Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
2011–03–09

1.

a. (i)-D, (ii)-B, (iii)-A, (iv)-C. Motivate e.g. by computing the equilibrium points of the four systems.

b. The directions are given in figure 1. Motivate e.g. by classifying all equilibrium points of the four systems.

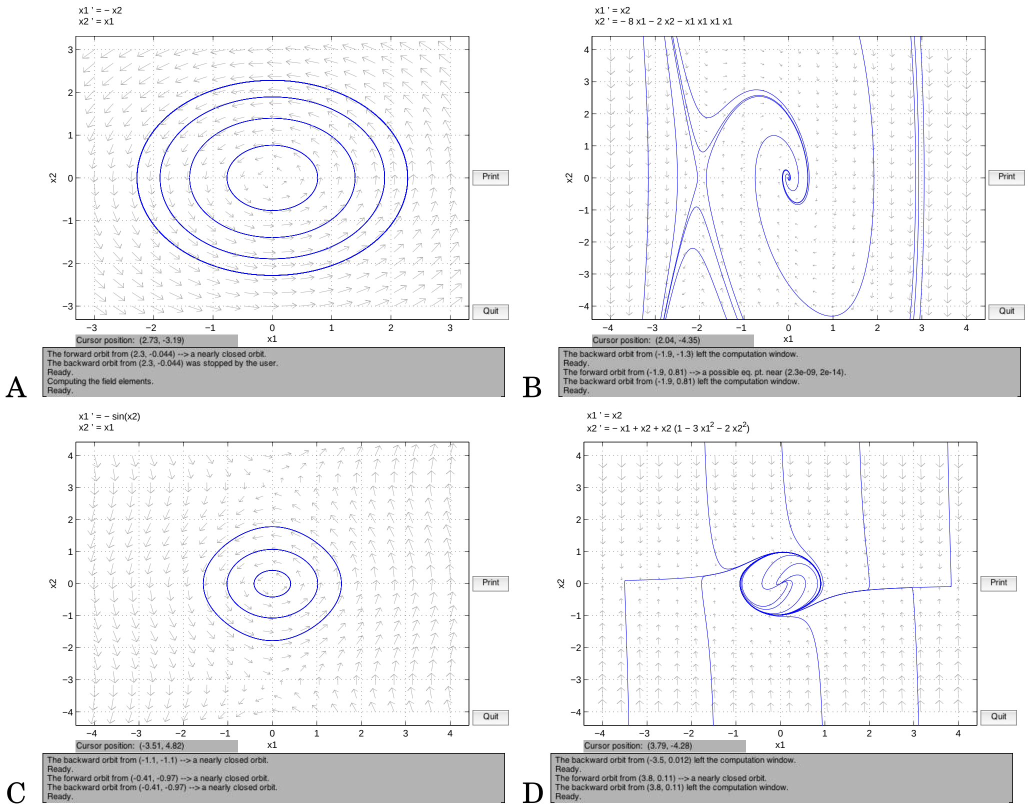

Figure 1 The phase portraits in problem 1

2.

a. The saturation nonlinearity commonly appears in actuators (limitation of control signal).

b. The nonlinearity belongs to the sector $[0, 1]$.

c. Since $P(s)$ is stable and $0 = k_1 < k_2 = 1$, the Nyquist curve should stay to the right of the line Re $s = -1/k_2 = -1$. In this case, the Nyquist curve does not stay to the right of Re $s = -1$ so we can not conclude stability of the closed loop system.

The closed loop system is locally stable when $r = 0$, since the nonlinearity is linear around 0 and the system $[1 + P(s)]^{-1}$ is stable according to the Nyquist theorem.

