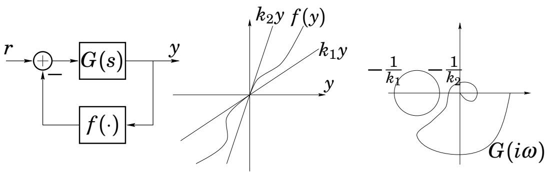
Lecture 5 — Input–output stability

or

"How to make a circle out of the point $-1+0i$, and different ways to stay away from it ..."



Today's Goal

To understand

- *signal norms*
- *system gain*
- *bounded input bounded output (BIBO) stability*

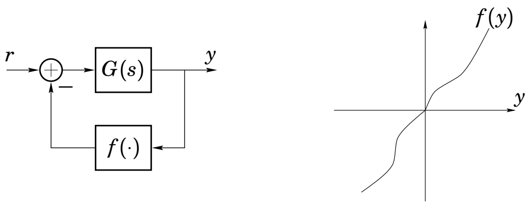
To be able to analyze stability using

- *the Small Gain Theorem,*
- *the Circle Criterion,*
- *Passivity*

Material

- *[Glad & Ljung]:* Ch 1.5-1.6, 12.3
  [Khalil]: Ch 5–7.1; *[Slotine & Li]:* Ch.4.7–4.8
- lecture slides

History

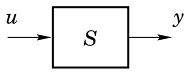


For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- Lur'e and Postnikov's problem (1944)
- Aizerman's conjecture (1949) (False!)
- Kalman's conjecture (1957) (False!)
- Solution by Popov (1960) (Led to the Circle Criterion)

Gain

Idea: Generalize static gain to nonlinear dynamical systems



The gain γ of S should tell what is the largest amplification from u to y

Here S can be a constant, a matrix, a linear time-invariant system, etc

Question: How should we measure the size of u and y?

Norms

A norm $\|\cdot\|$ measures size.

A **norm** is a function from a space Ω to $\mathbf{R}^+$, such that for all $x, y \in \Omega$

- $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
- $\|x+y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $\alpha \in \mathbf{R}$

Examples
Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$
Max norm: $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$

Signal Norms

A signal $x(t)$ is a function from $\mathbf{R}^+$ to $\mathbf{R}$.
A signal norm is a way to measure the size of $x(t)$.

Examples
2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}$
sup-norm: $\|x\|_\infty = \sup_{t\in\mathbf{R}^+} |x(t)|$

The space of signals with $\|x\|_2 < \infty$ is denoted $\mathcal{L}_2$.

Parseval's Theorem

Theorem If $x, y \in \mathcal{L}_2$ have the Fourier transforms

$$X(i\omega) = \int_0^\infty e^{-i\omega t} x(t) dt, \qquad Y(i\omega) = \int_0^\infty e^{-i\omega t} y(t) dt,$$

then

$$\int_0^\infty y^T(t) x(t) dt = \frac{1}{2\pi} \int_{-\infty}^\infty Y^*(i\omega) X(i\omega) d\omega.$$

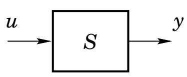
In particular,

$$\|x\|_2^2 = \int_0^\infty |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |X(i\omega)|^2 d\omega.$$

$\|x\|_2 < \infty$ corresponds to bounded energy.

System Gain

A system S is a map between two signal spaces: $y = S(u)$.

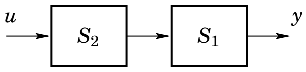


The gain of S is defined as $\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$

Example The gain of a static relation $y(t) = \alpha u(t)$ is

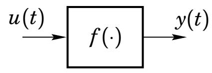
$$\gamma(\alpha) = \sup_{u\in\mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{|\alpha| \|u\|_2}{\|u\|_2} = |\alpha|$$

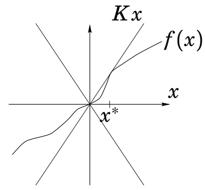
2 minute exercise: *Show that* $\gamma(S_1 S_2) \le \gamma(S_1)\gamma(S_2)$.



