LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - August 30, 2012 at 8–13

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: $12 - 16$ points

4: $16.5 - 20.5$ points

5: $21 - 25$ points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/”Collection of Formulae”. Pocket calculator.

Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact Anders Rantzer for checking your corrected exam.

Note!

In many cases the sub-problems can be solved independently of each other.

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
August, 2012.

1. Consider the control system

$$\ddot{x} - 2(\dot{x})^2 + x = u - 1 \tag{1}$$

a. Write the system in first-order state-space form. (1 p)

b. Suppose $u(t) \equiv 0$. Find all equilibria and determine if they are stable or asymptotically stable if possible. (2 p)

c. Show that Eq. (1) satisfies the periodic solution $x(t) = \cos(t)$, $u(t) = \cos(2t)$. Linearize the system around this solution. (2 p)

d. Design a state-feedback controller $u = u(x, \dot{x})$ for (1), such that the origin of the closed loop system is globally asymptotically stable. (1 p)

Solution

a. Introduce $x_1 = 1$, $x_2 = \dot{x}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2x_2^2 + u - 1 \end{aligned} \tag{2}$$

b. Let $\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow (x_1, x_2) = (-1, 0)$ is the only equilibrium. The linearization around this point is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (-1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The characteristic equation for the linearized system is $s^2 + 1 = 0 \Rightarrow s = \pm i$. We can not conclude stability of the nonlinear system from this.

c.

$$x = \cos(t) \Rightarrow \dot{x} = -\sin(t) \Rightarrow \ddot{x} = -\cos(t)$$

By inserting this in the system dynamics and using *e.g.*, $u = \cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$ we get

$$\ddot{x} - 2(\dot{x})^2 + x = -\cos(t) - 2\sin^2(t) + \cos(t) = 2 + \cos^2(t) - 2 = u - 1$$

which shows that the trajectory is a solution.
The linearized system is thus

$$\begin{aligned} \delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 4x_2 \end{bmatrix}_{(x_1^o, x_2^o) = (\cos(t), -\sin(t))} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -4\sin(t) \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \end{aligned} \tag{3}$$

where

$$\delta x = \begin{bmatrix} x_1(t) - \cos(t) \\ x_2(t) - (-\sin(t)) \end{bmatrix}, \quad \delta u = u(t) - \cos(2t)$$

d. The simplest way is to cancel the constant term and the nonlinearity with the control signal and introduce some linear feedback.

$$u = +1 - 2(\dot{x}_2)^2 - a\dot{x}, \quad a > 0 \Rightarrow \ddot{x} + a\dot{x} + x = 0$$

As the resulting system is linear and time invariant with poles in the left half plane for all $a > 0$ it is GAS.

2. Sketch the phase portrait for

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{4}$$

Illustrate the behavior of the system both for states close to and for states far from the origin (for large and small values of $|x|$). (2 p)

Solution

Local behavior around $(x_1, x_2) = (0, 0)$ (only equilibrium):

$$\begin{aligned} \dot{x}_1 &\approx -x_2 \\ \dot{x}_2 &\approx x_1 - x_2 \end{aligned} \tag{5}$$

Poles of linearized system: $s^2 + 2s + 1 = 0 \to s = (-1 \pm i\sqrt{3})/2$ (stable focus) However, for large values of both x_1 and x_2, we have to dominating dynamics $\dot{x}_1 = -x_2$ and $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \approx x_1^2 x_2$ From the simple table below that

$x_1 \gg 1,$	$x_2 \gg 0$	$\Rightarrow$	$\dot{x}_1 < 0,$	$\dot{x}_2 > 0$
$x_1 \gg 1,$	$x_2 \ll 0$	$\Rightarrow$	$\dot{x}_1 > 0,$	$\dot{x}_2 > 0$
$x_1 \ll -1,$	$x_2 \gg 0$	$\Rightarrow$	$\dot{x}_1 < 0,$	$\dot{x}_2 < 0$
$x_1 \ll -1,$	$x_2 \ll 0$	$\Rightarrow$	$\dot{x}_1 > 0,$	$\dot{x}_2 < 0$

we can see that the trajectories escape to infinity in (at least) the second and forth quadrant, from which we can conclude that the origin is not globally asymtotically stable. See Figure 1 for the complete phase plot.

