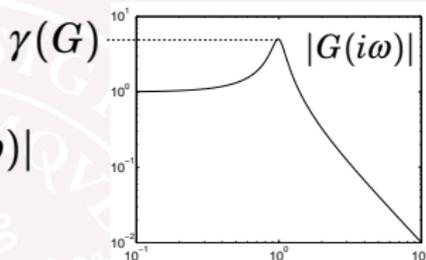
The gain of  $S$  is defined as 
$$\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$$

**Example** The gain of a static relation  $y(t) = \alpha u(t)$  is

$$\gamma(\alpha) = \sup_{u \in \mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{|\alpha| \|u\|_2}{\|u\|_2} = |\alpha|$$

# Example—Gain of a Stable Linear System

$$\gamma(G) = \sup_{u \in \mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in (0, \infty)} |G(i\omega)|$$



*Proof:* Assume  $|G(i\omega)| \leq K$  for  $\omega \in (0, \infty)$  and  $|G(i\omega_*)| = K$  for some  $\omega_*$ . Parseval's theorem gives

$$\begin{aligned} \|y\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \leq K^2 \|u\|_2^2 \end{aligned}$$

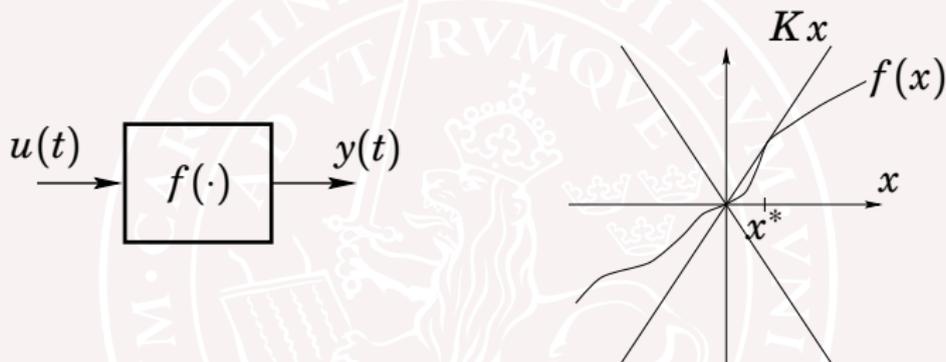
Equality by choosing  $u(t) = \sin \omega_* t$ .

**2 minute exercise:** Show that  $\gamma(S_1 S_2) \leq \gamma(S_1) \gamma(S_2)$ .



# Example—Gain of a Static Nonlinearity

$$|f(x)| \leq K|x|, \quad f(x^*) = Kx^*$$



$$\|y\|_2^2 = \int_0^\infty f^2(u(t)) dt \leq \int_0^\infty K^2 u^2(t) dt = K^2 \|u\|_2^2$$

$u(t) = x^*, t \in (0, \infty)$  gives equality  $\Rightarrow$

$$\gamma(f) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = K.$$

# BIBO Stability



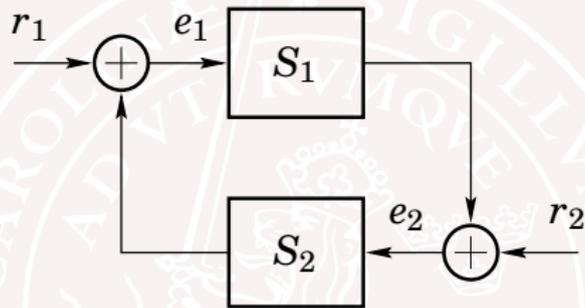
$$\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}$$

## Definition

$S$  is bounded-input bounded-output (BIBO) stable if  $\gamma(S) < \infty$ .

**Example:** If  $\dot{x} = Ax$  is asymptotically stable then  $G(s) = C(sI - A)^{-1}B + D$  is BIBO stable.

# The Small Gain Theorem



## Theorem

Assume  $S_1$  and  $S_2$  are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from  $(r_1, r_2)$  to  $(e_1, e_2)$  is BIBO stable.

