LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

# Nonlinear Control and Servo Systems (FRTN05)

**Exam - April 13, 2012 at 14:00 – 19:00**

### Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

*Preliminary* grades:

3: 12 – 16 points

4: 16.5 – 20.5 points

5: 21 – 25 points

### Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

### Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact Anders Rantzer for checking your corrected exam.

*Note!*

In many cases the sub-problems can be solved independently of each other.

**Good Luck!**

**0.** Do you want an e-mail with your result? If so, please confirm this and write your e-mail address where you want us to send it.

*Solution*

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**1.**

**a.** For each of the following systems, find and classify all equilibrium points for the nonlinear system. (2 p)

(a)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3/3 - x_2$$

(b)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + 2x_2(1 - 2x_1^2 - 3.5x_2^2)$$

**b.** The phase portraits of the above two systems are shown in Figure 1. Copy the phase plane and add the arrow heads and determine which phase plane corresponds to which system. (1 p)

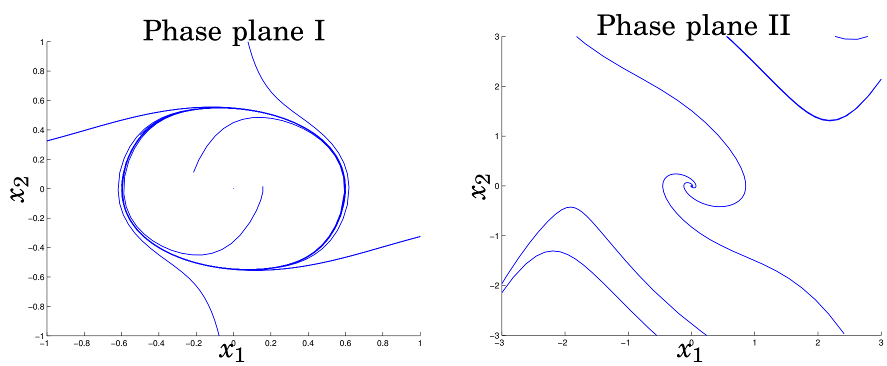


**Figure 1** Phase portraits for Exercise 1.

*Solution*

**a.** The equilibrium points for system (a) are

$$(x_1, x_2) = (0, 0), (\sqrt{3}, 0), (-\sqrt{3}, 0),$$

which are stable focus, saddle, and saddle, respectively.

The equilibrium for system (b) is

$$(x_1, x_2) = (0, 0),$$

which is an unstable focus.

**b.** The system (a) has three equilibrium points. The origin is a stable focus, while the other equilibria are saddles. The fact that $f_1 = x_2$ makes it simple to determine the directions of the arrow heads. The function $f_1$ is positive in the upper half of the plane, and negative in the lower half.

The system (b) has the origin as a unique equilibrium point, being a unstable focus. The direction of the arrow heads can be determined by inspection of the vector fields. In particular, notice that since $f_1(x) = x_2$ the function $f_1$ is positive in the upper half of the plane, and negative in the lower half.

System (a) must corresponds to phase plane II, due to the systems equilibrium points. System (b) must corresponds to phase plane I, due to the systems only equilibrium point and the possible limit cycle.

