Nonlinear Control and Servo Systems

Lecture 13

- Nonlinear control design methods cont'd

Today's Goal

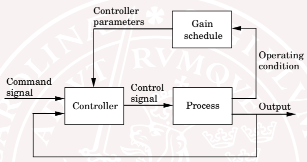
To be able to design controllers based on

- *Gain scheduling*
- *Internal model control*
- *Model predictive control*
- *Nonlinear observers*

Material

- Lecture notes
- Internal model, more info in e.g.,
  - Section 8.4 in [Glad&Ljung]
  - Ch 12.1 in [Khalil]

Gain Scheduling

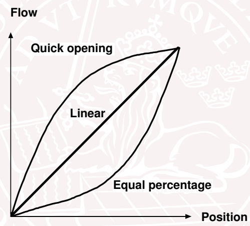


Example of scheduling variables

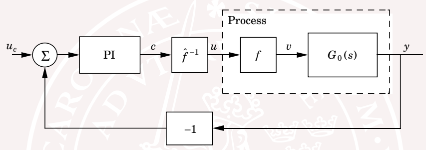
- Production rate
- Machine speed
- Mach number and dynamic pressure

Compare structure with adaptive control!

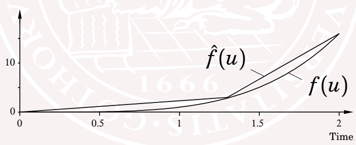
Valve Characteristics



Nonlinear Valve

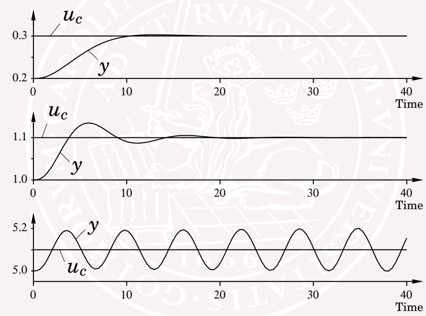


Valve characteristics



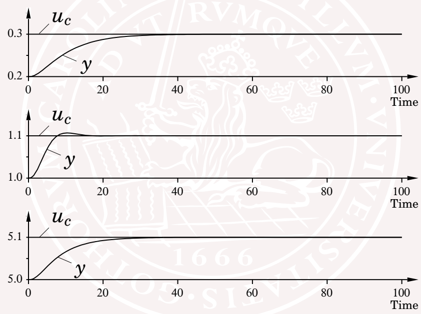
Results

Without gain scheduling



Results

With gain scheduling



Gain Scheduling

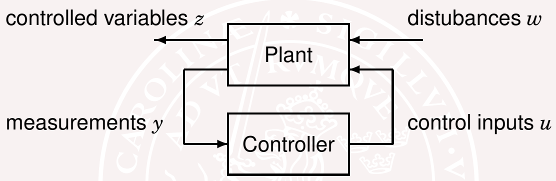
2 possibilities:

- state dependent controller parameters.
  - $K = K(q)$
- design controllers for a number of operating points.
  - use the closest controller.

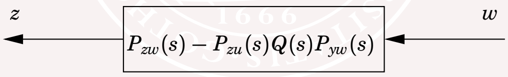
Problems:

- How should you switch between different controllers?
  - Bumpless transfer
- Switching between stabilizing controllers can cause instability.

The Q-parametrization (Youla)

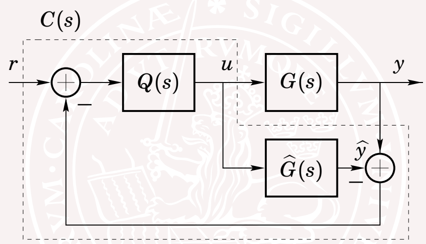


Basic Idea:
The choice of controller generally corresponds to finding $Q(s)$, to get desirable properties of the map from w to z:



Once $Q(s)$ is determined, a corresponding controller is found.