# “Proof” of the Small Gain Theorem

Existence of solution  $(e_1, e_2)$  for every  $(r_1, r_2)$  has to be verified separately. Then

$$\|e_1\|_2 \leq \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]$$

gives

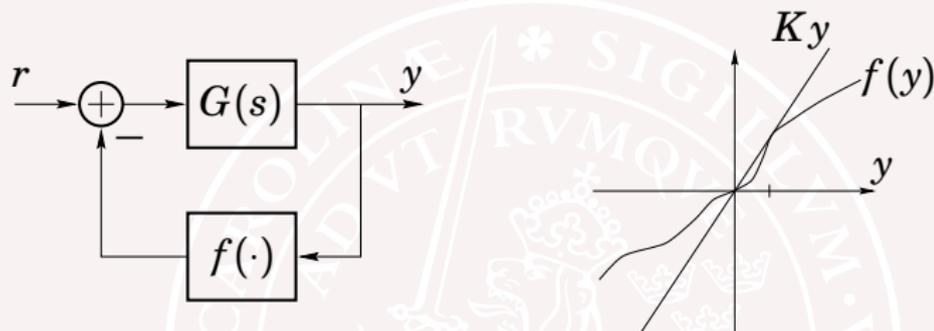
$$\|e_1\|_2 \leq \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1 - \gamma(S_2)\gamma(S_1)}$$

$\gamma(S_2)\gamma(S_1) < 1$ ,  $\|r_1\|_2 < \infty$ ,  $\|r_2\|_2 < \infty$  give  $\|e_1\|_2 < \infty$ .  
Similarly we get

$$\|e_2\|_2 \leq \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1 - \gamma(S_1)\gamma(S_2)}$$

so also  $e_2$  is bounded.

# Linear System with Static Nonlinear Feedback (1)

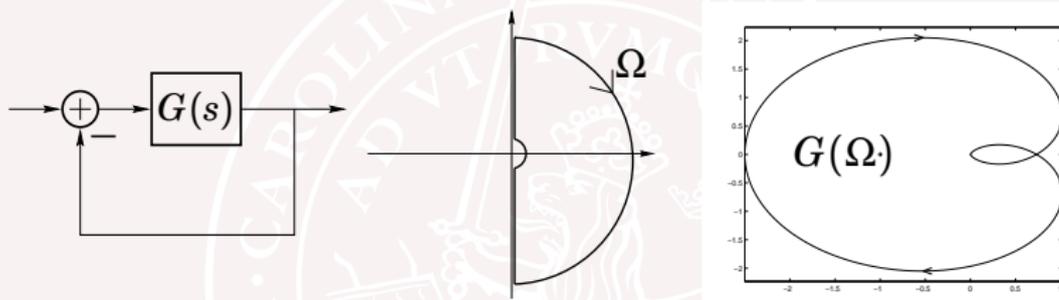


$$G(s) = \frac{2}{(s+1)^2} \quad \text{and} \quad 0 \leq \frac{f(y)}{y} \leq K$$

$$\gamma(G) = 2 \quad \text{and} \quad \gamma(f) \leq K.$$

The small gain theorem gives that  $K \in [0, 1/2)$  implies BIBO stability.

# The Nyquist Theorem

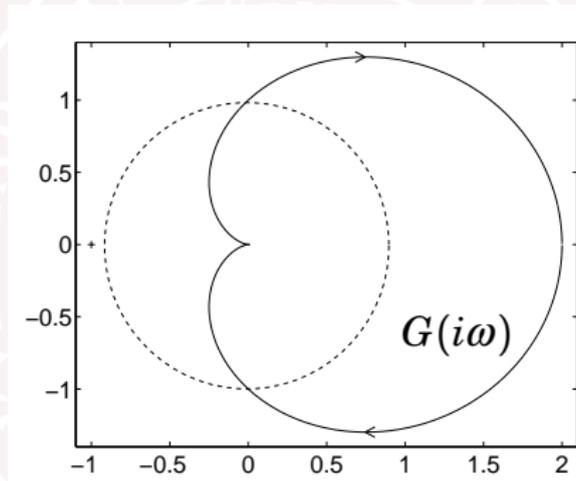


## Theorem

The closed loop system is stable iff the number of counter-clockwise encirclements of  $-1$  by  $G(\Omega)$  (note:  $\omega$  increasing) equals the number of open loop unstable poles.

