Nonlinear Control Design

Lecture 8

- Exact-linearization
- Lyapunov-based design
  - Lab 2
  - Adaptive control
  - Backstepping
- Hybrid / Piece-wise linear control
  - NOTE: Only overview!

Exact Feedback Linearization

Idea: Find state feedback $u = u(x, v)$ so that nonlinear system

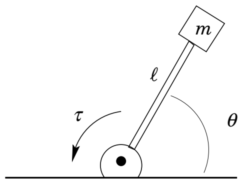
$$\dot{x} = f(x) + g(x)u$$

turns into linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

Exact linearization: example [one-link robot]



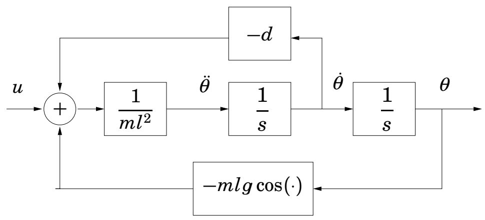
$$m\ell^2\ddot{\theta} + d\dot{\theta} + m\ell g\cos\theta = u$$

where d is the viscous damping.

The control $u = \tau$ is the applied torque

Design state feedback controller $u = u(x)$ with $x = (\theta, \dot{\theta})^T$

Block diagram



$$m\ell^2\ddot{\theta} + d\dot{\theta} + m\ell g\cos\theta = u$$

$$\ddot{\theta} = \frac{1}{m\ell^2}\left(-d\dot{\theta} - m\ell g\cos\theta + u\right)$$

Introduce new control variable v and let

$$u = m\ell^2 v + d\dot{\theta} + m\ell g\cos\theta$$

Then

$$\ddot{\theta} = v$$

Choose e.g. a PD-controller

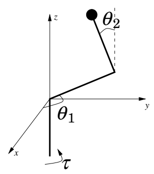
$$v = v(\theta, \dot{\theta}) = k_p(\theta_{\text{ref}} - \theta) - k_d\dot{\theta}$$

This gives the closed-loop system:

$$\ddot{\theta} + k_d\dot{\theta} + k_p\theta = k_p\theta_{\text{ref}}$$

Hence, $u = m\ell^2[k_p(\theta - \theta_{\text{ref}}) - k_d\dot{\theta}] + d\dot{\theta} + m\ell g\cos\theta$

Multi-link robot (n-joints)



General form

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

M	$n \times n$ inertia matrix, $M = M^T > 0$
$C\dot{\theta}$	$n \times 1$ vector of centrifugal and Coriolis forces
G	$n \times 1$ vector of gravitation terms

Computed torque

The computed torque
(also known as "Exact linearization", "dynamic inversion" , etc.)

$$\begin{aligned} u &= M(\theta)v + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) \\ v &= K_p(\theta_{ref} - \theta) - K_d\dot{\theta}, \end{aligned} \tag{1}$$

gives closed-loop system

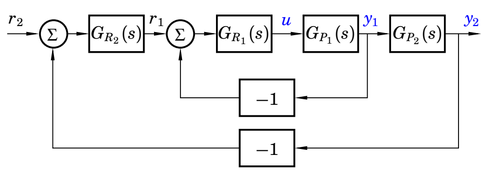
$$\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_{Ref}$$

The matrices K_d and K_p can be chosen diagonal (no cross-terms) and then this decouples into n independent second-order equations.

Identify FF and FB-part!

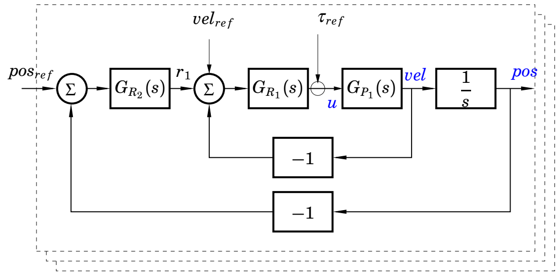
Cascade control - revisited

For systems with one control signal and many outputs:



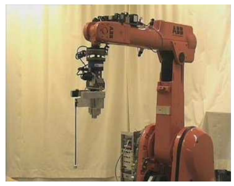
- $G_{R_1}(s)$ controls the subsystem $G_{P_1}(s)$ $(\Rightarrow G_{y_1 r_1}(s) \approx 1)$
- $G_{R_2}(s)$ controls the subsystem $G_{P_2}(s)$

Often used in motion control, e.g., robotics, with cascaded velocity and position controllers, BUT should have velocity reference feedforward!!