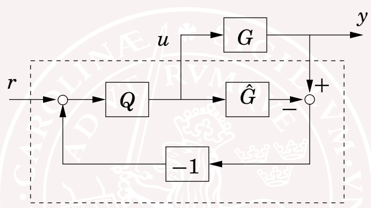
Internal Model Control

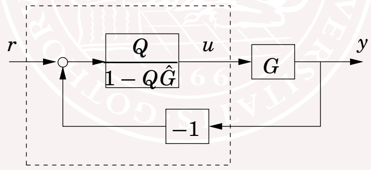


Feedback from model error $y - \widehat{y}$.

Design: Choose $\widehat{G} \approx G$ and Q stable with $Q \approx G^{-1}$.

Two equivalent diagrams

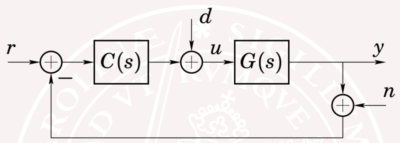




2 minute exercise: *Show that the internal model controller can be written as*

$$u = C(r - y) = (1 - Q\widehat{G})^{-1}Q(r - y)$$

Sensitivity S and Complementary Sensitivity T



$$S = (1 + GC)^{-1}, \qquad T = (1 + GC)^{-1}GC, \qquad S + T = 1$$

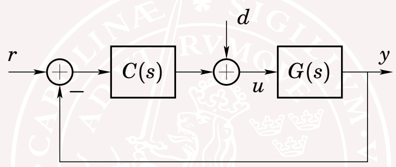
Assume $\widehat{G} = G$. The controller $C = (1 - QG)^{-1}Q$ gives

$$S^{-1} = 1 + G(1 - QG)^{-1}Q = 1 + GQ(1 - GQ)^{-1}$$
$$= [(1 - GQ) + GQ](1 - GQ)^{-1} = (1 - GQ)^{-1}$$

$$S = 1 - GQ \qquad \text{and} \qquad T = GQ$$

Perfect control if $Q = G^{-1}$. Seldom possible!

Stable Q and G Gives Stable Closed-Loop System

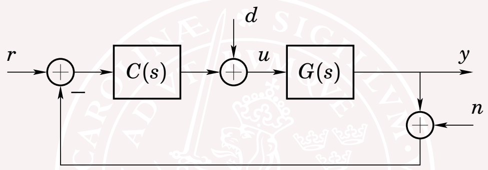


Assume G stable and $\widehat{G} = G$. The closed-loop system is

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} T & SG \\ SC & S \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} = \begin{pmatrix} GQ & (1 - GQ)G \\ Q & 1 - GQ \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix}$$

All elements are stable if Q is stable.

Sensitivity S and Complementary Sensitivity T



$|S(i\omega)| \ll 1$ gives good disturbance rejection.
$|T(i\omega)| \approx 1$ gives good reference response.
$|T(i\omega)| \ll 1$ gives good noise rejection.

$S + T = 1 \Longrightarrow$ different trade-offs for different frequencies.

Choose Q stable and proper. $Q \approx G^{-1}$ gives $|S(i\omega)| \ll 1$ and $|T(i\omega)| \approx 1$.

Example

$$G(s) = \frac{1}{1 + sT_1}$$

Choose

$$Q = \frac{1 + sT_1}{1 + \tau s}$$

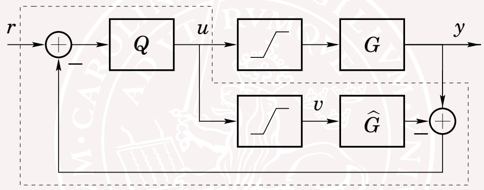
Gives the PI controller

$$C = \frac{1 + sT_1}{s\tau} = \frac{T_1}{\tau}\left(1 + \frac{1}{T_1 s}\right)$$

Internal Model Control Can Give Problems

- Unstable G
- $Q \not\approx G^{-1}$ due to RHP zeros
- Cancellation of process poles may show up in some signals

Internal Model Control with Static Nonlinearity



Include the nonlinearity in the model in the controller.

