Nonlinear Control and Servo System

Lecture 4, Lyapunov Stability

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Today's Goal

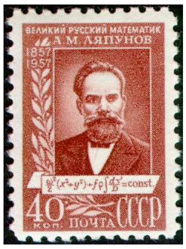
To be able to

- *prove local and global stability of an equilibrium point using Lyapunov's method*
- *show stability of a set (for example, a limit cycle) using La Salle's invariant set theorem.*

Material

- Glad & Ljung Ch. 12.2
- Khalil Ch. 4.1-4.3
- Slotine and Li: Chapter 3 (not 3.5.2-3.5.3)
- Lecture notes

Alexandr Mihailovich Lyapunov (1857–1918)



Master thesis "On the stability of ellipsoidal forms of equilibrium of rotating fluids," St. Petersburg University, 1884.

Doctoral thesis "The general problem of the stability of motion," 1892.

Main idea

Lyapunov formalized the idea:

If the total energy is dissipated, then the system must be stable.

Main benefit: By looking at **how** an energy-like function **V** (a so called *Lyapunov function*) **changes over time**, we might **conclude** that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.

Main question: **How to find** a Lyapunov function?

Examples

Start with a Lyapunov *candidate* V to measure e.g.,

- "size"[1] of state and/or output error,
- "size" of deviation from true parameters,
- energy difference from desired equilibrium,
- weighted combination of above
- ...

Example of common choice in adaptive control

$$V = \frac{1}{2}\left(e^2 + \gamma_a \tilde{a}^2 + \gamma_b \tilde{b}^2\right)$$

(here weighted sum of output error and parameter errors)

[1] Often a magnitude measure or (squared) norm like $|e|_2^2$, ...

Analysis: Check if V is decreasing with time

- Continuous time: $\frac{dV}{dt} < 0$
- Discrete time: $V(k+1) - V(k) < 0$

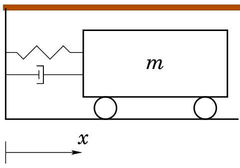
Synthesis: Choose e.g. control law and/or parameter update law to satisfy $\dot{V} \leq 0$

$$\frac{dV}{dt} = e\dot{e} + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} =$$
$$= \tilde{x}(-a\tilde{x} - \tilde{a}\hat{x} + \tilde{b}u) + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = ...$$

If a is constant and $\tilde{a} = a - \hat{a}$ then $\dot{\tilde{a}} = -\dot{\hat{a}}$.

Choose update law $\frac{d\hat{a}}{dt}$ in a "good way" to influence $\frac{dV}{dt}$.
(more on this later...)

A Motivating Example



$$m\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{damping}} - \underbrace{k_0 x - k_1 x^3}_{\text{spring}}$$

$$b, k_0, k_1 > 0$$

Total energy = kinetic + pot. energy: $V = \frac{mv^2}{2} + \int_0^x F_{spring}\, ds \Rightarrow$

$$V(x,\dot{x}) = m\dot{x}^2/2 + k_0 x^2/2 + k_1 x^4/4 > 0, \qquad V(0,0) = 0$$

$$\frac{d}{dt}V(x,\dot{x}) = m\ddot{x}\dot{x} + k_0 x\dot{x} + k_1 x^3 \dot{x} = \{\text{plug in system dynamics}^2\}$$
$$= -b|\dot{x}|^3 \quad < 0, \text{ for } \dot{x} \neq 0$$

What does this mean?

[2] Also referred to evaluate "along system trajectories".

Stability Definitions

An equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

locally stable, if for every $R > 0$ there exists $r > 0$, such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

locally asymptotically stable, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0$$

globally asymptotically stable, if asymptotically stable for all $x(0) \in \mathbf{R}^n$.

Lyapunov Theorem for Local Stability

Theorem Let $\dot{x} = f(x)$, $f(0) = 0$, and $0 \in \Omega \subset \mathbf{R}^n$. Assume that $V : \Omega \to \mathbf{R}$ is a C^1 function. If

(1) $V(0) = 0$

(2) $V(x) > 0$, for all $x \in \Omega$, $x \neq 0$

(3) $\frac{d}{dt}V(x) \leq 0$ along all trajectories of **the system** in Ω

then $x = 0$ is locally stable. Furthermore, if also

(4) $\frac{d}{dt}V(x) < 0$ for all $x \in \Omega$, $x \neq 0$

then $x = 0$ is locally asymptotically stable.