Example of couplings

Example: Couplings and interaction: "good"/"bad"



"Robot Furuta pendulum": Underactuated – coupling as control action
"Ordinary" Robot control:

Often cascaded PI-controllers for each joint
(inner velocity and outer position loop)

Feedforward for

- **disturbance rejection** between joints
- velocity and torque reference (improved tracking!)

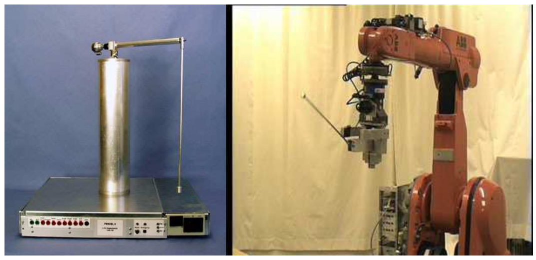
Lyapunov-Based Control Design Methods

$$\dot{x} = f(x, u)$$

- Find stabilizing state feedback $u = u(x)$
- Verify stability through Lyapunov function
- Methods depend on structure of f

Examples are energy shaping as in Lab 2 and e.g. **Back-stepping control design**, which require certain f discussed later.

Lab 2 : Energy shaping for swing-up control



[movie]

Use Lyapunov-based design for swing-up control.

Lab 2 : Energy shaping for swing-up control

Rough outline of method to get the pendulum to the upright position

- Find expression for total energy E of the pendulum (potential energy + kinetic energy)
- Let E_n be energy in upright position.
- Look at deviation $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find "swing strategy" of control torque u such that $\frac{dV}{dt} \leq 0$

Example of Lyapunov-based design

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2, \end{aligned} \qquad (2)$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

We try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0, 0) = 0$, and $V(x_1, x_2) > 0\ \forall (x_1, x_2) \neq (0, 0)$.

Example - cont'd

$$\begin{aligned} \dot{V} &= \dot{x}_1x_1 + \dot{x}_2x_2 = (-3x_1 + 2x_1x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4 \end{aligned}$$

We would like to have

$$\dot{V} < 0 \qquad \forall (x_1, x_2) \neq (0, 0)$$

Alt.1
Inserting the control law, $u = -x_1^3$, we get

$$\dot{V} = -3x_1^2 - x_2^2 - x_1^4 + 2x_1^2x_2^2 - x_2^4 = -3x_1^2 - x_2^2 - \left(x_1^2 - x_2^2\right)^2 < 0, \quad \forall x \neq 0$$

Alt.2
Inserting the control law, $u = -2x_1x_2^2$, we get

$$\dot{V} = -3x_1^2 - x_2^2 \underbrace{-2x_1^2x_2^2 + 2x_1^2x_2^2}_{=0} - x_2^4 = -3x_1^2 - x_2^2 - x_2^4 < 0, \quad \forall x \neq 0$$

Both control alternatives gives global asymptotic stability of the origin.

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned} \tag{3}$$

Find a globally asymptotically stabilizing control law $u = u(x)$.

Attempt 1: Try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \;\forall (x_1, x_2) \neq (0, 0)$.

$$\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = x_2^3 \cdot x_1 + u \cdot x_2 = x_2 \underbrace{(x_2^2 x_1 + u)}_{-x_2} = -x_2^2 \leq 0$$

where we chose

$$u = -x_2 - x_2^2 x_1$$

However $\dot{V} = 0$ as soon as $x_2 = 0$ (Note: x_1 could be anything).

According to LaSalle's theorem the set $E = \{x | \dot{V} = 0\} = \{(x_1, 0)\} \,\forall x_1$

What is the largest invariant set M?

Plugging in the control law $u = -x_2 - x_2^2 x_1$, we get

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_2 - x_2^2 x_1\end{aligned} \tag{4}$$

and we see that if we start anywhere on the line $\{(x_1, 0)\}$ we will stay in the same point as both $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, thus M=E and we will not converge to the origin, but get stuck on the line $x_2 = 0$.

Draw phase-plot with e.g., `pplane and study the behaviour.`

Attempt 2:

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned} \tag{5}$$

Try the Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4,$$

which satisfies

- $V(0,0) = 0$
- $V(x_1, x_2) > 0, \quad \forall (x_1, x_2) \neq (0, 0)$.
- radially unbounded,

$$\frac{dV}{dt} = \dot{x}_1 x_1 + \dot{x}_2 x_2^3 = x_2^3(x_1 + u) = -x_2^4 \leq 0$$

$u = -x_1 - x_2$if we use $u = -x_1 - x_2$

With

$$u = -x_1 - x_2$$

we get the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2\end{aligned} \tag{6}$$

$\dot{V} = 0$ if $x_2 = 0$, thus

$$E = \{x | \dot{V} = 0\} = \{(x_1, 0)\} \,\forall x_1$$

However, now the only possibility to stay on $x_2 = 0$ is if $x_1 = 0$, (else $\dot{x}_2 \neq 0$ and we will leave the line $x_2 = 0$).

