Lecture 2

- Linearization
- Stability definitions
- Simulation in Matlab/Simulink

Material

- Glad& Ljung Ch. 11, 12.1,
(Khalil Ch 2.3, part of 4.1, and 4.3)
(Slotine and Li, pp 40-57)
- Lecture slides

Today's Goal

To be able to

- *linearize, both around equilibria and trajectories,*
- *explain definitions of local and global stability,*
- *check local stability and local controllability at equilibria*
- *simulate in Simulink,*

Example - Linearization around equilibrium point

The linearization of

$$\ddot{x}(t) = f(x(t)) = \frac{g}{l}\sin x(t)$$

around the equilibrium $x_0 = n\pi$ is given by

$$\ddot{\tilde{x}}(t) = f'(x_0)\tilde{x}(t) = \frac{g}{l}(-1)^n\tilde{x}(t)$$

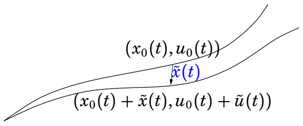
Linearization Around a Trajectory

Idea: Make Taylor-expansion around a known solution $\{x_0(t), u_0(t)\}$

Neglect small terms (*i.e.,* keep the linear terms, as these will locally dominate over the higher order terms).

Let $dx_0/dt = f(x_0(t), u_0(t))$ be a known solution. How will a small deviation $\{\tilde{x}, \tilde{u}\}$ from this solution behave?

$$\frac{d(x_0 + \tilde{x})}{dt} = f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$$



2 minute exercise: Linearize

$$\dot{x}^2 = 4x^3 + u$$

around the solution

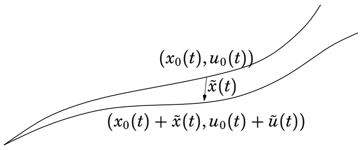
$$x_0(t) = \frac{1}{t^2} \qquad u_0(t) = 0$$

Hint:
Plug-in $x(t) = t^{-2} + \tilde{x}(t)$, expand the expressions, and finally remove higher order terms (≥ 2) of $\tilde{x}$.

Linearization Around a Trajectory, cont.

Let $(x_0(t), u_0(t))$ denote a solution to $\dot{x} = f(x, u)$ and consider another solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

$$\begin{aligned}\dot{x}(t) &= f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))\\ &= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t)\\ &\quad + \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + O(\|\tilde{x}, \tilde{u}\|^2)\end{aligned}$$



State-space form

Hence, for small $(\tilde{x}, \tilde{u})$, approximately

$$\dot{\tilde{x}}(t) = A(x_0(t), u_0(t))\tilde{x}(t) + B(x_0(t), u_0(t))\tilde{u}(t)$$

where (if dim $x = 2$, dim $u = 1$)

$$A(x_0(t), u_0(t)) = \frac{\partial f}{\partial x}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(x_0(t), u_0(t))}$$

$$B(x_0(t), u_0(t)) = \frac{\partial f}{\partial u}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\Bigg|_{(x_0(t), u_0(t))}$$

Note that A and B are **time dependent**! However, if we don't linearize around a trajectory but linearize around an equilibrium point $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ then A and B are constant.

Linearization, cont'd

The linearization of the output equation

$$y(t) = h(x(t), u(t))$$

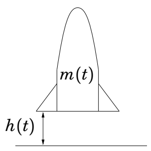
around the nominal output $y_0(t) = h(x_0(t), u_0(t))$ is given by

$$(y(t) - y_0(t)) = C(t)(x(t) - x_0(t)) + D(t)(u(t) - u_0(t))$$

where (if dim $y =$ dim $x = 2$, dim $u = 1$)

$$\begin{aligned} C(t) &= \frac{\partial h}{\partial x}\bigg|_{(x_0, u_0)} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix}\Bigg|_{(x_0(t), u_0(t))} \\ D(t) &= \frac{\partial h}{\partial u}\bigg|_{(x_0, u_0)} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial u_1} \end{bmatrix}\Bigg|_{(x_0(t), u_0(t))} \end{aligned}$$

Example: Rocket



$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g + \frac{v_e u(t)}{m(t)}$$
$$\dot{m}(t) = -u(t)$$

