Lecture 14 — Course Summary

- CEQ
- The exam
- Questions/review of the course

CEQ

You will get a mail regarding CEQ (Course evaluation) to be filled out via a web-page.

Please, fill it in, and **write your comments**.

Both Swedish and English versions are available!

Remember, without your feedback we teach in open-loop.

Question: What's on the exam?

Among old exam questions:

- Models, equilibria etc
- Linearization and stability
- Circle criterion
- Small gain
- Describing Functions
- Lyapunov functions
- ...

Old exams and solutions are available from the course home page.

Exam (in English)

Course Material Allowed:

- Lecture slides 1-15 (No exercises or old exams)
- Laboratory exercises 1, 2, and 3
- *Reglerteori* by Glad and Ljung
- *Applied Nonlinear Control* by Slotine and Li
- *Nonlinear Systems* by Khalil

You may bring everything on the list + "Collection of Formulae for Control" to the exam.

Question

Can I get different answers if use the Small Gain theorem and the Circle criterion? What does it mean?

- If the conditions for stability are not satisfied for one criterion it does not necessarily mean that the system is unstable. It just means that you can not use that method to guarantee stability. You can never 'prove' that a system is stable with one method and 'unstable' with another.
- Similarly, there are no general guaranteed methods to find a Lyapunov function (though some suggested good methods/candidates are worth to try).

Question

Please repeat the stability definitions and methods to prove stability.

Explain about invariant sets and when $\dot{V} = 0$.

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$ for some open set Ω. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$
- $\dot{V}(x) \leq 0$ along all trajectories in Ω

then $x = 0$ is locally stable. Furthermore, if also

- $\dot{V}(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

Lyapunov Theorem for Global Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$. Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is a C^1 function. If

- $V(0) = 0$
- $V(x) > 0$, for all $x \neq 0$
- $\dot{V}(x) < 0$ for all $x \neq 0$
- $V(x) \to \infty$ as $\|x\| \to \infty$ (radial unboundedness)

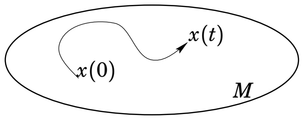
then $x = 0$ is globally asymptotically stable.

Invariant Sets

Definition A set M is called **invariant** if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.



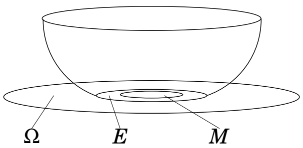
When finding Lyapunov function candidates with $\frac{dV}{dt} \leq 0$ we often want to use this like to show that the origin is the largest invariant set M.

Invariant Set Theorem

Theorem Let $\Omega \in \mathbf{R}^n$ be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^1 function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$



Example

Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - x_1^3$$

Try with

$$V(x) = x_1^2 + x_2^2 \qquad \text{(Alt. 1)}$$

or

$$V(x) = 0.5x_1^4 + x_2^2 \qquad \text{(Alt. 2)}$$

Alt 1:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}$$
$$= 2x_1x_2 + 2x_2(-x_2 - x_1^3) = 2(x_1 - x_1^3)x_2 - 2x_2^2 \qquad !??$$

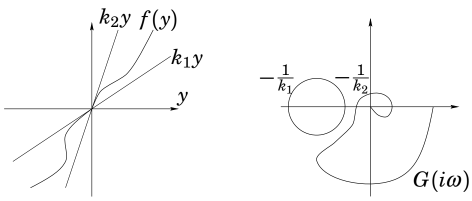
No information as indefinite.

Alt 2:

$$\frac{dV}{dt} = 0.5 \cdot 4x_1^3x_2 + 2x_2(-x_2 - x_1^3) = -2x_2^2 \leq 0$$

To show *asymptotic* stability we need to continue (Alt.2) and use LaSalle or the invariance set theorem!

The Circle Criterion, $0 < k_1 \leq k_2 < \infty$



Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y)$. Assume $G(s)$ is stable and that

$$k_1 \leq \frac{f(y)}{y} \leq k_2.$$

If the Nyquist curve of $G(s)$ stays outside the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable.