**2.** Consider the system in Figure 2.

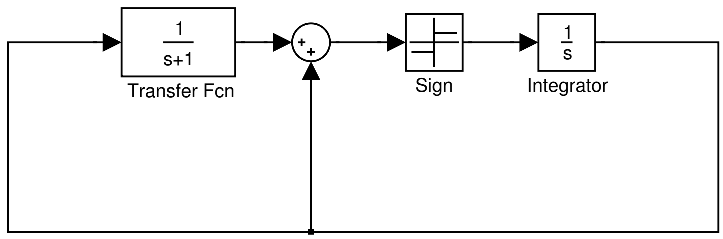


**Figure 2** System in Problem 2

**a.** Introduce states and write the system dynamics on state space form. (1 p)

**b.** Sketch the phase portrait for the system in Figure 2. (1 p)

*Solution*

**a.** Introduce the states $x_1$ and $x_2$ at the output of the linear blocks.

$$\dot{x}_1 = sign(x_1 + x_2)$$
$$\dot{x}_2 = +x_1 - x_2$$

**b.**

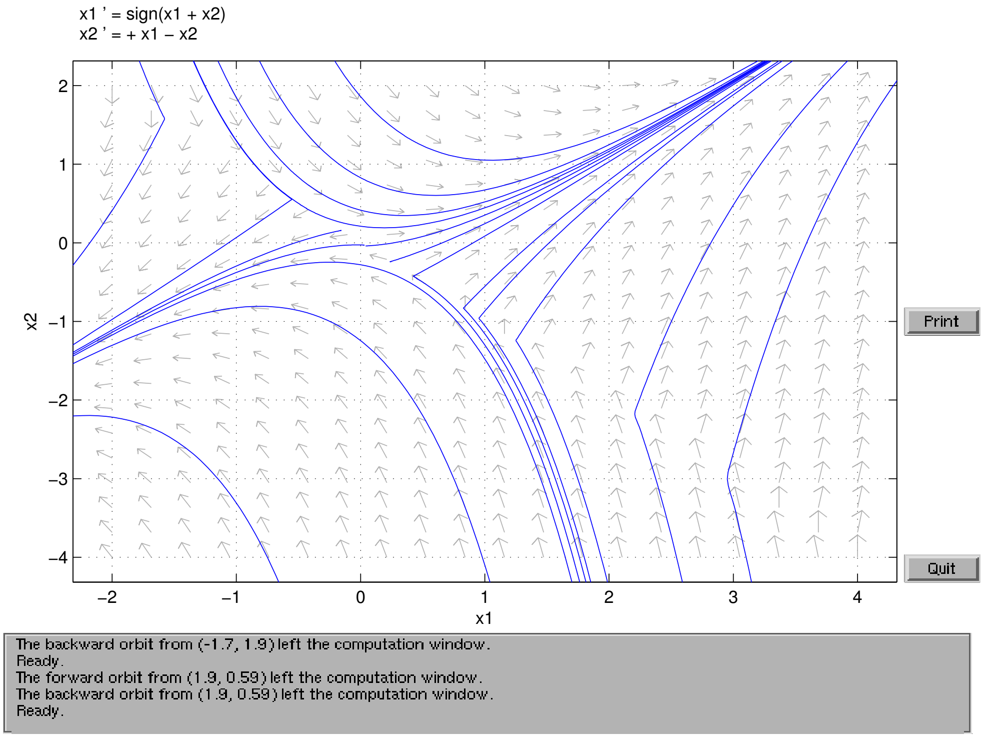


**Figure 3** Phase plane in Problem 2

**3.** Consider a DC-motor

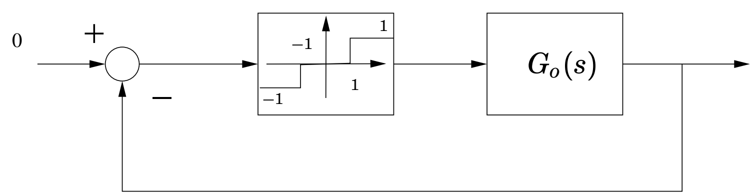


**Figure 4** Setup for problem 3

with transfer function

$$G_0(s) = \frac{K}{s(s+1)(s+4)}$$

that is controlled by a relay with dead-zone of size 1 as in Figure 4. A Bode diagram of $G(s)$ with $K = 1$ is given i Figure 5. The describing function for this relay with dead-zone is given by

$$\begin{cases} 0 & A < 1 \\ \dfrac{4}{\pi A}\sqrt{1 - \dfrac{1}{A^2}} & A \geq 1 \end{cases}$$

which is given in Figure 6.