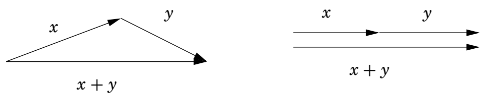
Let $u_0(t) \equiv u_0 > 0$; $x_0(t) = \begin{bmatrix} h_0(t) \\ v_0(t) \\ m_0(t) \end{bmatrix}$; $m_0(t) = m_0 - u_0 t.$

$$\text{Linearization:} \quad \dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-v_e u_0}{m_0(t)^2} \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ \frac{v_e}{m_0(t)} \\ -1 \end{bmatrix} \tilde{u}(t)$$

Part II: Stability definitions

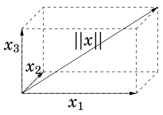
Definition: A *norm function* $||x|| : x \in R^n \to R$ satisfies the following three properties:

- $||x|| = 0$ if and only if (iff) $x = 0$, $\quad ||x|| > 0$ otherwise.
- $||\alpha x|| = \alpha ||x||$, for any positive α and any signal vector x.
- (The triangle inequality) $||x + y|| \le ||x|| + ||y||$



Definition: (Euclidean norm)

$$||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$



$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

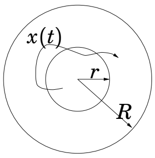
Local Stability

Consider $\dot{x} = f(x)$ where $f(0) = 0$

Definition The equilibrium $x = 0$ is **stable** if, for any $R > 0$, there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Longrightarrow \quad \|x(t)\| < R, \quad \text{for all } t \ge 0$$

Otherwise the equilibrium point is **unstable**.



Asymptotic Stability

Definition The equilibrium $x = 0$ is **locally asymptotically stable (LAS)** if it

1) is stable
2) there exists $r > 0$ so that if $\|x(0)\| < r$ then

$$x(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

(PhD-exercise: Show that 1) does not follow from 2))

Global Asymptotic Stability

Definition The equilibrium is said to be **globally asymptotically stable (GAS)** if it is LAS and for all $x(0)$ one has

$$x(t) \to 0 \text{ as } t \to \infty.$$

Part III: Check local stability and controllability

Lyapunov's Linearization Method

Theorem Assume

$$\dot{x} = f(x)$$

has the linearization

$$\frac{d}{dt}(x(t) - x_0) = A(x(t) - x_0)$$

around the equilibrium point x_0 and put

$$\alpha(A) = \max \text{Re}(\lambda(A))$$

- If $\alpha(A) < 0$, then $\dot{x} = f(x)$ is LAS at x_0,
- If $\alpha(A) > 0$, then $\dot{x} = f(x)$ is unstable at x_0,
- If $\alpha(A) = 0$, then no conclusion can be drawn.

(Proof in Lecture 4)

Example

The linearization of

$$\begin{aligned}\dot x_1 &= -x_1^2 + x_1 + \sin(x_2)\\ \dot x_2 &= \cos(x_2) - x_1^3 - 5x_2\end{aligned}$$

at $x_0 = \begin{pmatrix}1\\0\end{pmatrix}$ gives $A = \begin{pmatrix}-1 & 1\\ -3 & -5\end{pmatrix}$

Eigenvalues are given by the *characteristic equation*

$$0 = \det(\lambda I - A) = (\lambda+1)(\lambda+5)+3$$

This gives $\lambda = \{-2, -4\}$, which are both in the left half-plane, hence the *nonlinear system* is LAS around x_0.

Local Controllability

Theorem Assume

$$\dot x = f(x,u)$$

has the linearization

$$\frac{d}{dt}(x(t)-x_0) = A(x(t)-x_0) + B(u(t)-u_0)$$

around the equilibrium (x_0, u_0) then

- (A,B) controllable ⇒ $f(x,u)$ nonlinear locally controllable

Here **nonlinear locally controllable** is defined as:

For every $T > 0$ and $\varepsilon > 0$ the set of states $x(T)$ that can be reached from $x(0) = x_0$, by using controls satisfying $\|u(t)-u_0\| < \varepsilon$, contains a small ball around x_0.