The other cases

G**: stable system**

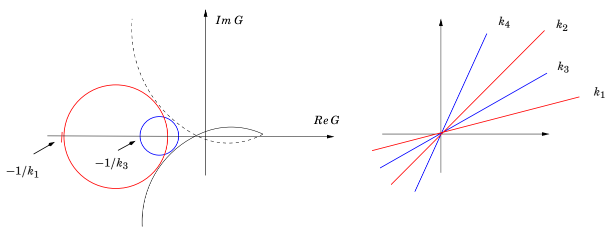
- $0 < k_1 < k_2$: Stay outside circle
- $0 = k_1 < k_2$: Stay to the right of the line Re $s = -1/k_2$
- $k_1 < 0 < k_2$: Stay inside the circle

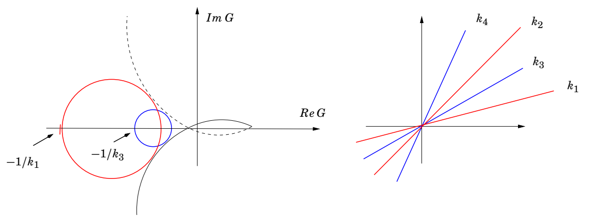
Other cases: Multiply f and G with -1.

Circle criterion / Sector conditions

What does it mean that we can get different sectors when using the circle criterion for a nonlinearity in feedback with a (fixed) linear system?

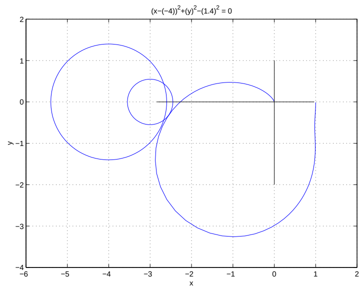
Can I have many different sector conditions, and what does that mean?



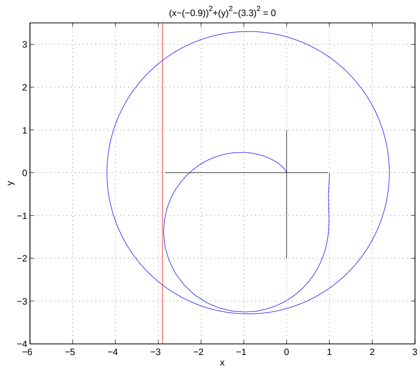


In the example above, the circle criterion can guarantee absolute stability for a nonlinearity which is bounded to **either** the sector $[k_1, k_2]$ **or** $[k_3, k_4]$ or in many other sectors, but NOT for a nonlinearity which is allowed to have a full variation within the sector $[k_1, k_4]$.

Example: $G(s) = \dfrac{1000}{(s+10)(s^2+2s+100)}$
in negative feedback with a sector bounded nonlinearity.



sector [0.28, 0.41] or [0.19, 0.38]



sector [0, 1/2.9] or [-0.41, 0.23]

Small gain theorem will give symmetric sector [-0.27, 0.27] as $||G|| = 3.69$.

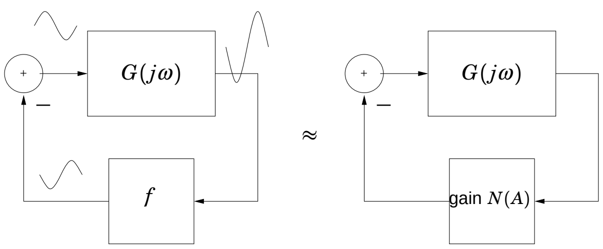
Questions

Is it possible to draw phase portraits for systems of higher order than two?

Can the describing function method be improved by including more coefficients from the Fourier series expansion?

Are there criteria to verify the low-pass character needed in a describing function argument?