# The Small Gain Theorem can be Conservative

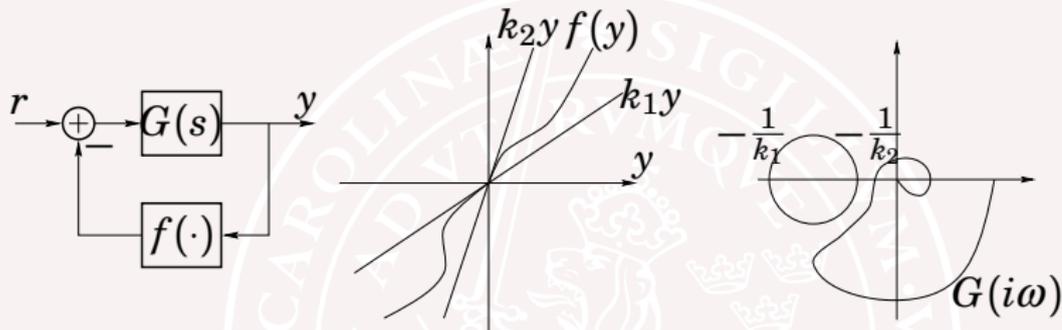
Let  $f(y) = Ky$  for the previous system.



The Nyquist Theorem proves stability when  $K \in [0, \infty)$ .  
The Small Gain Theorem proves stability when  $K \in [0, 1/2)$ .

# The Circle Criterion

**Case 1:**  $0 < k_1 \leq k_2 < \infty$



**Theorem** Consider a feedback loop with  $y = Gu$  and  $u = -f(y) + r$ . Assume  $G(s)$  is stable and that

$$0 < k_1 \leq \frac{f(y)}{y} \leq k_2.$$

If the Nyquist curve of  $G(s)$  does not intersect or encircle the circle defined by the points  $-1/k_1$  and  $-1/k_2$ , then the closed-loop system is BIBO stable from  $r$  to  $y$ .

# Other cases

## $G$ : stable system

- $0 < k_1 < k_2$ : Stay outside circle
- $0 = k_1 < k_2$ : Stay to the right of the line  $\text{Re } s = -1/k_2$
- $k_1 < 0 < k_2$ : Stay inside the circle

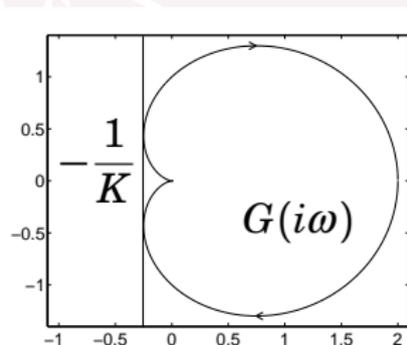
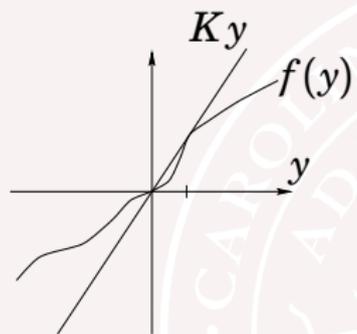
Other cases: Multiply  $f$  and  $G$  with  $-1$ .

## $G$ : Unstable system

To be able to guarantee stability,  $k_1$  and  $k_2$  must have same sign (otherwise unstable for  $k = 0$ )

- $0 < k_1 < k_2$ : Encircle the circle  $p$  times counter-clockwise (if  $\omega$  increasing)
- $k_1 < k_2 < 0$ : Encircle the circle  $p$  times counter-clockwise (if  $\omega$  increasing)

# Linear System with Static Nonlinear Feedback (2)



The “circle” is defined by  $-1/k_1 = -\infty$  and  $-1/k_2 = -1/K$ .