Example—Gain of a Static Nonlinearity

$$|f(x)| \le K|x|, \qquad f(x^*) = Kx^*$$



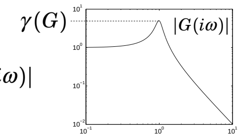


$$\|y\|_2^2 = \int_0^\infty f^2(u(t))dt \le \int_0^\infty K^2u^2(t)dt = K^2\|u\|_2^2$$

$u(t) = x^*$, $t \in (0,\infty)$ gives equality $\Rightarrow$
$\gamma(f) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = K$.

Example—Gain of a Stable Linear System

$$\gamma(G) = \sup_{u\in\mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega\in(0,\infty)} |G(i\omega)|$$

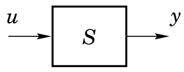


Proof: Assume $|G(i\omega)| \le K$ for $\omega \in (0,\infty)$ and $|G(i\omega_*)| = K$ for some ω_*. Parseval's theorem gives

$$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2|U(i\omega)|^2 d\omega \le K^2\|u\|_2^2$$

Equality by choosing $u(t) = \sin\omega_* t$.

BIBO Stability

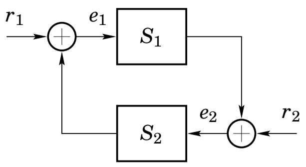


$$\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}$$

Definition
S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.

Example: If $\dot{x} = Ax$ is asymptotically stable then $G(s) = C(sI - A)^{-1}B + D$ is BIBO stable.

The Small Gain Theorem



Theorem
Assume S_1 and S_2 are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from (r_1, r_2) to (e_1, e_2) is BIBO stable.

"Proof" of the Small Gain Theorem

Existence of solution (e_1, e_2) for every (r_1, r_2) has to be verified separately. Then

$$\|e_1\|_2 \le \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]$$

gives

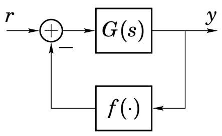
$$\|e_1\|_2 \le \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1 - \gamma(S_2)\gamma(S_1)}$$

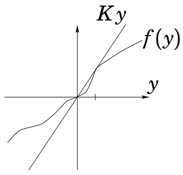
$\gamma(S_2)\gamma(S_1) < 1$, $\|r_1\|_2 < \infty$, $\|r_2\|_2 < \infty$ give $\|e_1\|_2 < \infty$.
Similarly we get

$$\|e_2\|_2 \le \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1 - \gamma(S_1)\gamma(S_2)}$$

so also e_2 is bounded.

Linear System with Static Nonlinear Feedback (1)



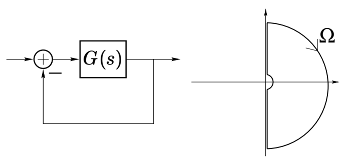


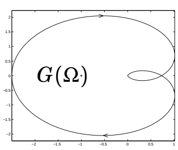
$$G(s) = \frac{2}{(s+1)^2} \quad \text{and} \quad 0 \le \frac{f(y)}{y} \le K$$

$\gamma(G) = 2$ and $\gamma(f) \le K$.

The small gain theorem gives that $K \in [0, 1/2)$ implies BIBO stability.

The Nyquist Theorem

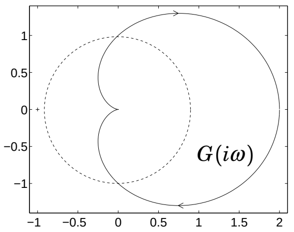




Theorem
The closed loop system is stable iff the number of counter-clockwise encirclements of -1 by $G(\Omega)$ (note: ω increasing) equals the number of open loop unstable poles.

The Small Gain Theorem can be Conservative

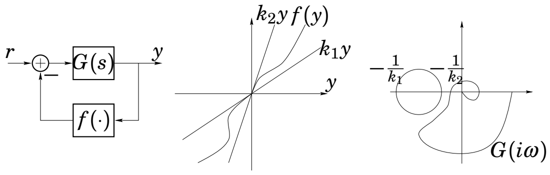
Let $f(y) = Ky$ for the previous system.



The Nyquist Theorem proves stability when $K \in [0, \infty)$.
The Small Gain Theorem proves stability when $K \in [0, 1/2)$.