Idea of Describing Function

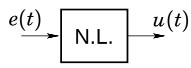


Find frequency ω and amplitude A such that

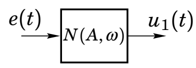
$$G(j\omega) \cdot N(A) = -1$$

Idea of Describing Function

$$e(t) = A \sin \omega t = \text{Im}\ (Ae^{i\omega t})$$



$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

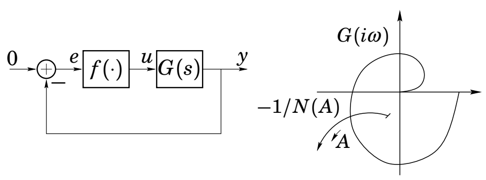


$$u_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t) = \text{Im}\ (N(A,\omega)Ae^{i\omega t})$$

where the **describing function** is defined as

$$N(A,\omega) = \frac{b_1(\omega) + ia_1(\omega)}{A} \implies U(i\omega) \approx N(A,\omega)E(i\omega)$$

Existence of Limit Cycles

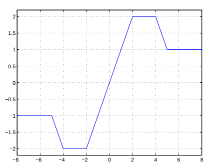


$$y = G(i\omega)u = -G(i\omega)N(A)y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of $G(i\omega)$ and $-1/N(A)$ give ω and A for *possible* limit cycles.

Harder if N is a function of both A and ω.

Example from exam 20090601 (a)



Which one of the three describing functions below corresponds to the nonlinearity $f(x)$ above?

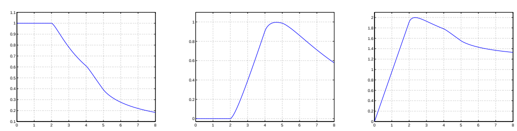
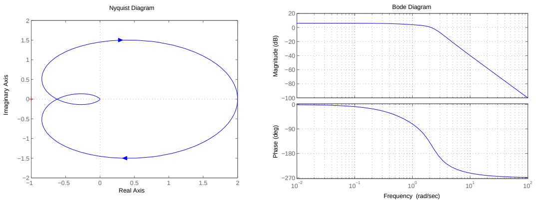


Figure: Describing functions $1 - -3$

Example from exam 20090601 (b)

Below we have the Nyquist and Bode curves of a linear system G. Assume that there exists non-linearities corresponding to to the three describing functions on previous page, and that each of these would be used in a negative feedback connection with G. For which do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?

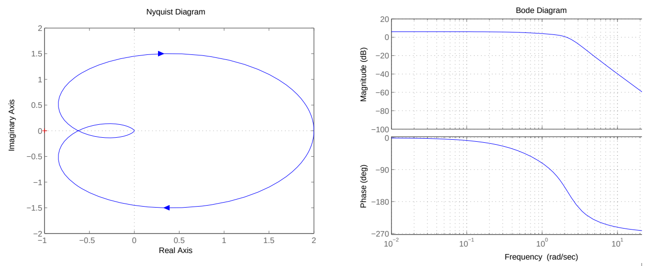


Since the third describing function fulfills that $-\frac{1}{N(2)} = -\frac{1}{2}$ and $G(i\omega_o) \approx -0.6$, we understand that we have **two** intersections. The first intersection occurs when $A \approx 1.8$ and the second intersection occurs when $A \approx 4.5$.

Examining the describing function around the first intersection, we see that $-\frac{1}{N(A)}$ goes from the outside of $G(i\omega)$ to the inside, with increasing A. Hence, we conclude that the possible limit cycle at $A \approx 1.8$ is unstable. By similar argument, we understand that the possible limit cycle at $A \approx 4.5$ is stable.

What would the corresponding frequency of the limit cycles in **(b)** be?

The frequency of all possible limit cycles is approximately 2.5 rad/s. To understand this, we see in the Bode plot that for $\omega \approx 2.5$ we have that $\arg(G(i\omega)) \approx -180$.

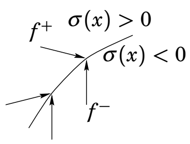


Question

Please repeat the most important facts about sliding modes.

Sliding Modes

$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



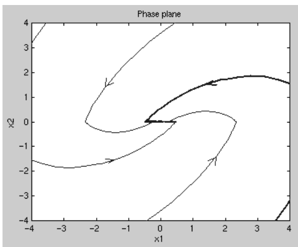
The **sliding set** is where $\sigma(x) = 0$ and f^+ and f^- point towards $\sigma(x) = 0$.