**a.** Which values of $K > 0$ results in possible limit cycles? (2 p)

**b.** For one such $K$, determine the amplitude and frequency of the stable oscillation. (2 p)

*Solution*

**a.**

$$G(i\omega) = -\frac{K}{\omega(-\omega^2 + 5\omega i + 4)} = -\frac{Ki(4-\omega^2-5\omega i)}{\omega((4-\omega^2)^2 + 25\omega^2)}$$

$$\mathrm{Im}(G(i\omega)) = 0 \Rightarrow \omega^* = 2$$

$$\mathrm{Re}(G(i\omega^*)) = -\frac{K}{20} = -\frac{1}{N(A)}$$

$$\max_A N(A) = \frac{\pi}{2} \approx 0.64$$

If

$$-\frac{K}{20} < -\frac{2}{\pi}$$

thus $K > 10\pi$ might give oscillations.

**b.** Choosing for example $K = 50$ gives

$$\mathrm{Re}(G(i\omega^*)) = -\frac{50}{20} = -2.5 \Rightarrow N(A) = -\frac{1}{-2.5} = 0.4$$

$$\Rightarrow A = \{1.1 \;\; 3\}$$

But the first operating point is unstable and thus $A = 3$ and $w^* = 2$.

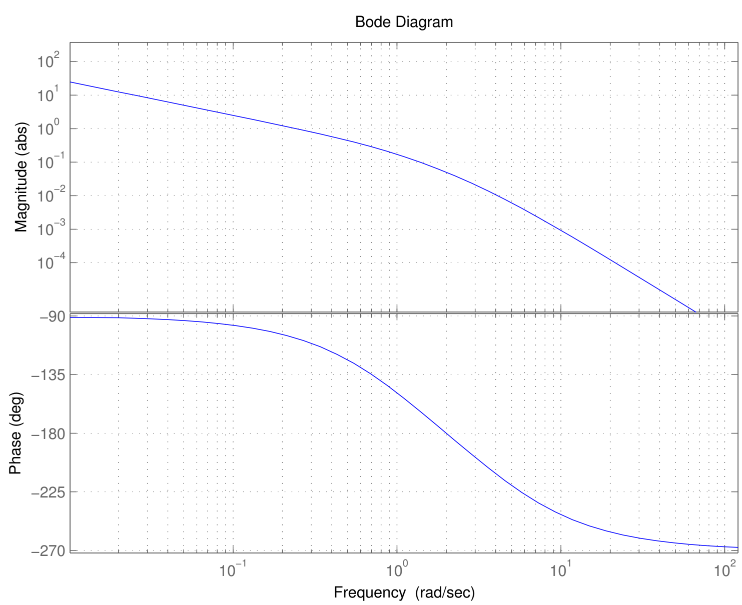


**Figure 5** Bode diagram for problem 3

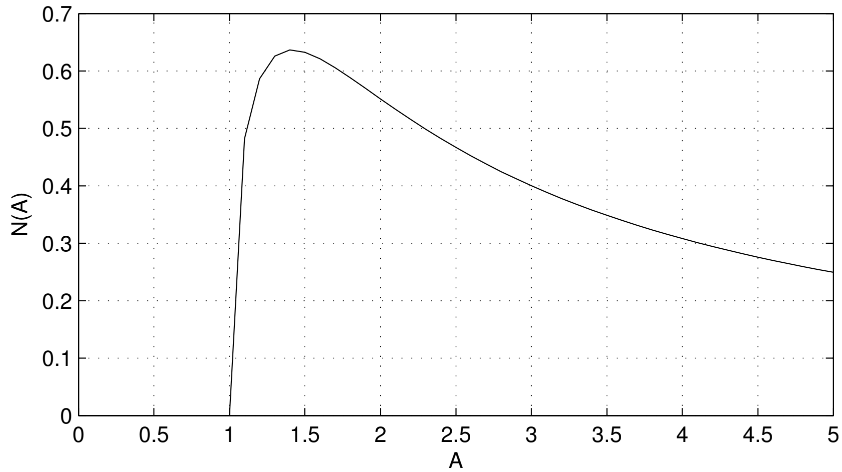


**Figure 6** Describing function for problem 3

**4.** Consider the nonlinear system

$$\dot{x}_1 = 3x_2^2 - x_1$$
$$\dot{x}_2 = 2x_1 - x_2^3 + u$$

**a.** Design a sliding mode controller with $\mu = 1$ and $\sigma(x) = x_2$. (1 p)

**b.** Determine the sliding set and the dynamics on the sliding set. (2 p)

*Solution*

**a.**

$$u = -\frac{[0\ 1]f(x)}{[0\ 1]g(x)} - \frac{\mu}{[0\ 1]g(x)}\text{sign}\sigma(x) = -x_2^3 - 2x_1 - \text{sign}(x_2)$$