Choose $Q \approx G^{-1}$.

Example (cont'd)

Assume $r = 0$ and $\widehat{G} = G$.

$$u = -Q(y - \widehat{G}v) = -\frac{1+sT_1}{1+\tau s}y + \frac{1}{1+\tau s}v$$

Same as above if $|u| \le u_{\max}$: Integrating controller.

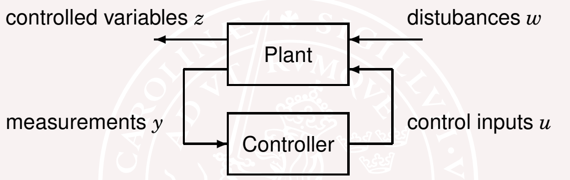
If $|u| > u_{\max}$ then

$$u = -\frac{1+sT_1}{1+\tau s}y \pm \frac{u_{\max}}{1+\tau s}$$

No integration.

A way to implement anti-windup.

The Q-parametrization (Youla)



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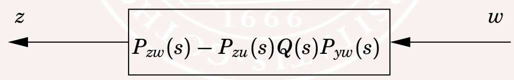
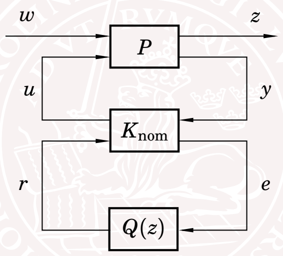


Once $Q(s)$ is determined, a corresponding controller is found.

Youla parametrization revisited

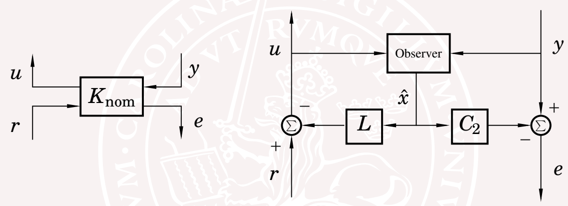
The Youla-parametrization:



where K_nom stabilizes the [P,K]-system and $Q(z)$ = is any stable transfer function.

Nominal Controller

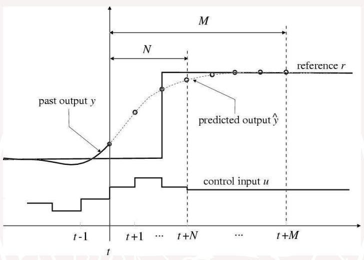
Linear system with observer



In equations

$$\hat{x} = A\hat{x} + Bu(k) + Ke(k)$$
$$u = r - L\hat{x}$$
$$e = y - C\hat{x}$$

Model Predictive Control – MPC



1. Derive the future controls $u(t+j), \quad j = 0, 1, \ldots, N-1$ that give an optimal predicted response.
2. Apply the first control $u(t)$.
3. Start over from 1 at next sample.

What is Optimal?

Minimize a cost function, V, of inputs and predicted outputs.

$$V = V(U_t, Y_t), \quad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \quad Y_t = \begin{bmatrix} \hat{y}(t+M|t) \\ \vdots \\ \hat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t \quad (1)$$

$\Longrightarrow$ linear controller

$$u(t) = -L\hat{x}(t|t)$$

Model Predictive Control

+ Flexible method
  * Many types of models for prediction:
    - state space, input–output, step response, FIR filters
  * MIMO
  * Time delays
+ Can include constraints on input signal and states
+ Can include future reference and disturbance information
– On-line optimization needed
– No stability (or performance) guarantees

Typical application: Chemical processes with slow sampling (minutes)

A predictor for Linear Systems

Discrete-time model

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_v v_1(t) \\ y(t) &= Cx(t) + v_2(t) \end{aligned} \qquad t = 0, 1, \ldots$$

Predictor (v unknown)

$$\begin{aligned} \widehat{x}(t+k+1|t) &= A\widehat{x}(t+k|t) + Bu(t+k) \\ \widehat{y}(t+k|t) &= C\widehat{x}(t+k|t) \end{aligned}$$

The M-step predictor for Linear Systems

$\widehat{x}(t|t)$ is predicted by a standard Kalman filter, using outputs up to time t, and inputs up to time $t-1$.