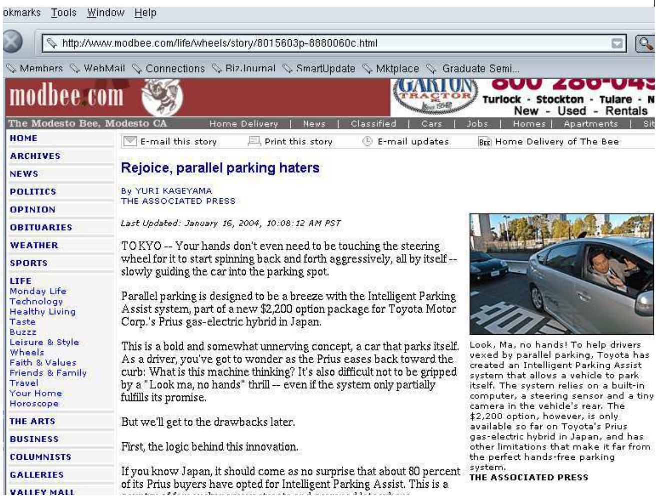
5 minute exercise:

Is the ball and beam

$$\frac{7}{5}\ddot x = x\dot\phi^2 + g\sin\phi + \frac{2r}{5}\ddot\phi$$

nonlinearly locally controllable around $\dot\phi = \phi = x = \dot x = 0$ (with $\ddot\phi$ as input)?

Remark: This is a bit bit more detailed model of the ball and beam than we saw in Lecture 1.



Rejoice, parallel parking haters

By YURI KAGEYAMA
THE ASSOCIATED PRESS

Last Updated: January 16, 2004, 10:08:12 AM PST

TOKYO -- Your hands don't even need to be touching the steering wheel for it to start spinning back and forth aggressively, all by itself -- slowly guiding the car into the parking spot.

Parallel parking is designed to be a breeze with the Intelligent Parking Assist system, part of a new $2,200 option package for Toyota Motor Corp.'s Prius gas-electric hybrid in Japan.

This is a bold and somewhat unnerving concept, a car that parks itself. As a driver, you've got to wonder as the Prius eases back toward the curb: What is this machine thinking? It's also difficult not to be gripped by a "Look ma, no hands" thrill -- even if the system only partially fulfills its promise.

But we'll get to the drawbacks later.

First, the logic behind this innovation.

If you know Japan, it should come as no surprise that about 80 percent of its Prius buyers have opted for Intelligent Parking Assist. This is a

Look, Ma, no hands! To help drivers vexed by parallel parking, Toyota has created an Intelligent Parking Assist system that allows a vehicle to park itself. The system relies on a built-in computer, a steering sensor and a tiny camera in the vehicle's rear. The $2,200 option, however, is only available so far on Toyota's Prius gas-electric hybrid in Japan, and has other limitations that make it far from the perfect hands-free parking system.
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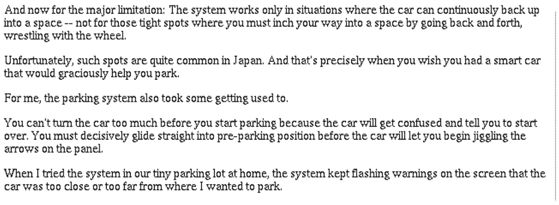
However...



And now for the major limitation: The system works only in situations where the car can continuously back up into a space -- not for those tight spots where you must inch your way into a space by going back and forth, wrestling with the wheel.

Unfortunately, such spots are quite common in Japan. And that's precisely when you wish you had a smart car that would graciously help you park.

For me, the parking system also took some getting used to.

You can't turn the car too much before you start parking because the car will get confused and tell you to start over. You must decisively glide straight into pre-parking position before the car will let you begin jiggling the arrows on the panel.

When I tried the system in our tiny parking lot at home, the system kept flashing warnings on the screen that the car was too close or too far from where I wanted to park.

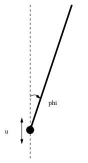
Bosch 2008 (Automatic parking assistance)

- Multiple turns
- parking lot $>$ car length + 80 cm

More parking in lecture 12

Example

An inverted pendulum with vertically moving pivot point



$$\ddot\phi(t) = \frac{1}{l}\left(g + u(t)\right)\sin(\phi(t)),$$

where $u(t)$ is acceleration, can be written as

$$\begin{aligned}\dot x_1 &= x_2\\ \dot x_2 &= \frac{1}{l}(g+u)\sin(x_1)\end{aligned}$$

Example, cont.