3.

a. Show that the origin of the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2^3 \\ \dot{x}_2 &= x_1 \end{aligned} \tag{6}$$

is globally asymptotically stable. Hint: You may use a Lyapunov function candidate of the form $V(x_1, x_2) = x_1^2 + a \cdot x_2^4$ for some proper value of a. (3 p)

b. What can be said about the stability of the system

$$\begin{aligned} \dot{x}_1 &= +x_1^2 + x_2^3 \\ \dot{x}_2 &= -x_1 \end{aligned} \tag{7}$$

Hint: compare the dynamics with the dynamics of the system in subproblem **a**). (1 p)

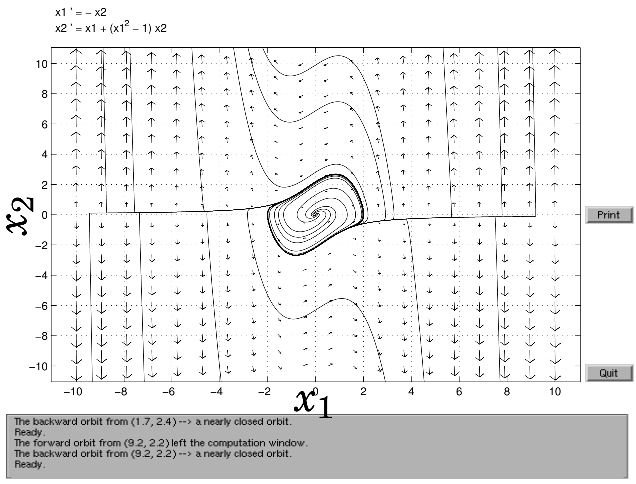

Figure 1 Phase plot for system in Problem 2. The origin is a stable focus.

Solution

a. $V(x_1, x_2) = x_1^2 + a \cdot x_2^4 > 0$ for $x \neq (0, 0)$ for $a > 0$. V is radially unbounded. $\dot{V}(x_1, x_2) = 2x_1\dot{x}_1 + 4ax_2^3\dot{x}_2 = 2x_1(-x_1^3 - x_2^3) + 4ax_2^3x_1 = \{choose\ a = 1/2\} = -2x_1^4 \leq 0$ V decreases for all values of $x_1 \neq 0$. To conclude asymptotic stability of the origin we need to use La Salle invariance theorem. If $x_1 \equiv 0 \Rightarrow x_2 \equiv 0$.

b. We see the the right hand side of the systems in problem **a)** and **b)** differ by sign. This is like considering the solutions to the same system in forward time and in backward time, respectively (do the varable change $t = -\tau$). As it is stated that the origin is globally asymptotically stable in subproblem **a)**, we can conclude that all solutions are escaping to infinity for all initial values $x \neq (0, 0)$.

4. We want to design an oscillator by the interconnection of a first order system with time delay, and a relay, see Figure 2. The system $G(s) = \dfrac{k}{s+1} \cdot e^{-Ls}$ and $\varphi(\cdot)$ is a relay with amplitude 1 (*i.e.,* $\varphi(z) = \text{sign}(z)$). We want to achieve an oscillation with amplitude = 2 Volts and a frequency of 4 Hz. Determine the parameters $k > 0$ and $L > 0$ to achieve this. Will the oscillation be stable? Motivate your answer. (3 p)

Solution

The describing function for a relay which switches between -1 and 1 is $N(A) = \dfrac{4}{\pi A}$. Find k and L such that $G(j\omega) = -1/N(A)$. $A = 2$ and $\omega = 2\pi \cdot 4 = 8\pi$

$$arg\{G(j\omega)\} = -atan(\omega/1) - \omega L = -\pi \Rightarrow L \approx 0.06$$

$$|G(j\omega)| = k/\sqrt{\omega^2 + 1^2} = \pi/2 \Rightarrow k \approx 39.5$$

It will be a stable limit cycle (see course litterature for condition).

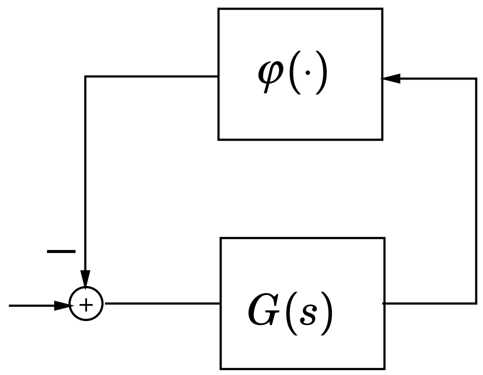

Figure 2 The block diagram for the oscillator in Problem 4. $\varphi(\cdot)$ is a relay with amplitude 1