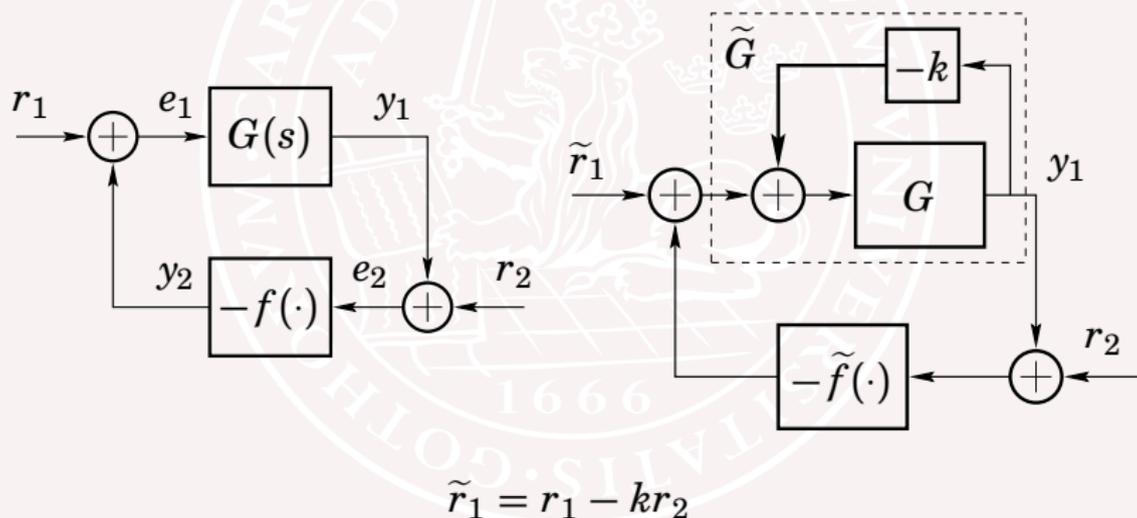
$$\min \operatorname{Re} G(i\omega) = -1/4$$

so the Circle Criterion gives that if  $K \in [0, 4)$  the system is BIBO stable.

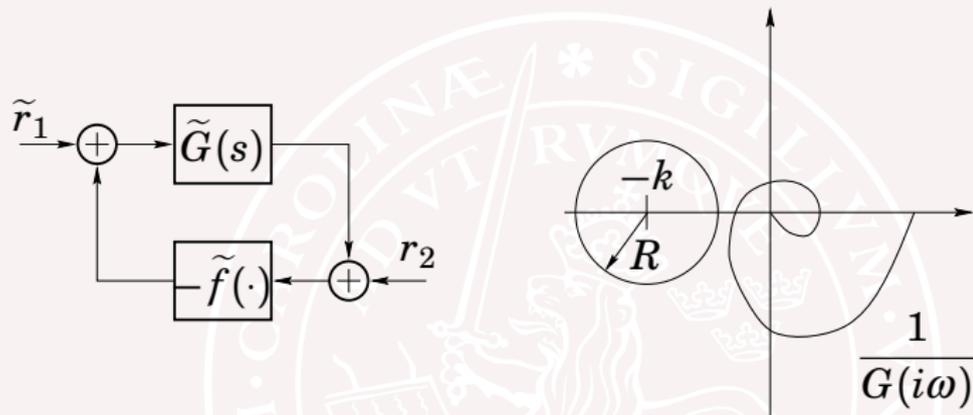
# Proof of the Circle Criterion

Let  $k = (k_1 + k_2)/2$  and  $\tilde{f}(y) = f(y) - ky$ . Then

$$\left| \frac{\tilde{f}(y)}{y} \right| \leq \frac{k_2 - k_1}{2} =: R$$



# Proof of the Circle Criterion (cont'd)



SGT gives stability for  $|\tilde{G}(i\omega)|R < 1$  with  $\tilde{G} = \frac{G}{1 + kG}$ .

$$R < \frac{1}{|\tilde{G}(i\omega)|} = \left| \frac{1}{G(i\omega)} + k \right|$$

Transform this expression through  $z \rightarrow 1/z$ .

