Optimal Control

Outline Lectures 10 and 11

- Introduction
- The rocket problem
- Optimal control problems
- The maximum principle
- A minimal time problem
- Numerical methods/Optimica
- Lab 3

with material from J. Åkesson, Dep of Automatic Control, LTH

Goal

- To be able to
  - solve simple optimal control problems by hand
  - design controllers

  using the maximum principle
- Rewrite a optimal control problem to "standard form" for numerical solvers

Material

- Lecture slides
- References to Glad & Ljung's *Control Theory - Multivariable and Nonlinear Methods (Reglerteori - Flervariabla och olinjära metoder)*, part of Chapter 18
  Note! page refs to Swedish edition

Optimal Control Problems

Idea: Formulate the design problem as optimization problem

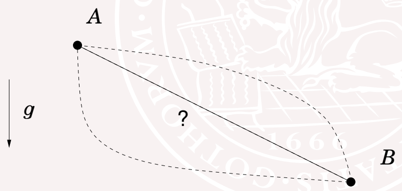
+ Gives systematic design procedure
+ Can use on nonlinear problems
+ Can capture limitations etc as constraints
− Hard to find suitable criterium?!
− Can be hard to find the optimal controller

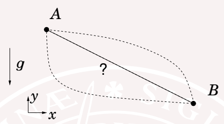
Solutions will often be of "bang-bang" character if control signal is bounded, compare lecture 10 on sliding mode controllers.

The beginning

- John Bernoulli: The bracistochrone problem 1696

  Let a particle slide along a frictionless curve. Find the curve that takes the particle from A to B in shortest time





$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{v} = \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

Find $y(x)$, with $y(0)$ and $y(1)$ given, that minimizes

$$J(y) = \int_0^1 \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

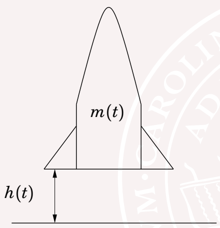
- Solved by John and James Bernoulli, Newton, l'Hospital
- Euler: Isoperimetric problems
  - Example: The largest area covered by a curve of given length is a circle [see also Dido/cow-skin/Carthage].

Optimal Control

- The space race (Sputnik 1957)
- Putting satellites in orbit
- Trajectory planning for interplanetary travel
- Reentry into atmosphere
- Minimum time problems
- LaSalle's bang-bang principle
- Tsien optimal trajectories
- The industrial labs
- Pontryagin's maximum principle, 1956
- Dynamic programming, Bellman 1957
- Vitalization of a classical field

An example: Goddard's Rocket Problem (1910)

How to send a rocket as high up in the air as possible?



where u = motor force, $D(v, h)$ = air resistance, m = mass.

Constraints

$$0 \le u \le u_{max}, \quad m(t_f) \ge m_1$$

Criterium

$$\text{Maximize } h(t_f), \qquad t_f \text{ given}$$

Goddard's Problem

Can you guess the solution when $D(v, h) = 0$?

Much harder when $D(v, h) \neq 0$

Can be optimal to have low v when air resistance is high. Burn fuel at higher level.

Took about 50 years before a complete solution was found.

Read more about Goddard at `http://www.nasa.gov/centers/goddard/`

Optimal Control Problem. Constituents

Control signal $u(t), 0 \le t \le t_f$

Criterium $h(t_f)$.

Differential equations relating $h(t_f)$ and u

Constraints on u

Constraints on $x(0)$ and $x(t_f)$

t_f can be fixed or a free variable

Preliminary: Static Optimization

Minimize $g_1(x,u)$, $x \in R^n$ and $u \in R^m$ subject to $g_2(x,u) = 0$
(Assume x can be solved for in g_2 given u)
Introduce the Hamiltonian

$$H(x,u,\lambda) = g_1(x,u) + \lambda^T g_2(x,u)$$

Consider variation of H

$$\delta g_1 = \delta H = \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u$$

where $\lambda \in R^n$ are the adjoined variables.

Necessary conditions for local minimum

$$\frac{\partial H}{\partial x} = 0 \qquad \frac{\partial H}{\partial u} = 0$$

Note: Difference if constrained control!

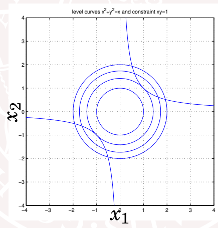
Example - static optimization

Minimize

$$g_1(x_1,x_2) = x_1^2 + x_2^2$$

with the constraint that

$$g_2(x_1,x_2) = x_1 \cdot x_2 - 1 = 0$$



Plot with level curves for $g_1 = constant$ and the constraint $g_2 = 0$, repectively.