The sliding dynamics are $\dot{x} = \alpha f^+ + (1-\alpha) f^-$, where α is obtained from $\dot{\sigma} = \frac{\partial \sigma}{\partial x}\dot{x} = 0$.
(More precisely, find α such that the components of f^+ and f^- perpendicular to the switching surface cancel.)

Sliding Mode Dynamics

The dynamics along the sliding surface $\sigma(x) = 0$ can also be obtained by setting $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.



Phase plane for example in lecture 12.

Example

$$\begin{aligned} \dot{x}_1 &= 1 - u/4 \\ \dot{x}_2 &= u, \\ u &= -\text{sign } x_2, \quad (\text{i.e.}, \sigma(x) = x_2) \end{aligned} \tag{1}$$

What is the *sliding set* and what is *the sliding dynamics* for the system above?

If

$$\sigma(x) > 0 \Rightarrow u = -1 \Rightarrow f^+ = \begin{bmatrix} 5/4 \\ -1 \end{bmatrix}$$

$$\sigma(x) < 0 \Rightarrow u = +1 \Rightarrow f^- = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

The sliding set:

Find those values of the states at the switching curve for which

$$\nabla\sigma \cdot f^+ < 0$$

and

$$\nabla\sigma \cdot f^- > 0$$

(means that the vector fields on either side of $\sigma(x)$ points towards $\sigma(x)$, i.e., the normal projection of f^+ is negative and the normal projection of f^- on $\sigma(x)$ is positive). If these conditions are not fulfilled we will just “flow through $\sigma(x)$...”

In this example all the values along $x_2 = 0$ will belong to the switching set. (Compare with example from lecture 9 where the switching set will be restricted to $x_2 = 0$ and $-1 \leq x_1 \leq 1$, see figure on slide above).

The sliding dynamics:

Alternative 1.a.: Solve via normal projection on σ:

Pick α such that for $\dot{x} = \alpha f^+ + (1-\alpha) f^-$, we get
$\dot{\sigma} = 0 \Rightarrow \dot{x}_2 = \alpha f_2^+ + (1-\alpha) f_2^- = 0$

Alternative 1.b.: (same thing put in other words)
(The normal component on either side of the switch curve should balance out each other).

$$f_n^+ = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; f_n^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\alpha f_n^+ + (1-\alpha) f_n^- = 0$ gives $\alpha = 1/2$,
hence $\dot{x} = \alpha f^+ + (1-\alpha) f^-$ and $\dot{x}_1 = 1$ is the sliding dynamics.

Alternative 2: Solve via Equivalent control

$\dot{\sigma}(x)_{u=u_{eq}} = 0$ and $\dot{\sigma} = \dot{x}_2 = u \Rightarrow u_{eq} = 0$.

Hence $\dot{x}_1 = 1 - u_{eq}/4 = 1$ is the sliding dynamics.

Problem Formulation (1)

$$\text{Minimize } \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t))$$
$$u(t) \in U, \quad 0 \le t \le t_f, \qquad t_f \text{ given}$$
$$x(0) = x_0$$

$x(t) \in R^n$, $u(t) \in R^m$

U control constraints

Problem Formulation (2)

As in (1) but with additions:

- r end constraints

$$\Psi(x(t_f)) = \begin{pmatrix} \Psi_1(x(t_f)) \\ \vdots \\ \Psi_r(x(t_f)) \end{pmatrix} = 0$$

- free end time t_f

Free end time t_f

If the choice of t_f is included in the optimization then there is an extra constraint:

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = -n_0 \phi_t(x^*(t_f)) - \mu^T \Psi_t(x^*(t_f))$$

Note that for the special case where ϕ and Ψ are time-invariant this reduces to

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$$

Question

Please repeat optimal control with some additional example

The Maximum Principle (18.2)

Introduce the **Hamiltonian**

$$H(x, u, \lambda) = L(x, u) + \underbrace{\lambda^T(t)}_{1 \times n} \underbrace{f(x, u)}_{n \times 1}.$$

Suppose optimization problem (1) has a solution $u^*(t), x^*(t)$.
Then the optimal solution must satisfy

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$\dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

where $H_x = \frac{\partial H}{\partial x} = [\frac{\partial H}{\partial x_1} \dots \frac{\partial H}{\partial x_n}]$, $\phi_x = \frac{\partial \phi}{\partial x}$.