# Lyapunov revisited

Original idea: “Energy is decreasing”

$$\begin{aligned} \dot{x} &= f(x), & x(0) &= x_0 \\ V(x(T)) - V(x(0)) &\leq 0 \\ &(+\text{some other conditions on } V) \end{aligned}$$

New idea: “Increase in stored energy  $\leq$  added energy”

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ y &= h(x) \end{aligned}$$
$$V(x(T)) - V(x(0)) \leq \int_0^T \underbrace{\varphi(y, u)}_{\text{external power}} dt \quad (1)$$

# Motivation

Will assume the external power has the form  $\phi(y, u) = y^T u$ .

Only interested in BIBO behavior. Note that

$$\exists V \geq 0 \text{ with } V(x(0)) = 0 \text{ and (1)}$$



$$\int_0^T y^T u dt \geq 0$$

Motivated by this we make the following definition

# Passive System



**Definition** The system  $S$  is **passive** from  $u$  to  $y$  if

$$\int_0^T y^T u dt \geq 0, \quad \text{for all } u \text{ and all } T > 0$$

and **strictly passive** from  $u$  to  $y$  if there  $\exists \epsilon > 0$  such that

$$\int_0^T y^T u dt \geq \epsilon(|y|_T^2 + |u|_T^2), \quad \text{for all } u \text{ and all } T > 0$$

# A Useful Notation

Define the **scalar product**

$$\langle y, u \rangle_T = \int_0^T y^T(t) u(t) dt$$

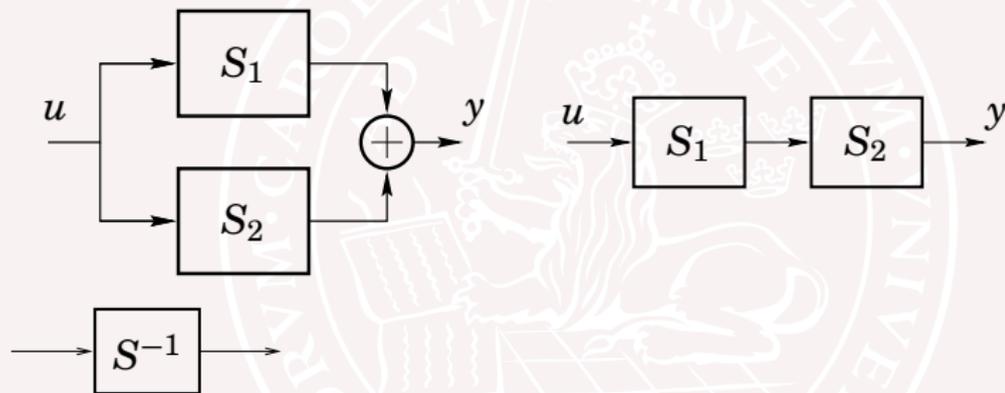


**Cauchy-Schwarz** inequality:

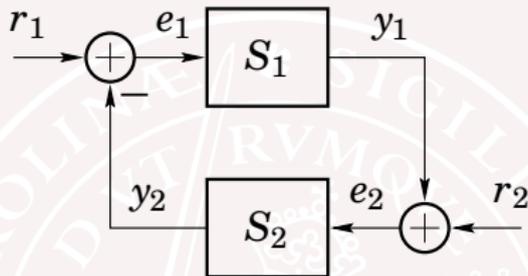
$$\langle y, u \rangle_T \leq |y|_T |u|_T$$

where  $|y|_T = \sqrt{\langle y, y \rangle_T}$ . Note that  $|y|_\infty = \|y\|_2$ .

## 2 minute exercise:



# Feedback of Passive Systems is Passive



If  $S_1$  and  $S_2$  are passive, then the closed-loop system from  $(r_1, r_2)$  to  $(y_1, y_2)$  is also passive.