3.

a. The describing function of a relay with amplitude 1 is $N(A) = \frac{4}{\pi A}$. Consequently, $-\frac{1}{N(A)}$ is on the negative real axis and there exists a limit cycle if the Nyquist curve of the system $G(s)$ intersects the negative real axis. The intersection must occur, since the system has three poles and no zeros. The frequency and amplitude of the limit cycle are thus given by:

$$\begin{aligned}
\arg G(iw_0) &= -\frac{\pi}{2} - \arctan(w_0) - \arctan(w_0/10) = -\pi \Rightarrow w_0 = 3.16\text{rad/s} \\
|G(iw_0)| &= \frac{50}{w_0\sqrt{w_0^2+1}\sqrt{w_0^2+10^2}} = 0.46 \\
-\frac{1}{N(A)} &= -\frac{\pi A}{4} = 0.46 \Rightarrow A = 0.58
\end{aligned} \tag{1}$$

b. The system in **a.** will give a limit cycle. The amplitude A of this limit cycle could be estimated from experiments, which gives gives the ultimate gain: $K_0 = \frac{1}{|G(iw_0|} = \frac{4}{\pi A}$ for the relay with amplitude 1. The period time T_0 of the limit cycle can be determined directly from the experiment. Figure 2 shows a simulation of the relay feedback system.

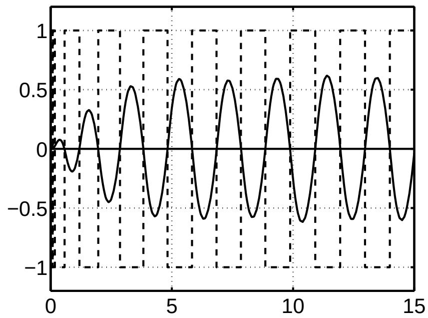

Figure 2 Simulation of the relay feedback system in problem 3.

4.

a. Introduce $x_1 = \theta$, $x_2 = \dot{\theta}$. We get:

$$\begin{aligned}
x_1 &= x_2 \\
x_2 &= \omega_0^2 \sin x_1 - \frac{\omega_0^2}{g} u \cos x_1
\end{aligned}$$

b. Swinging up the pendulum could be done by driving the energy of the pendulum to zero, which corresponds to being in the top position. Using the Lyapunov candidate $V(x) = \frac{1}{2}E(x)^2$ and the states chosen in **a.**, we get:

$$\dot{V}(x) = E(x)\dot{E}(x) = -\frac{1}{g}E(x)x_2\cos(x_1)u$$

Choose for example: $u = \text{sign}(Ex_2 \cos x_1) = \text{sign}[E(\theta, \dot{\theta})\dot{\theta}\cos(\theta)]$, which gives

$$\dot{V}(x) = -\frac{1}{g}Ex_2 \cos x_1 \text{sign}(E(x)x_2 \cos x_1) \leq 0 \quad \forall x$$

5. Study σ_1 first.

$$\sigma_1 = x_1 - x_2 = 0$$
$$\dot{\sigma}_1 = \dot{x}_1 - \dot{x}_2 = -x_1^2 + x_2 + u_{eq} - x_1 - x_2 = 0$$
$$u_{eq} = x_1 + x_1^2$$

With the calculated equivalent control u_{eq} inserted in the system we get

$$\begin{aligned} \dot{x}_1 = &\quad -x_1^2 + x_2 + u_{eq} = x_1 + x_2 = 2x_1 \\ \dot{x}_2 = &\quad x_1 + x_2 = 2x_2 \end{aligned}$$

Remember that along the sliding set where $\sigma_1(x) = 0$, we have $x_1 = x_2$. Thus the system is unstable along the sliding set.

Now study σ_2.

$$\sigma_2 = x_1 + 4x_2 = 0$$
$$\dot{\sigma}_2 = \dot{x}_1 + 4\dot{x}_2 = -x_1^2 + x_2 + u_{eq} + 4x_1 + 4x_2 = 0$$
$$u_{eq} = -4x_1 + x_1^2 - 5x_2$$

With the calculated equivalent control u_{eq} inserted in the system we get

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_2 + u_{eq} = -4x_1 - 4x_2 = -3x_1 \\ \dot{x}_2 &= x_1 + x_2 = -4x_2 + x_2 = -3x_2 \end{aligned}$$

Remember that along the sliding set where $\sigma_2(x) = 0$, we have $x_1 = -4x_2$. Thus the system is stable along the sliding set.