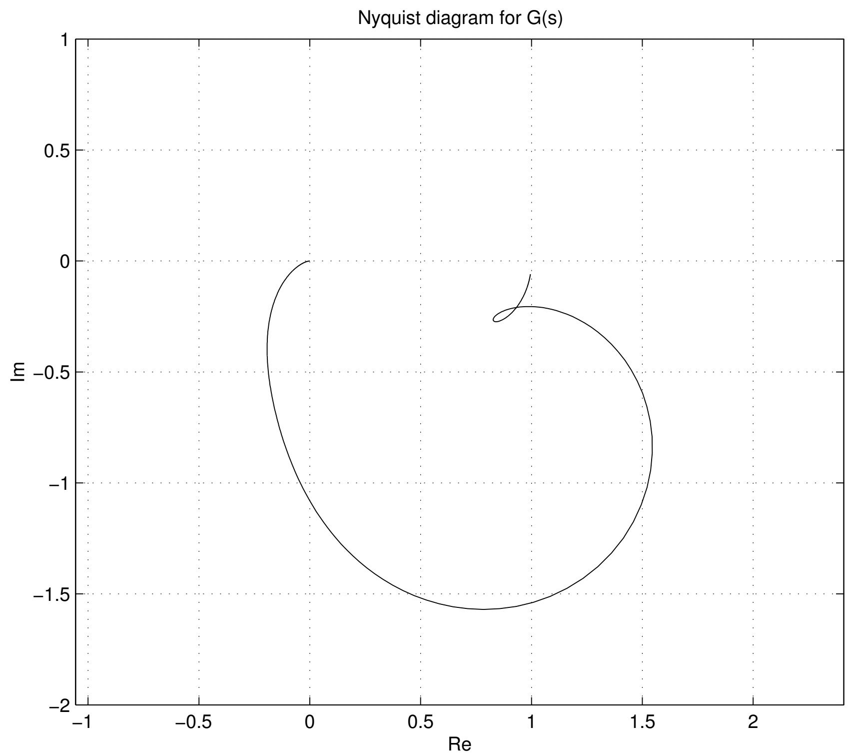

Figure 3 Nyquist diagram for linear system $G(s)$ in Problem 5.

5. An exponentially stable linear system $G(s)$ is negative feedback interconnected with a nonlinearity ψ. The Nyquist diagram of the linear system is shown in Figure 3.

a. What is the largest sector $\psi \in [0, \beta]$ for which *the circle criterion* guarantees stability for the closed loop? (1 p)

b. What is the larges sector $\psi \in [-k, k]$ for which *the small gain theorem* guarantees stability for the closed loop? (1 p)

Solution

a. According to the circle criterion, in this case the closed loop will be stable for the nonlinearity in the sector $[0, \beta]$ if the Nyquist curve stays to the right of the vertical line $-1/\beta$. From the Nyquist curve we see that we can take $\beta \approx 1/0.25 = 4$.

b. In this case we first want to find the maximum gain of the linear system which equals the largest magnitude ('radius') of the Nyquist curve. From the Nyquist curve we see that this is about 2. The small gain theorem then allows the sector to have $k < 1/2 = 0.5$.

6. Consider the following system describing stick-slip motion of a sliding mass with Coloumb friction, see also Figure 4.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \text{sign }(x_2 + 1)$$

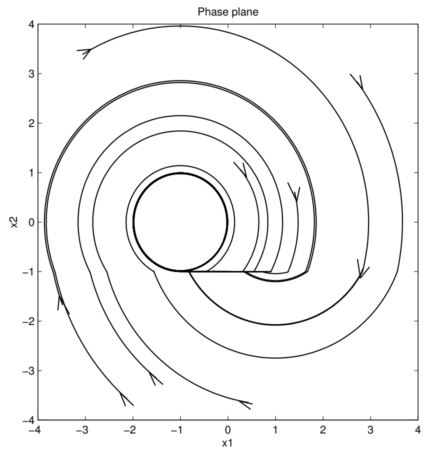

Figure 4 Phase plane diagram in Problem 5

a. Determine the region in which the sliding occurs and determine the dynamics on the sliding set. (3 p)

b. Check that $V = (x_1 + 1)^2 + x_2^2$ is a Lyapunov function for the system. (1 p)

c. What happens with $x(t)$ for large t? (1 p)

Solution

a. Sliding surface $\sigma(x) = x_2 + 1 = 0$. Dynamics on sliding surface is given by $x_2 = -1$ and $\dot{x}_1 = -1$.

b.

$$\dot{V}/2 = (x_1 + 2)x_2 + x_2(-x_1 - \text{sign}(x_2 + 1))$$
$$= x_2 - x_2\text{sign}(x_2 + 1)$$