Proof:

$$\begin{aligned}\langle y, r \rangle_T &= \langle y_1, r_1 \rangle_T + \langle y_2, r_2 \rangle_T \\ &= \langle y_1, r_1 - y_2 \rangle_T + \langle y_2, r_2 + y_1 \rangle_T \\ &= \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T \geq 0\end{aligned}$$

Hence,  $\langle y, r \rangle_T \geq 0$  if  $\langle y_1, e_1 \rangle_T \geq 0$  and  $\langle y_2, e_2 \rangle_T \geq 0$

# Passivity of Linear Systems

**Theorem** An asymptotically stable linear system  $G(s)$  is **passive** if and only if

$$\operatorname{Re} G(i\omega) \geq 0, \quad \forall \omega > 0$$

It is **strictly passive** if and only if there exists  $\epsilon > 0$  such that

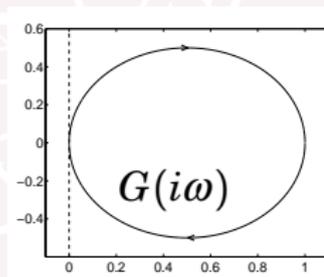
$$\operatorname{Re} G(i\omega) \geq \epsilon(1 + |G(i\omega)|^2), \quad \forall \omega > 0$$

*Proof:* See Slotine and Li p. 139 for the first part.

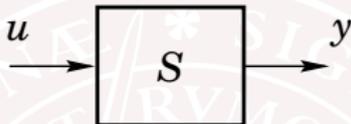
## Example

$G(s) = \frac{s+1}{s+2}$  is passive and strictly passive,

$G(s) = \frac{1}{s}$  is passive but not strictly passive.



# A Strictly Passive System Has Finite Gain



If  $S$  is strictly passive, then  $\gamma(S) < \infty$ .

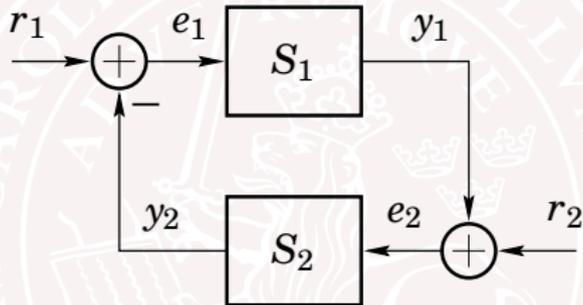
*Proof:* Note that  $\|y\|_2 = \lim_{T \rightarrow \infty} |y|_T$ .

$$\epsilon(|y|_T^2 + |u|_T^2) \leq \langle y, u \rangle_T \leq |y|_T \cdot |u|_T \leq \|y\|_2 \cdot \|u\|_2$$

Hence,  $\epsilon|y|_T^2 \leq \|y\|_2 \cdot \|u\|_2$ , so letting  $T \rightarrow \infty$  gives

$$\|y\|_2 \leq \frac{1}{\epsilon} \|u\|_2$$

# The Passivity Theorem



**Theorem** If  $S_1$  is strictly passive and  $S_2$  is passive, then the closed-loop system is BIBO stable from  $r$  to  $y$ .

# Proof of the Passivity Theorem

$S_1$  strictly passive and  $S_2$  passive give

$$\epsilon(|y_1|_T^2 + |e_1|_T^2) \leq \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T = \langle y, r \rangle_T$$

Therefore

$$|y_1|_T^2 + \langle r_1 - y_2, r_1 - y_2 \rangle_T \leq \frac{1}{\epsilon} \langle y, r \rangle_T$$