The Circle Criterion

Case 1: $0 < k_1 \le k_2 < \infty$



Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y) + r$. Assume $G(s)$ is stable and that

$$0 < k_1 \le \frac{f(y)}{y} \le k_2.$$

If the Nyquist curve of $G(s)$ does not intersect or encircle the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable from r to y.

Other cases

G: stable system

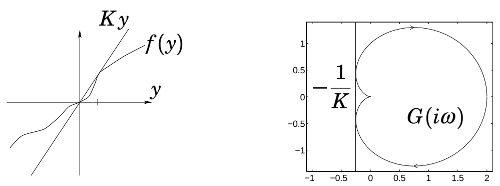
- $0 < k_1 < k_2$: Stay outside circle
- $0 = k_1 < k_2$: Stay to the right of the line Re $s = -1/k_2$
- $k_1 < 0 < k_2$: Stay inside the circle

Other cases: Multiply f and G with -1.

G: Unstable system
To be able to guarantee stability, k_1 and k_2 must have same sign (otherwise unstable for $k = 0$)

- $0 < k_1 < k_2$: Encircle the circle p times counter-clockwise (if ω increasing)
- $k_1 < k_2 < 0$: Encircle the circle p times counter-clockwise (if ω increasing)

Linear System with Static Nonlinear Feedback (2)



The "circle" is defined by $-1/k_1 = -\infty$ and $-1/k_2 = -1/K$.

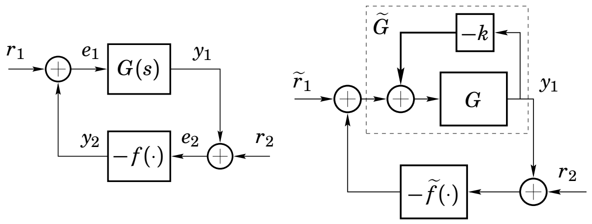
$$\min \text{Re}\, G(i\omega) = -1/4$$

so the Circle Criterion gives that if $K \in [0, 4)$ the system is BIBO stable.

Proof of the Circle Criterion

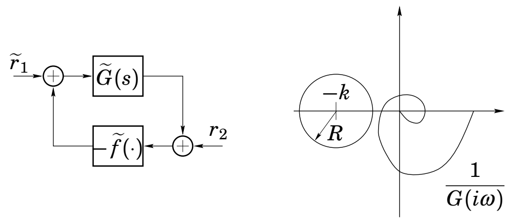
Let $k = (k_1 + k_2)/2$ and $\widetilde{f}(y) = f(y) - ky$. Then

$$\left| \frac{\widetilde{f}(y)}{y} \right| \le \frac{k_2 - k_1}{2} =: R$$



$$\widetilde{r}_1 = r_1 - kr_2$$

Proof of the Circle Criterion (cont'd)



SGT gives stability for $|\widetilde{G}(i\omega)|R < 1$ with $\widetilde{G} = \dfrac{G}{1 + kG}$.

$$R < \frac{1}{|\widetilde{G}(i\omega)|} = \left| \frac{1}{G(i\omega)} + k \right|$$

Transform this expression through $z \to 1/z$.

Lyapunov revisited

Original idea: "Energy is decreasing"

$$\begin{aligned} &\dot{x} = f(x), \qquad x(0) = x_0 \\ &V(x(T)) - V(x(0)) \le 0 \\ &(+\text{some other conditions on } V) \end{aligned}$$

New idea: "Increase in stored energy $\le$ added energy"

$$\begin{aligned} \dot{x} &= f(x,u), \qquad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

$$V(x(T)) - V(x(0)) \le \int_0^T \underbrace{\varphi(y,u)}_{\text{external power}} \, dt \qquad (1)$$

Motivation

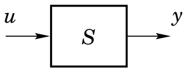
Will assume the external power has the form $\phi(y, u) = y^T u$.