Future predicted outputs are given by

$$\begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix} = \begin{bmatrix} CA^M \\ \vdots \\ CA \end{bmatrix} \widehat{x}(t|t) + \begin{bmatrix} CB & CAB & CA^2B & \ldots \\ 0 & CB & CAB & \ldots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u(t+M-1) \\ \vdots \\ u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}$$

$$Y_t = D_x \widehat{x}(t|t) + D_u U_t$$

2 minute exercise: Show that for a linear predictor, the quadratic cost function (1) can be written as

$$V(U_t) = U_t^T Q U_t + 2\widehat{x}(t|t)^T S U_t + \widehat{x}(t|t)^T R \widehat{x}(t|t)$$

This cost function is, for $Q > 0$, minimized by

$$U_t = -Q^{-1} S \widehat{x}(t|t)$$

and thus results in a constant linear controller.

Limitations

Limitations on control signals and outputs,

$$|u(t)| \le C_u \quad |y(t)| \le C_y,$$

leads to quadratic optimization and linear matrix inequalities (LMIs).

- Local minimum = global minimum

Efficient optimization software exists.

Design Parameters

- Model
- M (look on settling time)
- N as long as computational time allows
- If $N < M-1$ assumption on $u(t+N), \ldots, u(t+M-1)$ needed (e.g., $= 0$, $= u(t+N-1)$.)
- Q_y, Q_u (trade-offs between control effort etc)
- C_y, C_u limitations often given
- Sampling time

Product: ABB Advant

Example–Motor

$$A = \begin{pmatrix} 1 & 0.139 \\ 0 & 0.861 \end{pmatrix}, \quad B = \begin{pmatrix} 0.214 \\ 2.786 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Minimize $V(U_t) = \|Y_t - R\|$ where $R = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$, r=reference,

$M = 8$, $N = 2$, $u(t+2) = u(t+3) = u(t+7) = \ldots = 0$

Example–Motor

$$\begin{aligned} Y_t &= \begin{pmatrix} CA^8 \\ \vdots \\ CA \end{pmatrix} x(t) + \begin{pmatrix} CA^6B & CA^7B \\ \vdots & \vdots \\ 0 & CB \end{pmatrix} \begin{pmatrix} u(t+1) \\ u(t) \end{pmatrix} \\ &= D_x x(t) + D_u U_t \end{aligned}$$

Solution without control constraints

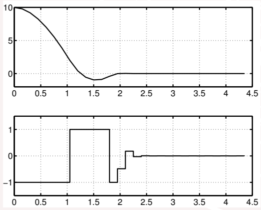
$$\begin{aligned} U_t &= -(D_u^T D_u)^{-1} D_u^T D_x x + (D_u^T D_u)^{-1} D_u^T R = \\ &= -\begin{pmatrix} -2.50 & -0.18 \\ 2.77 & 0.51 \end{pmatrix} \begin{pmatrix} x_1(t) - r \\ x_2(t) \end{pmatrix} \end{aligned}$$

Use

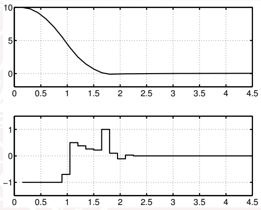
$$u(t) = -2.77(x_1(t) - r) - 0.51 x_2(t)$$

Example–Motor–Results

No control constraints in optimization (but in simulation)



Control constraints $|u(t)| \leq 1$ in optimization.