The linearization around $x_1 = x_2 = 0, u = 0$ is given by

$$\begin{aligned}\dot x_1 &= x_2\\ \dot x_2 &= \frac{g}{l}x_1\end{aligned}$$

It is not controllable, hence no conclusion can be drawn about nonlinear controllability

However, simulations show that the system is stabilized by

$$u(t) = \varepsilon\omega^2\sin(\omega t)$$

if ω is large enough !

$\boxed{\text{Demonstration}}$ We will come back to this example later.

Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at x^* if the linearization

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} \text{ has all eigenvalues in } |\lambda| < 1$$

(that is, within the unit circle).

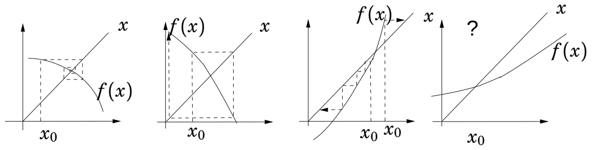
Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

$$x^* = f(x^*)$$

When does the iteration converge?



Part IV: Simulation

Often the only method

$\dot{x} = f(x)$

- ACSL
- Simnon
- Simulink

$F(\dot{x}, x) = 0$

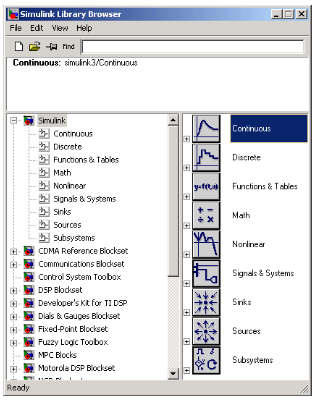
- Omsim
  `http://www.control.lth.se/~cace/omsim.html`
- Dymola `http://www.dynasim.se/`
- Modelica
  `http://www.dynasim.se/Modelica/index.html`

Special purpose

- Spice (electronics)
- EMTP (electromagnetic transients)
- Adams (mechanical systems)

Simulink

```
> matlab
>> simulink
```



Continuous: simulink3/Continuous

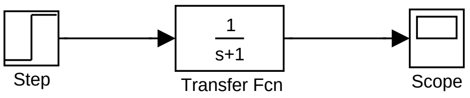
Simulink
Continuous
Discrete
Functions & Tables
Math
Nonlinear
Signals & Systems
Sinks
Sources
Subsystems
CDMA Reference Blockset
Communications Blockset
Control System Toolbox
DSP Blockset
Developer's Kit for TI DSP
Dials & Gauges Blockset
Fixed-Point Blockset
Fuzzy Logic Toolbox
MPC Blocks
Motorola DSP Blockset

Continuous
Discrete
Functions & Tables
Math
Nonlinear
Signals & Systems
Sinks
Sources
Subsystems

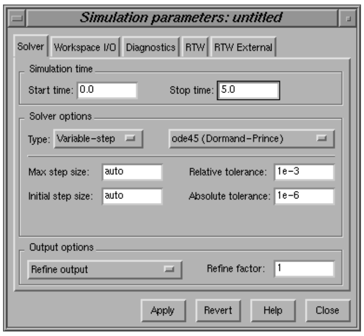
Ready

Simulink, An Example

```
File -> New -> Model
Double click on Continuous
Transfer Fcn
Step (in Sources)
Scope (in Sinks)
Connect (mouse-left)
Simulation->Parameters
```



Choose Simulation Parameters



Simulation time

Start time: 0.0 Stop time: 5.0

Solver options

Type: Variable−step ode45 (Dormand−Prince)

Max step size: auto Relative tolerance: 1e−3

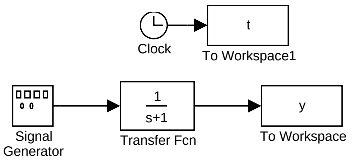
Initial step size: auto Absolute tolerance: 1e−6

Output options

Refine output Refine factor: 1

Don't forget "Apply"

Save Results to Workspace



Check "Save format" of output blocks ("Array" instead of "Structure")

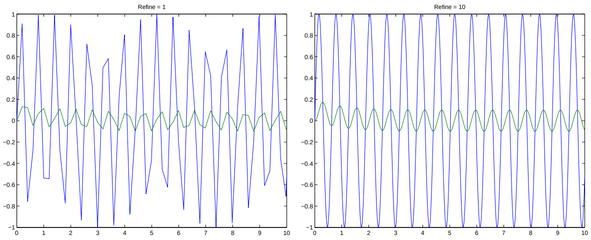
```
>> plot(t,y)
```

(or use "Structure" which also contains the time information.)