Only interested in BIBO behavior. Note that

$$\exists V \ge 0 \text{ with } V(x(0)) = 0 \text{ and } (\textbf{??})$$
$$\Longleftrightarrow$$
$$\int_0^T y^T u \, dt \ge 0$$

Motivated by this we make the following definition

Passive System



Definition The system S is **passive** from u to y if

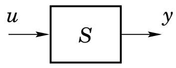
$$\int_0^T y^T u\,dt \ \ge\ 0, \quad \text{for all } u \text{ and all } T > 0$$

and **strictly passive** from u to y if there $\exists \epsilon > 0$ such that

$$\int_0^T y^T u\,dt \ \ge\ \epsilon(|y|_T^2 + |u|_T^2), \quad \text{for all } u \text{ and all } T > 0$$

A Useful Notation

Define the **scalar product**



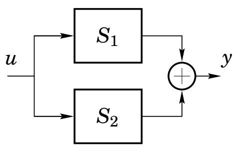
$$\langle y, u\rangle_T = \int_0^T y^T(t)u(t)\,dt$$

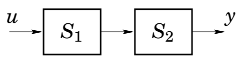
Cauchy-Schwarz inequality:

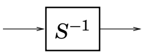
$$\langle y, u\rangle_T \le |y|_T |u|_T$$

where $|y|_T = \sqrt{\langle y, y\rangle_T}$. Note that $|y|_\infty = \|y\|_2$.

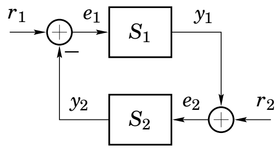
2 minute exercise:







Feedback of Passive Systems is Passive



If S_1 and S_2 are passive, then the closed-loop system from (r_1, r_2) to (y_1, y_2) is also passive.

Proof:
$$\begin{aligned}\langle y, r\rangle_T &= \langle y_1, r_1\rangle_T + \langle y_2, r_2\rangle_T \\ &= \langle y_1, r_1 - y_2\rangle_T + \langle y_2, r_2 + y_1\rangle_T \\ &= \langle y_1, e_1\rangle_T + \langle y_2, e_2\rangle_T \ge 0\end{aligned}$$

Hence, $\langle y, r\rangle_T \ge 0$ if $\langle y_1, e_1\rangle_T \ge 0$ and $\langle y_2, e_2\rangle_T \ge 0$

Passivity of Linear Systems

Theorem An asymptotically stable linear system $G(s)$ is **passive** if and only if

$$\text{Re}\, G(i\omega) \ge 0, \qquad \forall \omega > 0$$

It is **strictly passive** if and only if there exists $\epsilon > 0$ such that

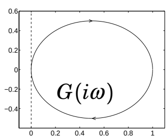
$$\text{Re}\, G(i\omega - \epsilon) \ge 0, \qquad \forall \omega > 0$$

Proof: See Slotine and Li p. 139 for the first part.

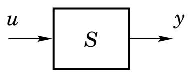
Example

$G(s) = \dfrac{1}{s+1}$ is passive and strictly passive,

$G(s) = \dfrac{1}{s}$ is passive but not strictly passive.



A Strictly Passive System Has Finite Gain



If S is strictly passive, then $\gamma(S) < \infty$.

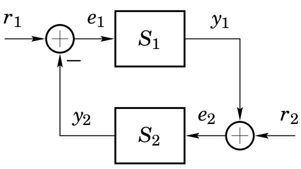
Proof: Note that $\|y\|_2 = \lim_{T\to\infty} |y|_T$.

$$\epsilon(|y|_T^2 + |u|_T^2) \le \langle y, u\rangle_T \le |y|_T \cdot |u|_T \le \|y\|_2 \cdot \|u\|_2$$

Hence, $\epsilon |y|_T^2 \le \|y\|_2 \cdot \|u\|_2$, so letting $T \to \infty$ gives

$$\|y\|_2 \le \frac{1}{\epsilon}\|u\|_2$$

The Passivity Theorem



Theorem If S_1 is strictly passive and S_2 is passive, then the closed-loop system is BIBO stable from r to y.

