LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

Nonlinear Control and Servo Systems (FRTN05)

Exam - May 4, 2011 at 08.00–13.00

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: $12 - 16.5$ points

4: $17 - 21.5$ points

5: $22 - 25$ points

Accepted aid

All course material, except for the exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”. Pocket calculator.

Results

The exam results will be posted on the notice-board at the Department of Automatic Control and on the course homepage
`http://www.control.lth.se/course/FRTN05` within a week of the exam date. You will have an opportunity to see your corrected exam. See the course homepage for an exact date.

Note!
In many cases the sub-problems can be solved independently of each other.

Good Luck!

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
2011–05–04

1.

a. The system

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$
$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

is given. Assume that the stability properties of the origin are of interest. With the help of $V(x_1, x_2) = x_1^2 + x_2^2$, perform a stability analysis of the origin.

(2 p)

b. In the Nonlinear Control course much attention has, of course, been drawn to the study of nonlinear systems. Give two examples of phenomena that can occur for nonlinear systems but not for linear ones. (2 p)

Solution

a.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0, \quad \forall 0 < x_1^2 + x_2^2 < 2,$$

hence the system is locally asymptotically stable around the origin.

b.
1. Nonlinear systems can be locally stable, as for the previous example, and yet still not be globally stable.
2. Nonlinear systems can have finite escape time.

2. Consider the system

$$\dot{x}_1 = -x_2 - 2x_1 + u$$
$$\dot{x}_2 = x_1$$

Let $u = -\text{sign}(x_1 + x_2)$. Determine the sliding set and the dynamics on it. (3 p)

Solution

Let $\sigma(x) = x_1 + x_2$. The switching curve is given by all x such that $\sigma(x) = 0$. We have that $\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = -\sigma - \text{sign}(x_1 + x_2) = -\sigma - \text{sign}(\sigma)$. Therefore the sliding set is given by $\sigma = x_1 + x_2 = 0$. The dynamics on the set is given by

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2$$

That is, the sliding dynamics is asymptotically stable.

3. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_1^2 x_2 \\ \dot{x}_2 &= \mathrm{sat}(x_1) + u\end{aligned}$$

a. Verify that the system is on strict feedback form. (1 p)

b. Design a controller using backstepping to globally stabilize the origin. (2 p)

Solution

a. With

$$\begin{aligned} f_1(x_1) &= x_1^3 \\ g_1(x_1) &= x_1^2 \\ f_2(x_1, x_2) &= \mathrm{sat}(x_1) \\ g_2(x_1, x_2) &= 1 \end{aligned}$$

the system can be written on the strict feedback form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

b. Start with the system $\dot{x}_1 = x_1^3 + x_1^2\phi(x_1)$ which can be stabilized using $\phi(x_1) = -2x_1$. Notice that $\phi(0) = 0$. Take $V_1(x_1) = x_1^2/2$. To backstep, define

$$\zeta_2 = (x_2 - \phi(x_1)) = x_2 + 2x_1,$$

to transfer the system into the form

$$\begin{aligned} \dot{x}_1 &= x_1^3 + x_1^2(\zeta_2 - 2x_1) = -x_1^3 + x_1^2\zeta_2 \\ \dot{\zeta}_2 &= \dot{x}_2 + 2\dot{x}_1 = \mathrm{sat}(x_1) + u - 2x_1^3 + 2x_1^2\zeta_2 \end{aligned}$$

Taking $V = V_1(x_1) + \zeta_2^2/2$ as a Lyapunov function gives

$$\dot{V} = x_1(-x_1^3 + x_1^2\zeta_2) + \zeta_2(u + \mathrm{sat}(x_1) - 2x_1^3 + 2x_1^2\zeta_2)$$

With

$$u = -\mathrm{sat}(x_1) + x_1^3 - 2x_1^2\zeta_2 - \zeta_2 = -\mathrm{sat}(x_1) - 3x_1^3 - 2x_1^2x_2 - 2x_1 - x_2$$

we get

$$\dot{V} = -x_1^4 - \zeta_2^2 < 0 \quad \forall (x_1, \zeta_2) \neq 0$$

The Lyapunov function is radially unbounded. Hence, the origin is globally asymptotically stable.