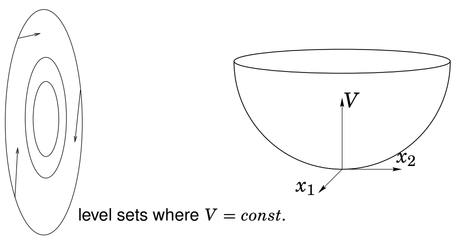
Lyapunov Functions (≈ Energy Functions)

A function V that fulfills (1)–(3) is called a *Lyapunov function*.

Condition (3) means that V is non-increasing along all trajectories in Ω:

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

where $\frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \dots & \frac{\partial V}{\partial x_n} \end{bmatrix}$



Conservation and Dissipation

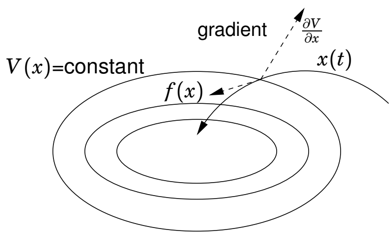
Conservation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 0$, i.e. the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.

Example: Total energy of a lossless mechanical system or total fluid in a closed system.

Dissipation of energy: $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, i.e. the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$ make an obtuse angle (Sw. "trubbig vinkel").

Example: Total energy of a mechanical system with damping or total fluid in a system that leaks.

Geometric interpretation



Vector field points into sublevel sets

Trajectories can only go to lower values of $V(x)$

Boundedness:

For an trajectory $x(t)$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)d\tau \leq V(x(0))$$

which means that the whole trajectory lies in the set

$$\{z \mid V(z) \leq V(x(0))\}$$

For stability it is thus important that the sublevel sets $\{z \mid V(z) \leq c)\}$ are locally bounded.

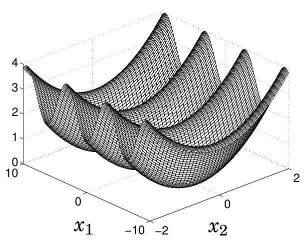
2 min exercise—Pendulum

Show that the origin is locally stable for a mathematical pendulum.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Use as a Lyapunov function candidate

$$V(x) = (1 - \cos x_1)g\ell + \ell^2 x_2^2/2$$



Lyapunov Theorem for Global Asymptotic Stability

Theorem Let $\dot{x} = f(x)$ and $f(0) = 0$.
If there exists a $\mathbb{C}^1$ function $V : \mathbf{R}^n \to \mathbf{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$, for all $x \neq 0$

(3) $\dot{V}(x) < 0$ for all $x \neq 0$

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

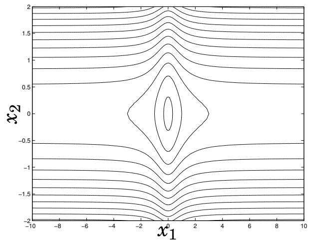
then $x = 0$ is globally asymptotically stable.

Radial Unboundedness is Necessary

If the condition $V(x) \to \infty$ as $\|x\| \to \infty$ is not fulfilled, then global stability cannot be guaranteed.

Example Assume $V(x) = x_1^2/(1+x_1^2) + x_2^2$ is a Lyapunov function for a system. Can have $\|x\| \to \infty$ even if $\dot{V}(x) < 0$.

Contour plot $V(x) = C$:



Example [Khalil]:

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2$$
$$\dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2}$$

Somewhat Stronger Assumptions

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(0) = 0$
(2) $V(x) > 0$ for all $x \neq 0$
(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x
(4) The sublevel sets $\{x | V(x) \leq c\}$ are bounded for all $c \geq 0$

then $x = 0$ is globally **exponentially** stable.

Proof Idea

Assume $x(t) \neq 0$ (otherwise we have $x(\tau) = 0$ for all $\tau > t$). Then

$$\frac{\dot{V}(x)}{V(x)} \leq -\alpha$$

Integrating from 0 to t gives

$$\log V(x(t)) - \log V(x(0) \leq -\alpha t \;\Rightarrow\; V(x(t)) \leq e^{-\alpha t} V(x(0))$$

Hence, $V(x(t)) \to 0, \, t \to \infty$.