If $x_2 > -1$ we get $\dot{V} = 0$. If $x_2 < -1$ we get $\dot{V} = 4x_2 < 0$. Hence V is a Lyapunov function. (One should really also check that V decreases on the sliding set. On the sliding set we have $x_2 = -1$, $\dot{x}_1 = -1$ and therefore $\dot{V} = 2(x_1 + 1)\dot{x}_1 = -2(x_1 + 1) \leq 0$ (since $|x_1| \leq 1$ there).

c. $\dot{V} = 0$ only when $x_2 \geq -1$. The largest invariant set there is the circle

$$M = \{(x_1 + 1)^2 + x_2^2 \leq 1\}.$$

hence $x(t)$ will approach M.

7. Consider the following mass-damper system system

$$\ddot{x} + d\dot{x} = u \qquad u \in [-1, 1] \tag{8}$$

where u is the control signal (the force), bounded to $[-1, 1]$. x is the position and $d > 0$ is the damping constant. The problem is to go from any initial state $x(0)$ to $x = 0$ in as short time as possible.

Write down the corresponding optimal control problem, and find the structure of the optimal control law. You do not have to solve for numeric constants. Sketch one optimal trajectory. (3 p)

Solution

Rewrite in state-space form as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -dx_2 + u, \quad |u| \le 1\end{aligned}$$

by taking $x = x_1$ and $\dot{x} = x_2$.

The problem can be written as

$$\min \int_0^{t_f} 1dt$$

with

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -dx_2 + u, \quad |u| \le 1\end{aligned}$$

$$x(0) = x^0 \quad x(t_f) = 0.$$

Introduce

$$\Psi_1 = x_1 \quad \Psi_2 = x_2,$$

and we require that $\Psi_1(t_f) = 0$ and $\Psi_2(t_f) = 0$.

The Hamiltonian is

$$H = n_0 + \lambda_1 x_2 + \lambda_2(u - dx_2).$$

The adjoint variables satisfy

$$\dot{\lambda}_1 = -H_{x_1} = 0,$$

$$\dot{\lambda}_2 = -H_{x_2} = -\lambda_1 + d\lambda_2,$$

$$\lambda_1(t_f) = n_0\phi_{x_1} + \mu_1\Psi_{1x_1} + \mu_2\Psi_{2x_1} = \mu_1,$$

and

$$\lambda_2(t_f) = \mu_2.$$

This means that $\lambda_1(t) = \mu_1$ is a constant. λ_2 acts like a first order unstable system (as $d > 0$) with constant input λ_1.

The maximum principle gives that u must minimize H at each point. Since H is linear in u, u will always be at either $u = 1$ or $u = -1$. More specifically

$$u = \begin{cases} 1 & \lambda_2 > 0 \\ -1 & \lambda_2 < 0 \end{cases}.$$

Since λ_2 is driven by a constant and is a first order system, it can only change sign at most once. The optimal control law must be

$$u(t) = \begin{cases} 1 & t \le t_{\text{switch}} \\ -1 & t > t_{\text{switch}} \end{cases}$$

or

$$u(t) = \begin{cases} -1 & t \le t_{\text{switch}} \\ 1 & t > t_{\text{switch}} \end{cases}$$

where t_{switch} depends on x^0 (and sometimes $t_{\text{switch}} = t_f$). See Figure 5 for the explicit switching line and an example trajectory.

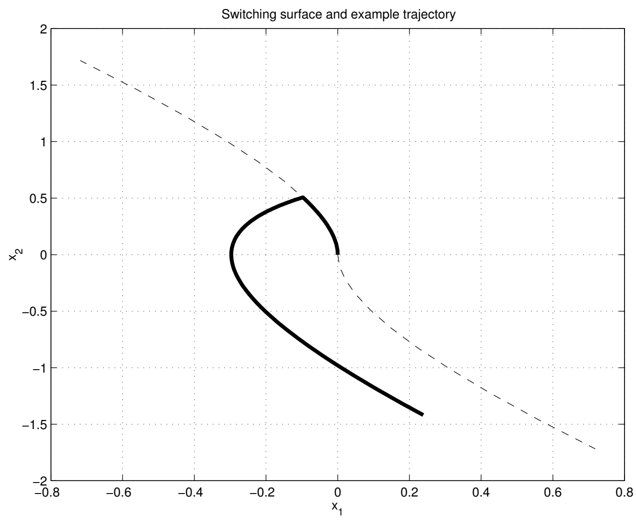

Figure 5 The switching line and example trajectory for Problem 7.