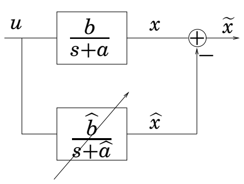
Thus, the largest invariant set

$$M = (0, 0)$$

According to the Invariant Set Theorem (LaSalle) all solutions will end up in M and so the origin is GAS.

Draw phase-plot with e.g., `pplane and study the behaviour.`

Adaptive Noise Cancellation Revisited



$$\begin{aligned}\dot{x} + ax &= bu \\ \dot{\hat{x}} + \hat{a}\hat{x} &= \hat{b}u\end{aligned}$$

Introduce $\tilde{x} = x - \hat{x}, \;\; \tilde{a} = a - \hat{a}, \;\; \tilde{b} = b - \hat{b}$.

Want to design adaptation law so that $\tilde{x} \to 0$

Let us try the Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}(\tilde{x}^2 + \gamma_a \tilde{a}^2 + \gamma_b \tilde{b}^2) \\ \dot{V} &= \tilde{x}\dot{\tilde{x}} + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = \\ &= \tilde{x}(-a\tilde{x} - \tilde{a}\hat{x} + \tilde{b}u) + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = -a\tilde{x}^2\end{aligned}$$

where the last equality follows if we choose

$$\dot{\tilde{a}} = -\dot{\hat{a}} = \frac{1}{\gamma_a}\tilde{x}\hat{x} \qquad \dot{\tilde{b}} = -\dot{\hat{b}} = -\frac{1}{\gamma_b}\tilde{x}u$$

Invariant set: $\tilde{x} = 0$.

This proves that $\tilde{x} \to 0$.

(The parameters $\tilde{a}$ and $\tilde{b}$ do not necessarily converge: $u \equiv 0$.)

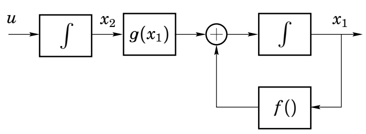
Back-Stepping Control Design

We want to design a state feedback $u = u(x)$ that stabilizes

$$\begin{aligned}\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u\end{aligned} \tag{7}$$

at $x = 0$ with $f(0) = 0$.

Idea: See the system as a cascade connection. Design controller first for the inner loop and then for the outer.



Suppose the partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\bar{v}$$

can be stabilized by $\bar{v} = \phi(x_1)$ and there exists Lyapunov fcn $V_1 = V_1(x_1)$ such that

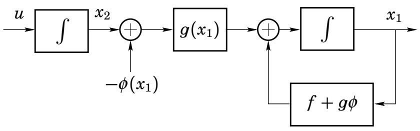
$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) \leq -W(x_1)$$

for some positive definite function W.

The Trick

Equation (**??**) can be rewritten as

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)[x_2 - \phi(x_1)]$$
$$\dot{x}_2 = u$$

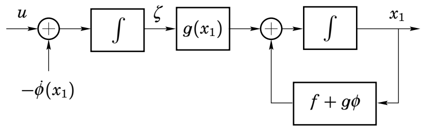


Introduce new state $\zeta = x_2 - \phi(x_1)$ and control $v = u - \dot{\phi}$:

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)\zeta$$
$$\dot{\zeta} = v$$

where

$$\dot{\phi}(x_1) = \frac{d\phi}{dx_1}\dot{x}_1 = \frac{d\phi}{dx_1}\Big(f(x_1) + g(x_1)x_2\Big)$$



Consider $V_2(x_1, x_2) = V_1(x_1) + \zeta^2/2$. Then,

$$\dot{V}_2(x_1,x_2) = \frac{dV_1}{dx_1}\Big(f(x_1) + g(x_1)\phi(x_1)\Big) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$
$$\le -W(x_1) + \frac{dV_1}{dx_1}g(x_1)\zeta + \zeta v$$

Choosing

$$v = -\frac{dV_1}{dx_1}g(x_1) - k\zeta, \qquad k > 0$$

gives

$$\dot{V}_2(x_1,x_2) \le -W(x_1) - k\zeta^2$$

Hence, $x = 0$ is asymptotically stable for (**??**) with control law $u(x) = \dot{\phi}(x) + v(x)$.

If V_1 radially unbounded, then global stability.