Static Optimization cont'd

Solving the equations

$$\frac{\partial H}{\partial x} = \frac{\partial g_1}{\partial x} + \lambda^T \frac{\partial g_2}{\partial x} = 0 \Rightarrow \lambda^T = -\frac{\partial g_1}{\partial x}\left(\frac{\partial g_2}{\partial x}\right)^{-1}$$

$$\frac{\partial H}{\partial u} = \frac{\partial g_1}{\partial u} + \lambda^T \frac{\partial g_2}{\partial u} = 0 \Rightarrow \frac{\partial g_1}{\partial u} - \frac{\partial g_1}{\partial x}\left(\frac{\partial g_2}{\partial x}\right)^{-1}\frac{\partial g_2}{\partial u} = 0$$

This gives m equations to solve for u. Note that $\frac{\partial g_2}{\partial x}$ must be non-singular (which it should be if u determines x through g_2).

Sufficient condition for local minimum

$$\frac{\partial^2 H}{\partial u^2} > 0$$

Optimization with Dynamic Constraint

Optimal Control Problem

$$\min_u J = \min_u \left\{ \phi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)\,dt \right\}$$

subject to

$$\dot{x} = f(x,u), \quad x(t_0) = x_0$$

Introduce *Hamiltonian*: $H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$

$$J = \phi(x(t_f)) + \int_{t_0}^{t_f} \left[L(x,u) + \lambda^T (f - \dot{x})\right] dt$$

$$= \phi(x(t_f)) - \left[\lambda^T x\right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[H + \dot{\lambda}^T x\right] dt$$

where the second equality is obtained from "integration by parts".

Optimization with Dynamic Constraint cont'd

Variation of J:

$$\delta J = \left[\left(\frac{\partial \phi}{\partial x} - \lambda^T\right)\delta x\right]_{t=t_f} + \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right] dt$$

Necessary conditions for local minimum ($\delta J = 0$)

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} \qquad \dot{x}^T = \frac{\partial H}{\partial \lambda} \quad \frac{\partial H}{\partial u} = 0$$

$$\lambda(t_f)^T = \left.\frac{\partial \phi}{\partial x}\right|_{t=t_f} \qquad x(t_0) = x_0$$

- Adjoined, or co-state, variables, $\lambda(t)$
- λ specified at $t = t_f$ and x at $t = t_0$
- Two Point Boundary Value Problem (TPBV)
- For sufficiency $\frac{\partial^2 H}{\partial u^2} \ge 0$

- Look at some common problems in optimal control
- Rewrite in "standard/canonical forms"

Problem Formulation (1)

(Re-write your problem to the) Standard form (1):

$$\text{Minimize} \int_0^{t_f} \overbrace{L(x(t),u(t))}^{Traj.cost}\,dt + \overbrace{\phi(x(t_f))}^{Finalcost}$$

$$\dot{x}(t) = f(x(t),u(t))$$

$$u(t) \in U, \quad 0 \le t \le t_f, \qquad t_f \text{ given}$$

$$x(0) = x_0$$

$x(t) \in R^n$, $u(t) \in R^m$

U control constraints

Here we have a fixed end-time t_f. This will be relaxed later on.

The Maximum Principle (18.2)

Introduce the **Hamiltonian**

$$H(x,u,\lambda) = L(x,u) + \lambda^T(t) f(x,u).$$

Suppose optimization problem (1) has a solution $\{u^*(t), x^*(t)\}$.
Then the optimal solution must satisfy

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$d\lambda(t)/dt = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \text{with} \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

Notation

$$H_x = \frac{\partial H}{\partial x} = \begin{pmatrix} \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} \dots \end{pmatrix}$$

Remarks

Proof: If you are theoretically interested look at proof in [Glad & Ljung].

The idea is simply to note that every change of $u(t)$ from the suggested optimal $u^*(t)$ must lead to larger value of the criterium. Then do clever Taylor expansions.

Should be called "minimum principle"

$\lambda(t)$ are called the **Lagrange multipliers** or the **adjoint variables**

Remarks

The MP gives **necessary** conditions

A pair $(u^*(\cdot), x^*(\cdot))$ is called **extremal** the conditions in the MP are satisfied. Many extremals can exist.

The maximum principle gives all possible candidates.