Now study σ_3.

$$\sigma_3 = x_1^2 - x_2 = 0$$
$$\dot{\sigma}_3 = 2x_1\dot{x}_1 - \dot{x}_2 = 2x_1(-x_1^2 + x_2 + u_{eq}) - x_1 - x_2 = 0$$
$$u_{eq} = \frac{1}{2x_1}(x_1 + x_2) + x_1^2 - x_2$$

We see directly that the control law is not well-defined for $x_1 = 0$ and therefore we do not need to check the sliding mode dynamics.

It is therefore decided that the second sliding set is the best of these three suggested sets. Then, according to the lecture notes, the complete sliding mode controller can be written as

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)}\text{sign } \sigma(x) = -\mu \text{ sign}(x_1 + 4x_2) + x_1^2 - 5x_2 - 4x_1$$

where μ is a arbitrary positive constant that decides the rate of convergence to the sliding set.

6. Introduce $x_1 = x$, $x_2 = \dot{x}$. We get

$$\min \frac{1}{2}\int_0^T u(t)^2 dt$$

when

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 + u \\ x_1(0) = 0, \quad x_2(0) = 0 \\ x_1(T) = 3, \quad x_2(T) = 0 \end{cases}$$

$$H = -\frac{1}{2}u^2 + \lambda_1 x_2 - 2\lambda_2 x_1 + \lambda_2 u$$
$$\hat{u}(t) = \lambda_2(t)$$

The adjoint equations become

$$\begin{cases} \dot{\lambda}_1 = 2\lambda_2 \\ \dot{\lambda}_2 = -2x_1 + u\lambda_1 \end{cases}$$

$\Rightarrow \ddot{\lambda}_2 = -\dot{\lambda}_1 = -2\lambda_2$ which gives

$$\hat{u}(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega = \sqrt{2}$. A and B are given from the boundary conditions on x_1 and x_2.

(3 p)

7. We see that the system consists of two separate subsystems:

$$\begin{cases} \dot{x}_1 = x_1^3 + 2x_1^2 x_2 \\ \dot{x}_2 = x_2^2 + u_1 \end{cases} \tag{2}$$

$$\begin{cases} \dot{x}_3 = -x_3 + 2x_4 \\ \dot{x}_4 = x_4 + u_2 \end{cases} \tag{3}$$

System (3) is a linear system. It can easily be stabilized by choosing e.g.

$$u_2 = -2x_4$$

System (2) is nonlinear and in strict feedback form. We therefore apply backstepping.

Start with the system $\dot{x}_1 = x_1^3 + 2x_1^2\phi(x_1)$ which can be stabilized using $\phi(x_1) = -x_1$. Notice that $\phi(0) = 0$. Take $V_1(x_1) = x_1^2/2$. To backstep, define

$$\zeta_2 = (x_2 - \phi(x_1)) = x_2 + x_1,$$

to transfer the system into the form

$$\begin{aligned} \dot{x}_1 &= x_1^3 + 2x_1^2(\zeta_2 - x_1) = -x_1^3 + 2x_1^2\zeta_2 \\ \dot{\zeta}_2 &= \dot{x}_2 + \dot{x}_1 = \zeta_2^2 - 2x_1\zeta_2 + x_1^2 + u_1 - x_1^3 + 2x_1^2\zeta_2 \end{aligned}$$

Taking $V = V_1(x_1) + \zeta_2^2/2$ as a Lyapunov function gives

$$\dot{V} = x_1(-x_1^3 + 2x_1^2\zeta_2) + \zeta_2(u_1 + \zeta_2^2 - 2x_1\zeta_2 + x_1^2 - x_1^3 + 2x_1^2\zeta_2)$$

With

$$u_1 = -\zeta_2^2 + 2x_1\zeta_2 - x_1^2 - x_1^3 - 2x_1^2\zeta_2 - \zeta_2 = -x_1 - x_2 - 3x_1^2 - 2x_1^2x_2 - x_2^2$$

we get

$$\dot{V} = -x_1^4 - \zeta_2^2 < 0 \quad \forall (x_1, \zeta_2) \neq 0$$

The Lyapunov function is radially unbounded. Hence, the origin is globally asymptotically stable.