4. Consider the equations of motion for a single linked robot:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u.$$

Assume that $M(q)$ is a positive scalar function for all $q \neq 0$, and that $\dot{M} - 2C = 0$.

a. Assuming that the gravity term $g(q) = 0$, show that the simple PD control law $u = -K(q - r) - K_d\dot{q}$, where r is constant, achieves asymptotic tracking (i.e. $\lim_{t\to\infty} q(t) = r$).
(*Hint* : Use the Lyapunov candidate $V = \frac{1}{2}\dot{q}^2 M(q) + \frac{1}{2}(q - r)^2 K$) (3 p)

b. Now assume that the gravitational term $g(q)$ is present. Can you conclude asymptotic tracking by using the same Lyapunov candidate? How can the PD control law from (**a**) be modified to achieve the same effect in (**b**)? (1 p)

c. Find a control law $u = \mu(q, \dot{q})$ such that

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

becomes a linear system. (1 p)

Solution

a. The time derivative of V is

$$\dot{V} = \dot{q}^T M\ddot{q} + 1/2\dot{q}^T \dot{M}\dot{q} + \dot{q}^T K(q-r)$$

Solving for $M\ddot{q}$ in the equations of motion formula and substituting the resulting expression into $\dot{V}$ yields

$$\dot{V} = \dot{q}^T(u + K(q-r)).$$

Substituting the PD control law for u now gives

$$\dot{V} = -\dot{q}^T K_d \dot{q} \leq 0. \qquad (1)$$

To conclude asymptotic stability, suppose that $\dot{V} = 0$. Then (1) implies that $\dot{q} = \ddot{q} = 0$. From the equations of motion with the PD control we must have that

$$0 = -K(q-r),$$

which implies that $(q-r) = 0$. From LaSalle's theorem the equilibrium is then asymptotically stable.

b. With gravitational term the PD control alone can not guarantee asymptotic tracking ($\dot{V} = \dot{q}(u-g+K(q-r))$. To remove the drawback of non asymptotic tracking, add the gravitational term in u.

c. Put

$$u = M(q)a_q + C(q,\dot{q})\dot{q} + g(q).$$

Then the system dynamics becomes $\ddot{q} = a_q$, where the term a_q is to be chosen.

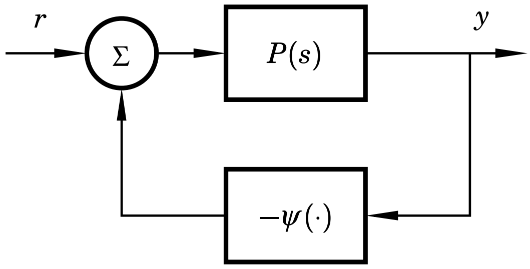
5. Consider the system

$$\frac{\mathrm{d}^3 \mathrm{z}}{\mathrm{d}t^3} + \frac{\mathrm{d}^2 \mathrm{z}}{\mathrm{d}t^2} + \frac{\mathrm{dz}}{\mathrm{d}t} = -\frac{1}{3}z^3 \qquad (2)$$

a. Show that the system can be written as a feedback connection as shown in Figure 1, where $P(s)$ is a transfer function and ψ is a static nonlinearity. (1 p)

b. Calculate the describing function of the nonlinearity $f(x) = \frac{1}{3}x^3$. (2 p)
(*Hint:* $\int_0^{2\pi} \sin(x)^4 \mathrm{d}x = \frac{3\pi}{4}$)

c. Analyse the existence, amplitude and frequency of possible limit cycles. (2 p)



Figur 1 Figure for problem 5

Solution

a. Let $\psi = 1/3z^3$. Then a Laplace transform between ψ and z results in

$$P = \frac{1}{s(s^2+s+1)}.$$

The nonlinearity is $\psi = 1/3z^3$.

b. The function is odd, which implies that it is real.

$$b_1 = \frac{A^3}{3\pi}\int_0^{2\pi} \sin(\phi)^4 \mathrm{d}\phi = \frac{A^3}{4},$$

which gives that the describing function

$$N(A) = \frac{A^2}{4}.$$

c. We want to find out the points where $\mathrm{Im}P(i\omega) = 0$. Some calculations gives that

$$\mathrm{Im}P(i\omega) = \frac{-(1-\omega^2)}{\omega((1-\omega)^2+\omega^2)},$$

which in its turn gives that $\omega = 1$. Finally, this yields that

$$P(i) = -1 = -\frac{1}{N(A)} = -\frac{4}{A^2} \Rightarrow A = 2.$$

To conclude: The frequency of the limit cycle is $\omega = 1$ rad/s and its amplitude is $A = 2$.