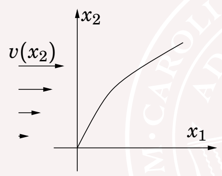
However, there might not exist a minimum!

Example

$$\text{Min } x(1), \quad \dot{x}(t) = u(t) \quad x(0) = 0, \quad u(t) \text{ free}$$

Why doesn't there exist a minimum?

Example–Boat in Stream



$$\begin{aligned}
&\min \; -x_1(T)\\
&\dot{x}_1 = v(x_2) + u_1\\
&\dot{x}_2 = u_2\\
&x_1(0) = 0\\
&x_2(0) = 0\\
&u_1^2 + u_2^2 = 1
\end{aligned}$$

Speed of water $v(x_2)$ in x_1 direction. Move maximum distance in x_1-direction in fixed time T

Assume v linear so that $v'(x_2) = 1$

Solution

Adjoint equation

$$\dot{\lambda} = -H_x = -[\partial H/\partial x_1, \; \partial H/\partial x_2]^T$$

where $H = 0 + \lambda^T f = [\lambda_1 \quad \lambda_2] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \lambda_1(v(x_2) + u_1) + \lambda_2 u_2$

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -v'(x_2)\lambda_1 = -\lambda_1$$

with boundary conditions
$\lambda_1(T) = \partial\phi/\partial x_1|_{x=x^*(t_f)} = -1, \quad \lambda_2(T) = 0.$

Gives $\lambda_1(t) = -1, \quad \lambda_2(t) = t - T$

Control signal should solve

$$\min_{u_1^2+u_2^2=1} \lambda_1(v(x_2) + u_1) + \lambda_2 u_2$$

Solution

Minimize $\lambda_1 u_1 + \lambda_2 u_2$ so that (u_1, u_2) has length 1

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}$$

$$u_1(t) = \frac{1}{\sqrt{1 + (t-T)^2}}, \quad u_2(t) = \frac{T-t}{\sqrt{1 + (t-T)^2}}$$

See fig 18.1 for plots

Remark: It can be shown that this optimal control problem has a minimum. Hence it must be the one we found, since this was the only solution to MP

5 min exercise

Solve the optimal control problem

$$\begin{aligned}
&\min \int_0^1 u^4 dt + x(1)\\
&\dot{x} = -x + u\\
&x(0) = 0
\end{aligned}$$

Problem Formulation (2)

We can extend the problem formulation (1) with extra conditions.

Possible additions (one or many of):

- r end constraints

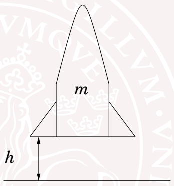
$$\Psi(t_f, x(t_f)) = \begin{pmatrix} \Psi_1(t_f, x(t_f)) \\ \vdots \\ \Psi_r(t_f, x(t_f)) \end{pmatrix} = 0$$

- t_f free variable (i.e., not specified *a priori*)
- time varying final penalty, $\phi(t_f, x(t_f))$

Goddard's Rocket Problem revisited

How to send a rocket as high up in the air as possible?

$$\frac{d}{dt}\begin{pmatrix} v \\ h \\ m \end{pmatrix} = \begin{pmatrix} \frac{u-D}{m} - g \\ v \\ -\gamma u \end{pmatrix}$$



$(v(0), h(0), m(0)) = (0, 0, m_0)$, $g, \gamma > 0$
u motor force, $D = D(v, h)$ air resistance

Constraints: $0 \le u \le u_{max}$ and $m(t_f) = m_1$ (empty)

Optimization criterion: $\max_u h(t_f)$

Generalized form:

$$\min_{u:[0,t_f]\to U} \int_0^{t_f} L(x(t), u(t))\, dt + \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$
$$\psi(x(t_f)) = 0$$

Note the diffences compared to standard form:

- End time t_f is free
- Final state is constrained: $\psi(x(t_f)) = x_3(t_f) - m_1 = 0$

Solution to Goddard's Problem

Goddard's problem is on generalized form with

$$x = (v, h, m)^T, \quad L \equiv 0, \quad \phi(x) = -x_2, \quad \psi(x) = x_3 - m_1$$

$D(v, h) \equiv 0$:

- Easy: let $u(t) = u_{max}$ until $m(t) = m_1$
- Burn fuel as fast as possible, because it costs energy to lift it

$D(v, h) \not\equiv 0$:

- Hard: e.g., it can be optimal to have low speed when air resistance is high, in order to burn fuel at higher level
- Took 50 years before a complete solution was presented

The Maximum Principle–General Case (18.4)

Introduce the Hamiltonian

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

Suppose optimization problem (2) has a solution $u^*(t), x^*(t)$. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$, and a vector $\mu \in R^r$ so that $[n_0 \;\; \mu^T] \ne 0$ and

$$\min_{u\in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where

$$\dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0)$$
$$\lambda(t_f) = n_0\phi_x^T(x^*(t_f)) + \Psi_x^T(t_f, x^*(t_f))\mu$$

Free end time t_f

If the choice of t_f is included in the optimization then there is an extra constraint:

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = -n_0\phi_t(t_f, x^*(t_f)) - \mu^T\Psi_t(t_f, x^*(t_f))$$

Note that for the special case where ϕ and Ψ are time-invariant this reduces to

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$$

Normal/abnormal cases

Can scale $n_0, \mu, \lambda(t)$ by the same constant

Can reduce to two cases

- $n_0 = 1$ (normal)
- $n_0 = 0$ (abnormal)

As we saw before (18.2): fixed time t_f and no end constraints $\Rightarrow$ normal case

Hamilton function is constant

H is constant along extremals (x^*, u^*)

Proof:

$$\frac{d}{dt}H = H_x\dot{x} + H_\lambda\dot{\lambda} + H_u\dot{u} = H_x f - f^T H_x^T + 0 = 0$$

Example: Optimal heating

T temperature, P heat effect

$$\dot{T} = P - T$$
$$0 \le P \le P_{max}$$
$$T(0) = 0, \quad T(1) = 1$$

$$\text{minimize} \int_0^{tf=1} P(t)\, dt$$

Solution

$$H = n_0 P + \lambda P - \lambda T$$

Adjoint equation $\dot{\lambda}^T = -H_T = -\frac{\partial H}{\partial T}$

$$\dot{\lambda}_1 = \lambda_1, \quad \lambda_1(1) = \mu_1$$

Solution

$$\lambda_1(t) = \mu_1 e^{t-1}$$

$$H = \underbrace{(n_0 + \mu_1 e^{t-1})}_{\sigma(t)} P - \lambda^T T$$

Solution

$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{max}, & \sigma(t) < 0 \end{cases}$$

$\mu \geq 0 \Rightarrow P(t) = 0$
$\mu < 0 \; n_0 = 0 \Rightarrow P(t) = P_{max}$
$\mu < 0, \; n_0 = 1 \Rightarrow$ Switching solution

$$\mu_1 < 0 \quad \Rightarrow \quad \sigma(t) \text{ decreasing}$$

Hence

$$P^*(t) = \begin{cases} 0, & 0 \leq t \leq t_1 \\ P_{max}, & t_1 < t \leq 1 \end{cases}$$

Solution

$$T(t) = \begin{cases} 0, & 0 \leq t \leq t_1 \\ \int_{t_1}^{1} e^{-(t-\tau)} P_{max}\, d\tau = \left(e^{-(t-1)} - e^{-(t-t_1)}\right) P_{max}, & t_1 < t \leq 1 \end{cases}$$

Time t_1 is given by

$$T(1) = \left(1 - e^{-(1-t_1)}\right) P_{max} = 1$$

Has solution $0 \leq t_1 \leq 1$ if

$$P_{max} \geq \frac{1}{1 - e^{-1}}$$

Second Variations

By expanding the criterion, J, to second order one can see that

$$\delta^2 J = \frac{1}{2}\delta x^T \phi_{xx}\, \delta x + \frac{1}{2}\int_{t_0}^{t_f} \begin{pmatrix} \delta x \\ \delta u \end{pmatrix}^T \begin{pmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta u \end{pmatrix} dt$$

$$\delta \dot{x} = f_x \delta x + f_u \delta u$$

where $J = J^* + \delta^2 J + \ldots$ is a Taylor expansion of the criterion and $\delta_x = x - x^*$ and $\delta_u = u - u^*$.