or

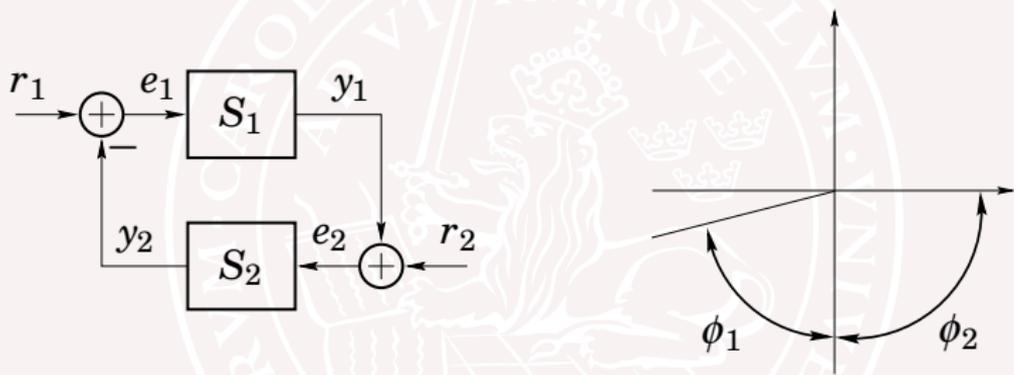
$$|y_1|_T^2 + |y_2|_T^2 - 2\langle y_2, r_2 \rangle_T + |r_1|_T^2 \leq \frac{1}{\epsilon} \langle y, r \rangle_T$$

Finally

$$|y|_T^2 \leq 2\langle y_2, r_2 \rangle_T + \frac{1}{\epsilon} \langle y, r \rangle_T \leq \left(2 + \frac{1}{\epsilon}\right) |y|_T |r|_T$$

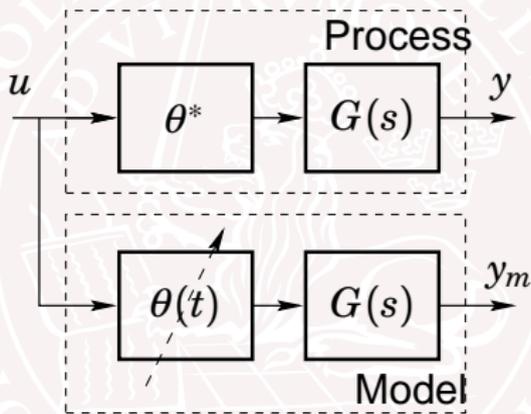
Letting  $T \rightarrow \infty$  gives  $\|y\|_2 \leq C\|r\|_2$  and the result follows

# Passivity Theorem is a “Small Phase Theorem”



# Example—Gain Adaptation

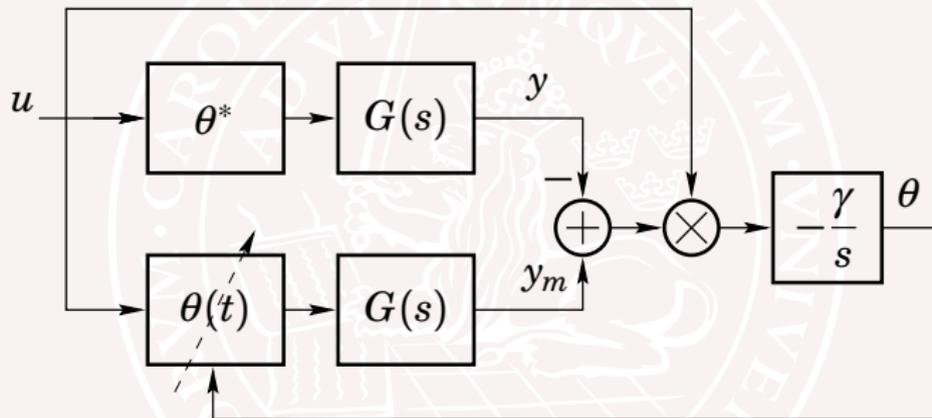
Applications in channel estimation in telecommunication, noise cancelling etc.



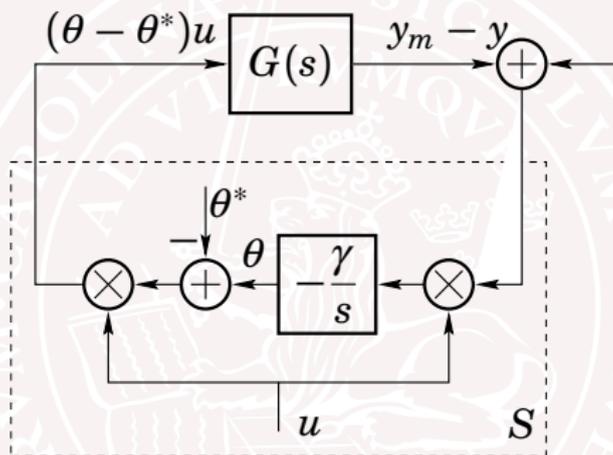
Adaptation law:

$$\frac{d\theta}{dt} = -\gamma u(t)[y_m(t) - y(t)], \quad \gamma > 0.$$

# Gain Adaptation—Closed-Loop System



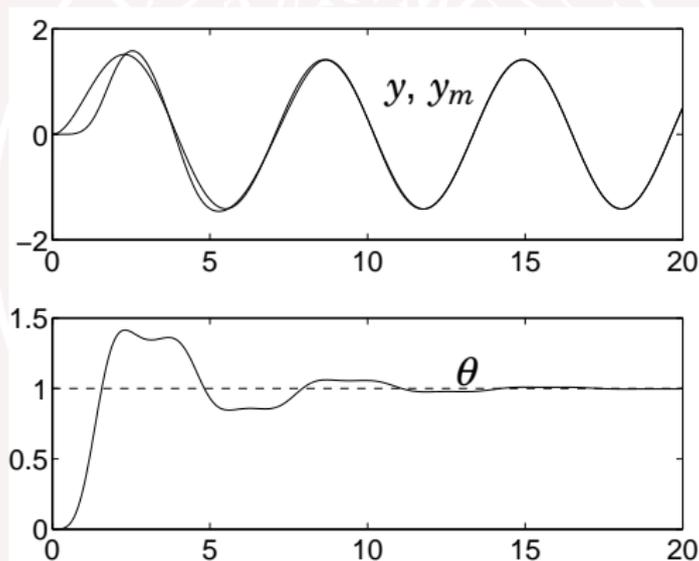
# Gain Adaptation is BIBO Stable



$S$  is passive (Exercise 4.12), so the closed-loop system is BIBO stable if  $G(s)$  is strictly passive.

# Simulation of Gain Adaptation

Let  $G(s) = \frac{1}{s+1} + \epsilon$ ,  $\gamma = 1$ ,  $u = \sin t$ ,  $\theta(0) = 0$  and  $\gamma^* = 1$



# Storage Function

Consider the nonlinear control system

$$\dot{x} = f(x, u), \quad y = h(x)$$

A **storage function** is a  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

- $V(0) = 0$  and  $V(x) \geq 0, \quad \forall x \neq 0$
- $\dot{V}(x) \leq u^T y, \quad \forall x, u$

**Remark:**

- $V(T)$  represents the stored energy in the system

$$\underbrace{V(x(T))}_{\text{stored energy at } t = T} \leq \underbrace{\int_0^T y(t)u(t)dt}_{\text{absorbed energy}} + \underbrace{V(x(0))}_{\text{stored energy at } t = 0},$$

$$\forall T > 0$$

# Storage Function and Passivity

**Lemma:** If there exists a storage function  $V$  for a system

$$\dot{x} = f(x, u), \quad y = h(x)$$

with  $x(0) = 0$ , then the system is passive.

*Proof:* For all  $T > 0$ ,

$$\langle y, u \rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$

# Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

**Lyapunov idea:** “Energy is decreasing”

$$\dot{V} \leq 0$$

**Passivity idea:** “Increase in stored energy  $\leq$  Added energy”

$$\dot{V} \leq u^T y$$

## Example KYP Lemma

Consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Assume there exists positive definite symmetric matrices  $P, Q$  such that

$$A^T P + PA = -Q, \quad \text{and} \quad B^T P = C$$

Consider  $V = 0.5x^T P x$ . Then

$$\begin{aligned} \dot{V} &= 0.5(\dot{x}^T P x + x^T P \dot{x}) = 0.5x^T (A^T P + PA)x + u^T B^T P x \\ &= -0.5x^T Q x + u^T y < u^T y, \quad x \neq 0 \end{aligned} \quad (2)$$

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.

# Next Lecture

- Describing functions (analysis of oscillations)