6. Consider the van der Pol equation with driving term

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (1-x_1^2)x_2 - x_1 + q\cos(\omega t)\\ y &= x_1\end{aligned}$$

A reduced order observer can be constructed by introducing the new variable

$$z = x_2 - 2y + y^3/3,$$

and its estimate update law is

$$\dot{\hat{z}} = -\hat{z} - 3y + y^3/3 + q\cos(\omega t).$$

(The estimate of x_2 will then be given by $\hat{x}_2 = \hat{z} + 2y - y^3/3$).
Show that the error dynamics for $e_r = z - \hat{z}$ is linear and asymptotically stable (i.e., exponentially stable). (2 p)

Solution

$$\begin{aligned}
&e_r = z - \hat{z} \Rightarrow \dot{e}_r = \dot{z} - \dot{\hat{z}} \\
&\dot{z} = \dot{x}_2 - 2\dot{y} + 3\frac{y^2}{3}\dot{y} = (1 - x_1^2)x_2 - x_1 + q\cos(\omega t) - 2x_2 + x_1^2 x_2 \\
&\quad = -x_1 - x_2 + q\cos(\omega t) \\
&\dot{\hat{z}} = -\hat{z} - 3y + y^3/3 + q\cos(\omega t) \\
&\Rightarrow \dot{e}_r = \dot{z} - \dot{\hat{z}} = (1 - x_1^2)x_2 - x_1 - 2x_2 + x_1^2 x_2 + \hat{z} + 3x_1 - x_1^3/3 = \ldots = \\
&\quad = -z + \hat{z} = -e_r
\end{aligned}$$

7. A body under influence of a force obeys the equation

$$m\ddot{x} = F, \quad F_{min} \le F \le F_{max}.$$

Assume for simplicity that $m = 1$, $F_{min} = -1 = -F_{max}$, and put $F = u$. Investigate how $u = u(x, \dot{x})$ should be chosen to move the body in shortest possible time from an arbitrary state $(x, \dot{x})$ to rest in the origin. Also, draw a phase plane diagram. (3 p)

Solution

The equations of motion are

$$\begin{aligned}
&\dot{x}_1 = x_2, \quad x_1(0) = x_0, \quad x_1(T) = 0, \\
&\dot{x}_2 = u, \quad x_2(0) = v_0, \quad x_2(T) = 0, \\
&u \in [-1, 1].
\end{aligned}$$

The problem to solve is

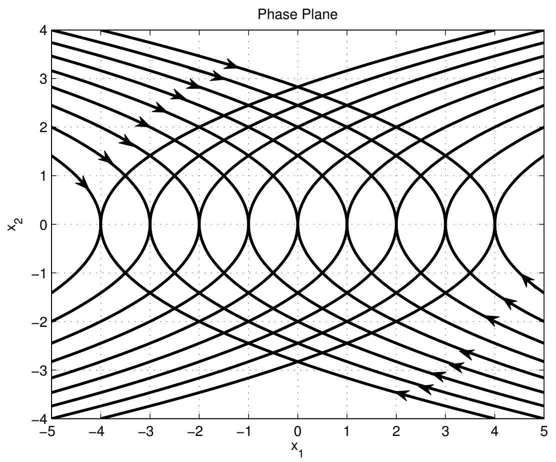
$$\int_0^T \mathrm{d}t.$$

The Hamilton function is

$$H = 1 + \lambda_1 x_2 + \lambda_2 u,$$

which implies that the adjoint equations are

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0, \Rightarrow \lambda_1 = \lambda_1^0, \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1, \Rightarrow \lambda_2 = \lambda_2^0 - \lambda_1^0 t.
\end{aligned}$$



Figur 2 Phase plane for problem 7.

We now see that $\sigma = \lambda_2$, which means that the control signal changes sign at most one time.

Through

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = \frac{x_2}{u}, \Rightarrow x_1 = \frac{x_2^2}{u} + C$$

the switching curve can be decided, and is $x_1 = -\mathrm{sign}(x_2)(x_2^2/2)$ (draw the phase plane curve). This implies that the control signal can be written as

$$u = -\mathrm{sign}(x_1 + \mathrm{sign}(x_2)(x_2^2/2)).$$

A phase plane is shown in Figure 1.