**b.** First determine the equivalent control signal on the sliding set.

$$\dot{\sigma} = \dot{x}_2 = u_{eq} \Rightarrow u_{eq} = 0$$

So the sliding set is the whole line $x_2 = 0$.
The dynamics on the sliding set is given by

$$\dot{x}_1 = 3x_2^2 - x_1 = -x_1$$
$$\dot{x}_2 = 2x_1 - x_2^3 + u = 2x_1 - x_2^3 - 2x_1 + x_2^3 + u_{eq} = u_{eq} = 0$$

Thus the system will slide to the origin.

**5.** A simple model for an exothermic chemical reaction is

$$\frac{dT}{dt} = f(T) \tag{1}$$

where the function $f(T)$ is plotted in Figure 7.

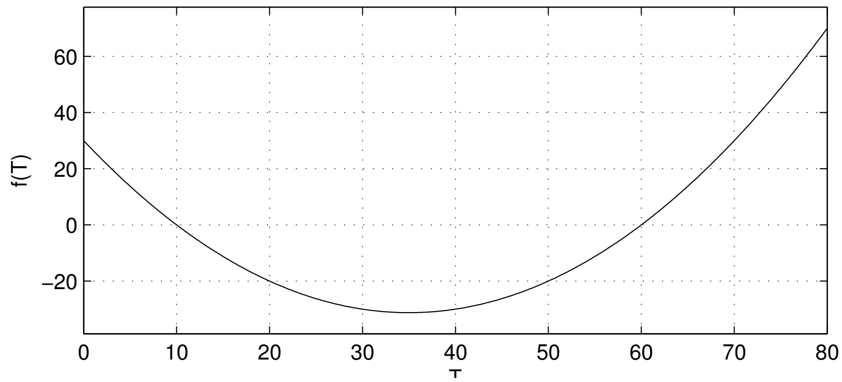


**Figure 7** The derivative of the temperature as a function of the temperature in problem 5

**a.** Determine all equilibrium points and their stability properties. (1 p)

**b.** For which initial values $T(0)$ will the temperature stay bounded? (1 p)

*Solution*

**a.** Equilibrium points $\Leftrightarrow$ $f(T) = 0 \Rightarrow T^0 = [10\ 60]$
The slope for $T = 10$ is negative $\Rightarrow$ stable
The slope for $T = 60$ is positive $\Rightarrow$ unstable

**b.** For $T < 10$, $T$ will increase towards $T = 10$.
For $10 < T < 60$, $T$ will decrease towards $T = 10$.
For $T \geq 60$, $T$ will increase towards positive infinity.
Thus $T$ stays bounded for $T(0) < 60$.

**6.** Consider the interconnection in Figure 8. The output $y$ from a linear time-invariant system $G(s)$ is multiplied with a time-varying function, $h(y) = (2+k\sin(t))\cdot y$, and interconnected in negative feedback with a static gain $b > 0$, see Figure 8.