Proof of the Passivity Theorem

S_1 strictly passive and S_2 passive give

$$\epsilon\left(|y_1|_T^2 + |e_1|_T^2\right) \le \langle y_1, e_1\rangle_T + \langle y_2, e_2\rangle_T = \langle y, r\rangle_T$$

Therefore

$$|y_1|_T^2 + \langle r_1 - y_2, r_1 - y_2\rangle_T \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

or

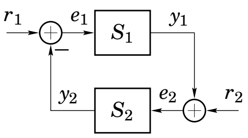
$$|y_1|_T^2 + |y_2|_T^2 - 2\langle y_2, r_2\rangle_T + |r_1|_T^2 \le \frac{1}{\epsilon}\langle y, r\rangle_T$$

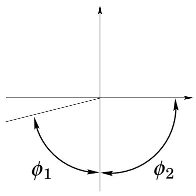
Finally

$$|y|_T^2 \le 2\langle y_2, r_2\rangle_T + \frac{1}{\epsilon}\langle y, r\rangle_T \le \left(2 + \frac{1}{\epsilon}\right)|y|_T |r|_T$$

Letting $T \to \infty$ gives $\|y\|_2 \le C\|r\|_2$ and the result follows

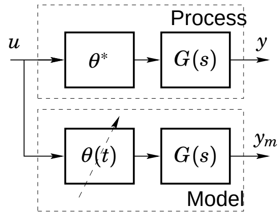
Passivity Theorem is a "Small Phase Theorem"





Example—Gain Adaptation

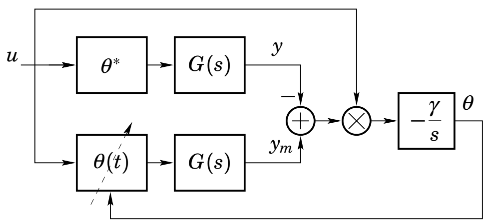
Applications in channel estimation in telecommunication, noise cancelling etc.



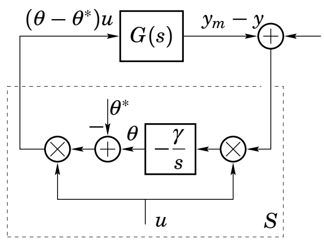
Adaptation law:

$$\frac{d\theta}{dt} = -\gamma u(t)[y_m(t) - y(t)], \qquad \gamma > 0.$$

Gain Adaptation—Closed-Loop System



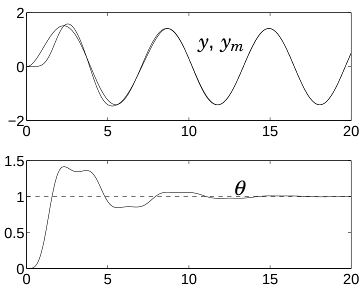
Gain Adaptation is BIBO Stable



S is passive (Exercise 4.12), so the closed-loop system is BIBO stable if $G(s)$ is strictly passive.

Simulation of Gain Adaptation

Let $G(s) = \dfrac{1}{s+1} + \epsilon$, $\gamma = 1$, $u = \sin t$, $\theta(0) = 0$ and $\gamma^* = 1$



Storage Function

Consider the nonlinear control system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

A **storage function** is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(0) = 0$ and $V(x) \geq 0, \quad \forall x \neq 0$
- $\dot{V}(x) \leq u^T y, \quad \forall x, u$

Remark:

- $V(T)$ represents the stored energy in the system
- $$\underbrace{V(x(T))}_{\text{stored energy at } t=T} \leq \underbrace{\int_0^T y(t)u(t)dt}_{\text{absorbed energy}} + \underbrace{V(x(0))}_{\text{stored energy at } t=0}, \quad \forall T > 0$$

Storage Function and Passivity

Lemma: If there exists a storage function V for a system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

with $x(0) = 0$, then the system is passive.

Proof: For all $T > 0$,

$$\langle y, u \rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$

Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: "Energy is decreasing"

$$\dot{V} \leq 0$$

Passivity idea: "Increase in stored energy $\leq$ Added energy"

$$\dot{V} \leq u^T y$$

Example KYP Lemma

Consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Assume there exists positive definite symmetric matrices P, Q such that

$$A^T P + PA = -Q, \text{ and } B^T P = C$$

Consider $V = 0.5x^T Px$. Then

$$\begin{aligned} \dot{V} &= 0.5(\dot{x}^T Px + x^T P\dot{x}) = 0.5x^T(A^T P + PA)x + u^T B^T Px \\ &= -0.5x^T Qx + u^T y < u^T y, \quad x \neq 0 \end{aligned} \tag{2}$$

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.

Next Lecture

- Describing functions (analysis of oscillations)