The Maximum Principle–General Case (18.4)

Introduce the Hamiltonian

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

Suppose optimization problem (2) has a solution $u^*(t), x^*(t)$.
Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$, and a vector $\mu \in R^r$ so that $[n_0 \;\; \mu^T] \ne 0$ and

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where

$$\dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0)$$
$$\lambda(t_f) = n_0 \phi_x^T(x^*(t_f)) + \Psi_x^T(x^*(t_f)) \mu$$

Two cases: $n_0 = 1$ or $n_0 = 0$

Example: Optimal storage control

$$\text{Minimize } \int_0^{t_f} [u(t) e^{rt} + c x(t)] dt$$

$$\text{subject to } \begin{cases} \dot{x} = u & 0 \le u \le M \\ x(0) = 0 \\ x(t_f) \ge A \end{cases}$$

$x =$ stock size
$u =$ production rate
$r =$ production cost growth rate
$c =$ storage cost

Optimal storage control II

Hamiltonian

$$H = L(x,u) + \lambda(t)^T f(x,u) = ue^{rt} + cx + \lambda u$$

Adjoint equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -c$$
$$\lambda = b - ct$$

At optimality

$$H(x^*, u^*, \lambda) = \min_u H(x^*, u, \lambda) = cx + \min_u [u(e^{rt} + b - ct)]$$

$$u^* = \begin{cases} M \text{ if } e^{rt} + b - ct < 0 \\ 0 \text{ if } e^{rt} + b - ct > 0 \end{cases}$$

Optimal storage control III

$$u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\ 0 & \text{if } e^{rt} + b - ct > 0 \end{cases} = \begin{cases} 0 & 0 \le t \le t_1 \\ M & t_1 \le t \le t_2 \\ 0 & t_2 \le t \le t_f \end{cases}$$

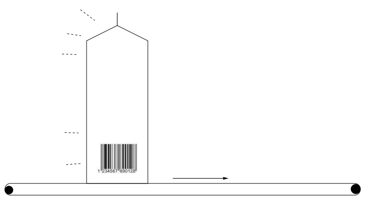
The condition $x(t_f) \ge A$ gives that $M(t_2 - t_1) = A$.

What remains is the scalar optimization problem

$$\min_{0 \le s \le A/M} \left[\int_s^{s+A/M} M(e^{rt} + ct)dt + \int_{s+A/M}^{t_f} cAdt \right]$$

Optimal Control: tuning at Tuna

Former economy student Cecilia is now working as controller at ICA Tuna. She notices that there are some problems with the new conveyor belts. To save time at the cashier the new conveyor belts were set up to be much faster than the old ones, but instead of increasing the flow there are now problems with the milk containers which are tipping over [1] and this slows down the procedure at the cashier.



[1](which in some rare cases indeed increases the flow)

Her old engineering friend H A Milton heard about the problem and told her about a "beautiful principle" which could be used for solving these kinds of optimization problems.

The new laser scanners can detect the EAN code (so called "streckkod") if the container passes at speeds below 0.2 m/s.

The conveyor belt is torque controlled, so it can be accelerated directly by the control signal u according to

$$\ddot{x} = u$$

where x is the position.

A crude approximation gives that the control signal should be bounded by $|u| < 5$ for the container not to tip over.

(a) Help Cecilia to formulate the optimization problem of moving a container from $x(0) = \dot{x}(0) = 0$ to $x(T) = 0.5\,m$, $\dot{x}(T) = 0.2\,m/s$ as fast as possible, without tipping over the container.