Treat this as a new optimization problem. Linear time-varying system and quadratic criterion. Gives an optimal controller of the form

$$u - u^* = L(t)(x - x^*)$$

Linear Quadratic Control

$$\text{minimize } x^T(t_f) Q_N x(t_f) + \int_0^{t_f} \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

where

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Optimal solution if $t_f = \infty$, $Q_N = 0$, all matrices constant, and x measurable:

$$u = -Lx$$

where $L = Q_{22}^{-1}(Q_{12} + B^T S)$ and S is the positive definite solution to

$$SA + A^T S + Q_{11} - (Q_{12} + SB) Q_{22}^{-1} (Q_{12} + B^T S) = 0$$

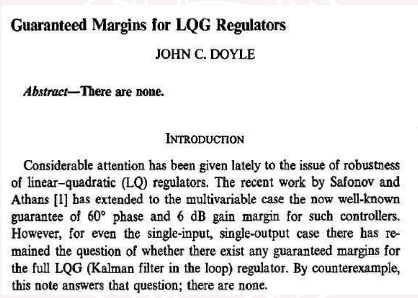
Robustness of LQ-controller

- Always stabilizing
- Loop gain $L(sI - A + BL)^{-1}B$ lies outside the circle $|s + 1| \leq 1$. Circle criteria says that the closed system is stable even if control signal is changed to

  $$u = -\alpha(t) L x$$

  with $\alpha(t) \in [1/2, \infty]$.
- phase margin 60 degrees

Robustness with Kalman filter - none:
If x is not measurable one must use a Kalman filter. Warning! Still stability but all robustness can be lost!


Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear-quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

[IEEE Trans. Automat. Contr., vol. 23, pp. 756–757, August 1978]

Feedback or feed-forward?

Example:

$$\frac{dx}{dt} = u, \qquad x(0) = 1$$
$$\text{minimize } J = \int_0^\infty \left(x^2 + u^2\right) dt \qquad (1)$$

$J_{min} = 1$ is achieved for

$$u(t) = -e^{-t} \qquad \text{open loop} \qquad (2)$$

or

$$u(t) = -x(t) \qquad \text{closed loop} \qquad (3)$$

(**??**) $\Longrightarrow$ stable system
(**??**) $\Longrightarrow$ asympt. stable system

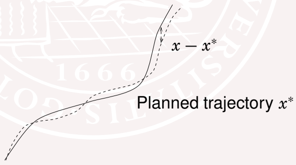
Sensitivity for noise and disturbances differ!!

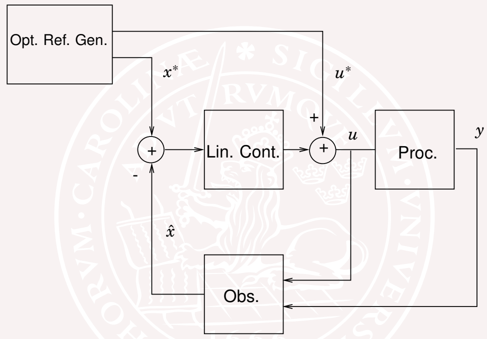
Reference generation using optimal control

Note that the optimization problem makes no distinction between open loop control $u^*(t)$ and closed loop control $u^*(t, x)$. Feedback is needed to take care of disturbances and model errors.

Idea: Use the optimal open loop solution $u^*(t), x^*(t)$ as reference values to a linear regulator that keeps the system close to the wanted trajectory

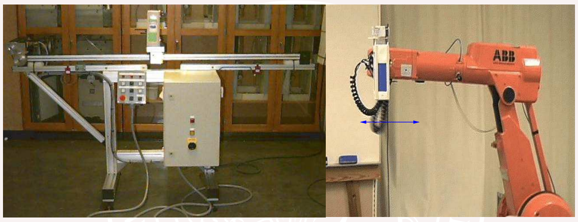
Efficient for large setpoint changes.





Take care of deviations with linear controller

Example – The Milk Race



Move milk in minimum time without spilling!
[M. Grundelius – Methods for Control of Liquid Slosh]

[movie]

Minimal Time Problem

NOTE! Common trick to rewrite criterion into "standard form"!!

$$\text{minimize } t_f = \text{minimize } \int_0^{t_f} 1\, dt$$

Control constraints

$$|u(t)| \le u_i^{max}$$

No spilling

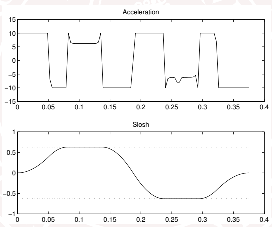
$$|Cx(t)| \le h$$

Optimal controller has been found for the milk race

Minimal time problem for linear system $\dot{x} = Ax + Bu$, $y = Cx$ with control constraints $|u_i(t)| \le u_i^{max}$. Often bang-bang control as solution

Results- milk race

Maximum slosh $\phi_{max} = 0.63$
Maximum acceleration = 10 m/s^2
Time optimal acceleration profile



Optimal time = 375 ms, industrial = 540ms

Extra example—Minimum Time Control

Problem: Use bounded control, $u \in [-1, 1]$, and bring the states of the double integrator to the origin as fast as possible.