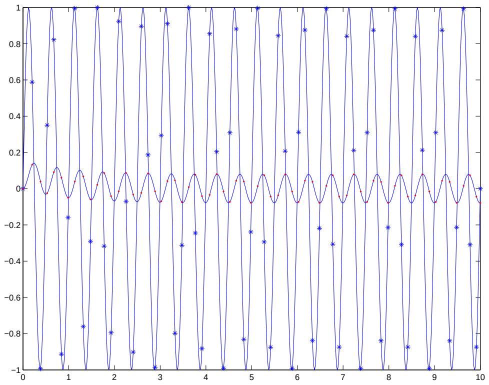
How To Get Better Accuracy

Modify Refine, Absolute and Relative Tolerances, Integration method

Refine adds interpolation points:



Points were correct, only the plot was bad



Use Scripts to Document Simulations

If the block-diagram is saved to `stepmodel.mdl`,
the following Script-file `simstepmodel.m` simulates the system:

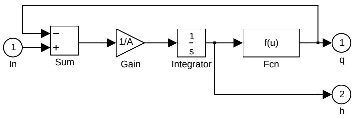
```
open_system('stepmodel')
set_param('stepmodel','RelTol','1e-3')
set_param('stepmodel','AbsTol','1e-6')
set_param('stepmodel','Refine','1')
tic
sim('stepmodel',6)
toc
subplot(2,1,1),plot(t,y),title('y')
subplot(2,1,2),plot(t,u),title('u')
```

Submodels, Example: Water tanks

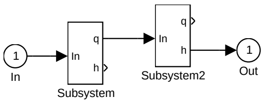
Equation for one water tank:

$$\begin{aligned} \dot{h} &= (u-q)/A \\ q &= a\sqrt{2g}\sqrt{h} \end{aligned}$$

Corresponding Simulink model:



Make a subsystem and connect two water tanks in series.



Linearization in Simulink

Use the command `trim` to find *e.g.,* stationary points to a system `>> A=2.7e-3;a=7e-6,g=9.8;`

```
>> % Example to find input u for desired states/output
>> [x0,u0,y0]=trim('flow',[0.1 0.1]',[],0.1)
x0 =
    0.1000
    0.1000
u0 =
    8.3996e-06
y0 =
    0.1000
```

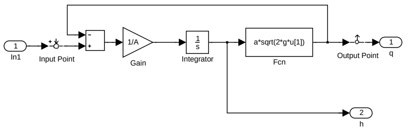
Linearization in Simulink, cont.

Use the command `linmod` to find a linear approximation of the system around an operating point:

```
>> [aa,bb,cc,dd]=linmod('flow',x0,u0);
>> sys=ss(aa,bb,cc,dd);
>> bode(sys)
```

Linearization in Simulink; Alternative

By right-clicking on a signal connector in a Simulink model you can add “Linearization points” (inputs and/or outputs).

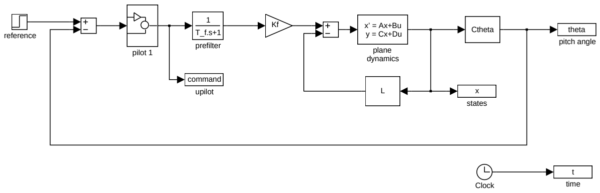


Start a “Control and Estimation Tool Manager” to get a linearized model by
`Tools -> Control Design ->Linear analysis ...`

where you can set the operating points, export linearized model to Workspace (Model-> Export to Workspace) and much more.

Computer exercise

Simulation of JAS 39 Gripen



- Simulation
- Analysis of PIO using describing functions
- Improve design

Today

- Linearization, both around equilibria and trajectories,
- Definitions of local and global stability,
- How to check local stability and local controllability at equilibria
- Simulation tool: Simulink,

Next Lecture

- Phase plane analysis
- Classification of equilibria