Nonlinear Observers

What if x is not measurable?

$$\dot{x} = f(x,u), \quad y = h(x)$$

Simplest observer (open loop – only works for as. stable systems).

$$\dot{\hat{x}} = f(\hat{x},u)$$

Correction, as in linear case,

$$\dot{\hat{x}} = f(\hat{x},u) + K(y - h(\hat{x}))$$

Choices of K

- Linearize f at x_0, find K for the linearization
- Linearize f at $\hat{x}(t)$, find $K(t)$ for the linearization

Second case is called *Extended Kalman Filter*

A Nonlinear Observer for the Pendulum

Control tasks:

1. Swing up
2. Catch
3. Stabilize in upward position

The observer must to be valid for a complete revolution

A Nonlinear Observer for the Pendulum

$$\frac{d^2\theta}{dt^2} = \sin\theta + u\cos\theta$$

$$x_1 = \theta,\ x_2 = \frac{d\theta}{dt} \Longrightarrow$$

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \sin x_1 + u\cos x_1$$

Observer structure:

$$\frac{d\hat{x}_1}{dt} = \hat{x}_2 \qquad + k_1(x_1 - \hat{x}_1)$$
$$\frac{d\hat{x}_2}{dt} = \sin\hat{x}_1 + u\cos\hat{x}_1 \qquad + k_2(x_1 - \hat{x}_1)$$

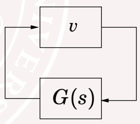
A Nonlinear Observer for the Pendulum

Introduce the error $\tilde{x} = \hat{x} - x$

$$\begin{cases} \frac{d\tilde{x}_1}{dt} = -k_1\tilde{x}_1 + \tilde{x}_2 \\ \frac{d\tilde{x}_2}{dt} = \sin\hat{x}_1 - \sin x_1 + u(\cos\hat{x}_1 - \cos x_1) - k_2\tilde{x}_1 \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} = \begin{bmatrix}-k_1 & 1\\ -k_2 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} v$$

$$v = 2\sin\frac{\tilde{x}_1}{2}\Big(\cos\big(x_1 + \frac{\tilde{x}_1}{2}\big) - u\sin\big(x_1 + \frac{\tilde{x}_1}{2}\big)\Big)$$



Stability with Small Gain Theorem

The linear block:

$$G(s) = \frac{1}{s^2 + k_1 s + k_2}$$

$$\Big|\frac{1}{G(i\omega)}\Big|^2 = \omega^4 + (k_1^2 - 2k_2)\omega^2 + k_2^2$$
$$= (\omega^2 - k_2 + k_1^2/2)^2 - k_1^4/4 + k_1^2 k_2$$

$$\gamma_G = \max G(i\omega) = \begin{cases} \frac{1}{\sqrt{k_1^2k_2 - k_1^4/4}}, & \text{if } k_1^2 < 2k_2 \\ \frac{1}{k_2}, & \text{if } k_1^2 \geq 2k_2 \end{cases}$$

Stability with Small Gain Theorem

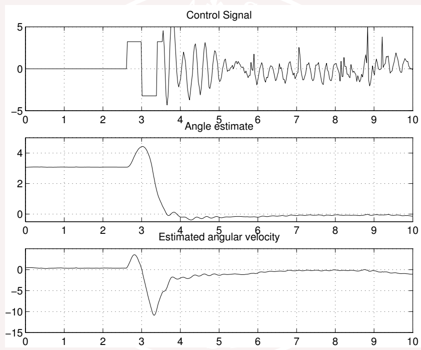
$$v = 2\sin\frac{\tilde{x}_1}{2}\Big(\cos\big(x_1 + \frac{\tilde{x}_1}{2}\big) - u\sin\big(x_1 + \frac{\tilde{x}_1}{2}\big)\Big)$$

$$|v| \leq |\tilde{x}_1|\sqrt{1 + u_{max}^2} = \beta|\tilde{x}_1|$$

The observer is stable if $\gamma_G\beta < 1$

$$\Longrightarrow \quad k_2 > \begin{cases} \beta^2 k_1^{-2} + k_1^2/4, & \text{if } k_1 < \sqrt{2\beta}, \\ \beta, & \text{if } k_1 \geq \sqrt{2\beta} \end{cases}$$

A Nonlinear Observer for the Pendulum



Next Lecture

- High-gain design methods and sliding mode controllers