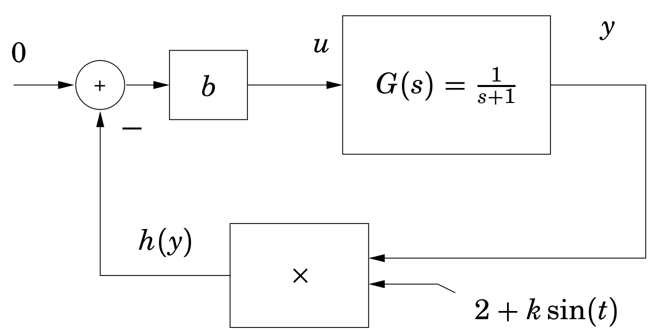


**Figure 8** Block diagram in problem 6.

**a.** Show that the closed loop system is stable for $k = 0$ (0.5 p)

**b.** Determine the largest value of $k$ (as a function of $b$) for which you can guarantee stability of the closed loop system. (1.5 p)

*Solution*

The function $h(y)$ consists of one linear time invariant part and one time-varying function which is bounded in the sector [-$k$, $k$].

**a.** For $k = 0$ we have that the closed loop system is given by

$$\frac{\frac{b}{s+1}}{1 + 2\frac{b}{s+1}} = \frac{b}{s + 1 + 2b}$$

which is stable for $b > -1/2$ and thus also for $b > 0$.

**b.** The maximum gain for $\dfrac{b}{s + 1 + 2b}$ is $\frac{b}{1+2b}$ which means that $|k| < 2 + \frac{1}{b}$ according to the small gain theorem.

**7.** Solve the optimal control problem

$$\min_u \int_0^1 u^2(t)\,dt + x^2(1)$$

$$\frac{d}{dt}x(t) = u \cdot t$$

$$x(0) = 1$$

$$u \in [-\alpha, \alpha]$$

Solve both for the case when the constant $\alpha$ is very large so that the control does not hit the constraint and for the case when $\alpha$ is so small that it will change the solution. (3 p)

*Solution*

The final time $T = 1$ is fixed. The system is normal so we can put $n_0 = 1$. The Hamiltonian is

$$H = u^2 + \lambda t u$$

Minimization with respect to $u$ gives $u = -sat_\alpha(\lambda(t)t/2)$, where

$$sat_\alpha(x) = \begin{cases} \alpha, & x \geq \alpha \\ x\ , & -\alpha \leq x \leq \alpha \\ -\alpha, & x \leq \alpha \end{cases}$$

The adjoint equation is

$$\dot{\lambda} = -H_x = 0, \quad \lambda(1) = 2x(1) \rightarrow \lambda(t) = 2\,x(1)$$

**a.** This gives $u = -x(1) \cdot t$. If this is put into the system equation we get

$$x(T) - x(0) = \int_0^T -t^2 x(T)\,dt = x(1) \cdot (-1)^3/3$$

and hence $x(1) = x(0)/(1 + 1^3/3) = \frac{3}{4}$. The optimal control signal is hence

$$u = -\frac{3t}{4}$$

if we do not hit any constraints, that is, if $\alpha > 3 \cdot 1/4$.

**b.** In the case were the saturation affects the solution the integral can be split in two parts.

$$x(1) = 1 - \int_0^{\frac{\alpha}{x(1)}} x(1)t^2\,\mathrm{d}t - \int_{\frac{\alpha}{x(1)}}^{1} \alpha t\,\mathrm{d}t = 1 - \frac{\alpha}{2} + \frac{\alpha^3}{6x(1)^2} \rightarrow x(1)^3 + \frac{\alpha-2}{2}x(1)^2 - \frac{\alpha^3}{6} = 0$$

Solving this equation for $x(1)$ gives the needed constant for finding $u(t)$ Reasonability of this solution can be verified by seeing that using the conditation $\alpha = 3 \cdot 1/4$ from the previous excercise we recover the same solution and by letting $\alpha$ approach zero we se that $x(1)$ approaches $x(0)$.