Heuristic reasoning to get bounds on the optimal time: **(b)** What would the control signal look like if we were to move the container 0.5 m as fast as possible? How could it look like if you were to move it from "rest to rest" (zero velocity at both start and end of motion)?

(c)

Solve for u which optimizes the criterion in **(a)**. For full point you should characterize how u depend on time. The optimal time should of course be within the bounds you found in **(b)**, but you do not need to explicitly solve for T here.

solution of Tuning at Tuna

We take the state vector to be $(x_1, x_2) = (x, \dot{x})$ then

$$\min_u \int_0^T 1dt$$

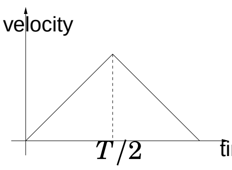
subject to $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x_1(0) = x_2(0) = 0, \; |u| < 5$

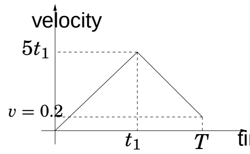
and $x_1(T) = 0.5$, $x_2(T) = 0.2$,
i.e. $\phi(x) = 0$, $\psi_1(x) = x_1 - 0.5$, $\psi_2(x) = x_2 - 0.2$

(a) To move the distance 0.5 m in as short time as possible, we should of course use the maximal acceleration all the time.
$\ddot{x} = 5$, $x(0) = 0$, $\dot{x}(0) = 0 \implies x(t) = 5t^2/2$.
$x(T) = 0.5 \implies 5T^2/2 = 0.5 \implies T \approx 0.45 sec$.

To instead get to rest at the end, we could first use max acceleration for half the time and then max retardation.

The velocity profile would then look like Fig. 50, where the area under the triangle ($= u_{max}T^2/4$) corresponds to the distance. The slope of the velocity is the max acceleration $u_{max} = 5$. It will thus take $T = \sqrt{2/5} \approx 0.63$ seconds.





(c) We first consider the normal case, i.e. $n_0 = 1$. The Hamiltonian is given by

$$H(y, u, \lambda) = L + \lambda^T f(x) = 1 + \lambda_1 x_2 + \lambda_2 u$$

At optimality we should minimize H with respect to u within its bounds. We see that if $\lambda_2 < 0$ we should choose $u = +5$ and if $\lambda_2 > 0$ we should choose $u = -5$. (same solution in this case if $n_0 = 0$). The adjoint equations are given by

$$\dot{\lambda} = -\frac{\partial H(x, u, \lambda)^T}{\partial x}, \qquad \lambda(T) = \frac{\partial \phi}{\partial x} + \mu^T \frac{\partial \Psi(T, x(T))}{\partial x}$$

$$\begin{aligned} \dot{\lambda}_1 &= 0, & \lambda_1(T) &= 0 + \mu_1 \cdot 1 + \mu_2 \cdot 0 \\ \dot{\lambda}_2 &= -\lambda_1, & \lambda_2(T) &= 0 + \mu_1 \cdot 0 + \mu_2 \cdot 1 \end{aligned} \quad (2)$$

Integrating, we get

$$\lambda_1(t) = \mu_1$$

Then

$$\lambda_2(t) = -\mu_1 t + C, \quad \lambda_2(T) = -\mu_1 \cdot T + C = \mu_2 \implies \lambda_2(t) = \mu_1(T - t) + \mu_2$$

Note that λ_2 increases or decreases linearly with time and may shift sign at most once. From the minimization of H we thus can conclude that u should shift sign when $\mu_1(T - t_1) + \mu_2 = 0$.

The velocity profile is showed in Fig. 50, where the switch from $u = 5$ to $u = -5$ occurs at time t_1. The total area under the curve should be 0.5 and the velocity $v_{end} = 0.2$

$$\begin{gathered} 0.5 = 5t_1^2/2 + (5t_1 - v_{end})(T - t_1)/2 + v_{end}(T - t_1) \\ v_{end} = 0.2 = 5t_1 - 5(T - t_1)5 \\ \implies t_1 \approx 0.32,\ T \approx 0.59 \end{gathered} \quad (3)$$