Free end-time t_f

$$\min_{u:[0,t_f]\to[-1,1]} t_f = \min_{u:[0,t_f]\to[-1,1]} \int_0^{t_f} 1\, dt$$
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$\psi(x(t_f)) = (x_1(t_f), x_2(t_f))^T = (0, 0)^T$$

Hamiltonian

$$H = n_0 L + \lambda^T f = n_0 \cdot 1 + [\lambda_1\, \lambda_2] \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

The optimal control is the control which minimizes

$$u^*(t) = \arg \min_{u\in[-1,1]} H = \arg \min_{u\in[-1,1]} n_0 + \lambda_1(t) x_2^*(t) + \lambda_2(t) u$$
$$= \begin{cases} 1, & \lambda_2(t) < 0 \\ -1, & \lambda_2(t) \ge 0 \end{cases}$$

This is called "bang-bang control".

Remark: Here "indep." of n_0 which is usually NOT the case, see exercises.

Adjoint equations $\dot{\lambda}_1(t) = 0$, $\dot{\lambda}_2(t) = -\lambda_1(t)$ gives

$$\lambda_1(t) = c_1, \quad \lambda_2(t) = c_2 - c_1 t$$

Use $\lambda(t_f) = -n_0 \phi_t(t_f, x^*(t_f)) - \mu^T \Psi_t(t_f, x^*(t_f))$, and $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$ to find c_1 and c_2.

"Alternative":
With $u(t) = \zeta = \pm 1$, we have

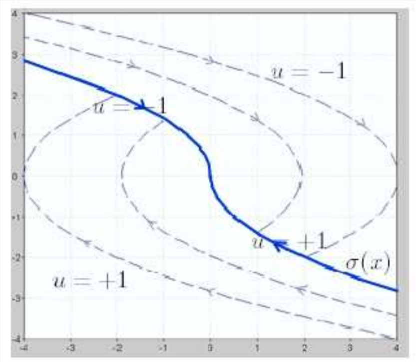
$$x_1(t) = x_1(0) + x_2(0)t + \zeta t^2/2$$
$$x_2(t) = x_2(0) + \zeta t$$

Eliminating t gives curves (parabolas)

$$x_1(t) \pm x_2(t)^2/2 = \text{const}$$

These define the *switch curves*, where the optimal control switch.

Remark: See also solution to exam March 8, 2005 for more details



Combination of solution curves: Follow one parabola using maximum or minimum control signal until you hit the switch curve σ, switch to minimum/maximum control and then follow that parabola to the origin.

For a "feedforward solution" one can use

$$u = \begin{cases} 1, & \lambda_2(t) < 0 \\ -1, & \lambda_2(t) \geq 0 \end{cases}$$

(and thus only use a "time-driven" expression for $\lambda_2(t)$) when to switch. This expression needs to be calculated for each different initial condition $x(0)$ and is not robust to disturbances).

For a "feedback solution" we use the switch curves

$$x_1(t) \pm x_2(t)^2/2 = 0$$

which is easy to apply for (robust) feedback.

Numerical Methods for Optimal Control

Several packages exist, *e.g.,*

- RIOTS, free software for Matlab
- Optimica
  - Used in this weeks computer exercise
    `http://www.control.lth.se/user/johan.akesson/`
    `http://www.control.lth.se/project/Langopt/`
  - `See extra hand-out`

Also exists special software for e.g., robotics

Numerical Methods for Optimal Control

Maximum principle gives so called two point boundary problems: $x(t_0)$, $\lambda(t_f)$ given. Can solve by “shooting method”:

- Guess $\lambda(t_0)$
- Solve ODEs for $x(t)$ and $\lambda(t)$ forward in time
- Determine how final conditions $x(t_f)$ and $\lambda(t_f)$ are changed when $\lambda(t_0)$ changes
- Iterate on $\lambda(t_0)$ until final conditions are ok

Local convergence only. Methods with better stability properties exist. Gradient methods and second order methods exist

Today’s Goal

You should be able to

- Design controllers based on optimal control theory for
  - Standard form
  - Generalized form
- Understand possibilities and limitations of optimal control