**8.** Consider the system

$$\begin{aligned} \dot{x}_1 &= 2x_1^2 x_2 - \alpha(x_1)(3x_1^2 + x_2^2 - 3) \\ \dot{x}_2 &= -6x_1^3 - \beta(x_2)(3x_1^2 + x_2^2 - 3), \end{aligned}$$

where the continuous functions $\alpha$ and $\beta$ have the same sign as their arguments, i.e., $x_1\alpha(x_1) \geq 0 \quad x_2\beta(x_2) \geq 0$ (note that $\alpha(0) = 0 \quad \beta(0) = 0$).

**a.** Show that $3x_1^2 + x_2^2 = 3$ is an invariant set and determine the motion **on** the invariant set. (2 p)

**b.** How will the trajectories of the system behave? (*Hint: Is the set in a) attractive?*) (2 p)

**c.** What equilibrium points does the system have? (1 p)

**d.** Will the set in a) be a limit cycle? (1 p)

*Solution*

**a.** The set $3x_1^2 + x_2^2 - 3 = 0$ is invariant since on trajectories

$$\frac{d}{dt}(3x_1^2 + x_2^2 - 3) = 6x_1^3 2x_2 + 2x_2(-6x_1^3) = 0.$$

The motion on this invariant set is given by

$$\begin{aligned} \dot{x}_1 &= 2x_2 \cdot x_1^2 \\ \dot{x}_2 &= -6x_1 \cdot x_1^2 \end{aligned}$$

**b.** Is the invariant set asymptotically stable?
Use the squared distance as Lyapunov candidate

$$V(x) = (3x_1^2 + x_2^2 - 3)^2.$$

Since $\dot{V} = \ldots = 4(3x_1^2 + x_2^2 - 3)^2(-3x_1\alpha(x_1) - x_2\beta(x_2))$ we conclude that $\dot{V} \leq 0$. We have $\dot{V} < 0$ unless we are on the invariant set or if $x_i f(x_i) = 0$. If we assume $x_1\alpha(x_1) > 0, x_2\beta(x_2) > 0$ for $x_i \neq 0$ we see that the invariant set is the set $3x_1^2 + x_2^2 - 3 = 0$ and the origin $x_1 = x_2 = 0$.

The state converges to either of the two equilibrium points $(x_1, x_2) = (0, +\sqrt{3})$ or $(x_1, x_2) = (0, -\sqrt{3})$ from any initial condition except from the origin.

**c.** The equilibrium points for the system.

$$\dot{x}_1 = \mathbf{0} = 2x_1^2 x_2 - \alpha(x_1)(3x_1^2 + x_2^2 - 3)$$
$$\dot{x}_2 = \mathbf{0} = -6x_1^3 - \beta(x_2)(3x_1^2 + x_2^2 - 3),$$

For general functions $f_i$, the only possibility is that

$x_1^2 x_2 = 0$, $\alpha(x_1)\cdot(3x_1^2 + x_2^2 - 3) = 0$, $\beta(x_2)\cdot(3x_1^2 + x_2^2 - 3) = 0$ and $-6x_1^3 = 0$

at the same time. We thus get that $(x_1, x_2) = (0,0)$ and $(x_1, x_2) = (0, \pm\sqrt{3})$ are the equilibrium points.

**d.** The set is however NOT a limit cycle, as the two equilibrium points $(x_1, x_2) = (0, \pm\sqrt{3})$ belongs to this set. Any trajectory moving along the invariant set (the state vector moves clockwise) will eventually end up in either $(x_1, x_2) = (0, +\sqrt{3})$ or $(x_1, x_2) = (0, -\sqrt{3})$ .