Back-Stepping Lemma

Lemma: Let $z = (x_1, \dots, x_{k-1})^T$ and

$$\dot{z} = f(z) + g(z)x_k$$
$$\dot{x}_k = u$$

Assume $\phi(0) = 0$, $f(0) = 0$,

$$\dot{z} = f(z) + g(z)\phi(z)$$

stable, and $V(z)$ a Lyapunov fcn (with $\dot{V} \le -W$). Then,

$$u = \frac{d\phi}{dz}\Big(f(z) + g(z)x_k\Big) - \frac{dV}{dz}g(z) - (x_k - \phi(z))$$

stabilizes $x = 0$ with $V(z) + (x_k - \phi(z))^2/2$ being a Lyapunov fcn.

Strict Feedback Systems

Back-stepping Lemma can be applied to stabilize systems on strict feedback form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\
\dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)x_4 \\
&\vdots \\
\dot{x}_n &= f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)u
\end{aligned}$$

where $g_k \neq 0$

Note: $x_1, \dots, x_k$ do not depend on $x_{k+2}, \dots, x_n$.

Back-Stepping

Back-Stepping Lemma can be applied **recursively** to a system

$$\dot{x} = f(x) + g(x)u$$

on strict feedback form.

Back-stepping generates stabilizing feedbacks $\phi_k(x_1, \dots, x_k)$ (equal to u in Back-Stepping Lemma) and Lyapunov functions

$$V_k(x_1,\dots,x_k) = V_{k-1}(x_1,\dots,x_{k-1}) + [x_k - \phi_{k-1}]^2/2$$

by "stepping back" from x_1 to u

Back-stepping results in the final state feedback

$$u = \phi_n(x_1, \dots, x_n)$$

Example

Design back-stepping controller for

$$\dot{x}_1 = x_1^2 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

Step 0 Verify strict feedback form
Step 1 Consider first subsystem

$$\dot{x}_1 = x_1^2 + \phi_1(x_1), \quad \dot{x}_2 = u_1$$

where $\phi_1(x_1) = -x_1^2 - x_1$ stabilizes the first equation. With $V_1(x_1) = x_1^2/2$, Back-Stepping Lemma gives

$$u_1 = (-2x_1 - 1)(x_1^2 + x_2) - x_1 - (x_2 + x_1^2 + x_1) = \phi_2(x_1, x_2)$$
$$V_2 = x_1^2/2 + (x_2 + x_1^2 + x_1)^2/2$$

Step 2 Applying Back-Stepping Lemma on

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

gives

$$\begin{aligned}
u = u_2 &= \frac{d\phi_2}{dz}\Big(f(z) + g(z)x_n\Big) - \frac{dV_2}{dz}g(z) - (x_n - \phi_2(z)) \\
&= \frac{\partial\phi_2}{\partial x_1}(x_1^2 + x_2) + \frac{\partial\phi_2}{\partial x_2}x_3 - \frac{\partial V_2}{\partial x_2} - (x_3 - \phi_2(x_1,x_2))
\end{aligned}$$

which globally stabilizes the system.

Hybrid Control

Control problems where there is a mixture between continuous states and discrete state variables.

Continuous states: position, velocity, temperature, pressure

Discrete states: on/off variables, controller modes, loss of actuators, loss of sensors, relays, etc

Discontinuous differential equations

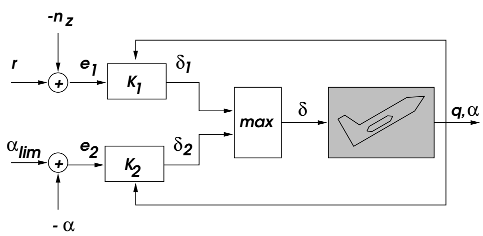
Much active field, much left to understand

Example of hybrid control

Control law that switches between different modes, e.g. between

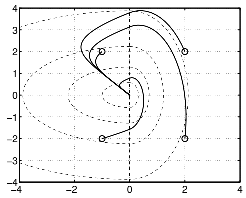
- Time optimal control – during large set point changes
- Linear control – close to set point

Aircraft Example



(Branicky, 1993)

Phase Plane



No common *quadratic* Lyapunov function exists.

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

Piecewise quadratic Lyapunov functions

$$V(x) = \begin{cases} x^* P x & \text{if } x_1 < 0 \\ x^* P x + \eta x_1^2 & \text{if } x_1 \geq 0 \end{cases}$$

The matrix inequalities

$$\begin{aligned} A_1^* P + P A_1 &< 0 \\ P &> 0 \\ A_2^*(P + \eta E^* E) + (P + \eta E^* E) A_2 &< 0 \\ P + \eta E^* E &> 0 \end{aligned}$$

with $E = [1\ 0]$, have the solution $P = \text{diag}\{1, 3\}$, $\eta = 7$.

Flower Example