Using the properties of V it follows that $x(t) \to 0, \, t \to \infty$.

Positive Definite Matrices

Definition: A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$. It is **positive semidefinite** if $x^T M x \geq 0$ for all x.

Lemma:

- $M = M^T$ is positive definite $\iff \lambda_i(M) > 0, \, \forall i$
- $M = M^T$ is positive semidefinite $\iff \lambda_i(M) \geq 0, \, \forall i$

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0, \, \forall x \neq 0$.

Positive Definite Matrices

A matrix M is **positive definite** if $x^T M x > 0$ for all $x \neq 0$.
It is **positive semidefinite** if $x^T M x \geq 0$ for all x.

A symmetric matrix $M = M^T$ is positive definite if and only if its eigenvalues $\lambda_i > 0$. (*semidefinite* $\Leftrightarrow \lambda_i \geq 0$)

Note that if $M = M^T$ is positive definite, then the Lyapunov function candidate $V(x) = x^T M x$ fulfills $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

More matrix results

A symmetric matrix $M = M^T$ satisfies the inequalities

$$\lambda_{\min}(M)\|x\|^2 \;\leq\; x^T M x \;\leq\; \lambda_{\max}(M)\|x\|^2$$

(To show it, use the factorization $M = U \Lambda U^*$, where U is a unitary matrix, $\|Ux\| = \|x\|$, U^* is complex conjugate transpose, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.)

For any matrix M one also has

$$\|Mx\| \;\leq\; \lambda_{\max}^{1/2}(M^T M)\|x\|$$

Lyapunov Stability for Linear Systems

Linear system: $\dot{x} = Ax$

Lyapunov equation: Let Q be a positive definite symmetric matrix and solve

$$PA + A^T P = -Q$$

with respect to the symmetric matrix P.

Lyapunov function: $V(x) = x^T P x, \Rightarrow$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x < 0$$

Global Asymptotic Stability: If P is positive definite, then the Lyapunov Stability Theorem implies global asymptotic stability, and hence the eigenvalues of A must satisfy $\text{Re}\,\lambda_k(A) < 0$ for all k

Converse Theorem for Linear Systems

If $\text{Re}\,\lambda_k(A) < 0$, then for every symmetric positive definite Q there exist a symmetric positive definite matrix $P =$ such that $PA + A^T P = -Q$

Proof: Choose $P = \int_0^\infty e^{A^T t} Q e^{At} dt$. Then

$$A^T P + PA = \int_0^\infty \left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} \right) dt$$
$$0 = \int_0^\infty \left(\frac{d}{dt} e^{A^T t} Q e^{At} \right) dt = \left[e^{A^T t} Q e^{At} \right]_0^\infty = -Q$$

Interpretation

Assume $\dot{x} = Ax$, $x(0) = z$. Then

$$\int_0^\infty x^T(t)Qx(t)dt = z^T \left(\int_0^\infty e^{A^Tt}Qe^{At}dt \right) z = z^T Pz$$

Thus $v(z)$ is the cost-to-go from the point z (with no input) and integral quadratic cost function with weighting matrix Q.

Lyapunov's Linearization Method

Recall from Lecture 2:

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $x = 0$ is an equilibrium point and that

$$\dot{x} = Ax + g(x)$$

is a linearization.

(1) If Re $\lambda_k(A) < 0$ for all k, then $x = 0$ is locally asymptotically stable.

(2) If there exists k such that $\lambda_k(A) > 0$, then $x = 0$ is unstable.

Proof of (1) in Lyapunov's Linearization Method

Lyapunov function candidate $V(x) = x^T Px$. $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and

$$\begin{aligned} \dot{V}(x) &= x^T Pf(x) + f^T(x)Px \\ &= x^T P[Ax + g(x)] + [x^T A + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T Pg(x) = -x^T Qx + 2x^T Pg(x) \end{aligned}$$

$$x^T Qx \geq \lambda_{\min}(Q)\|x\|^2$$

and for all $\gamma > 0$ there exists $r > 0$ such that

$$\|g(x)\| < \gamma\|x\|, \qquad \forall \|x\| < r$$

Thus, choosing γ sufficiently small gives

$$\dot{V}(x) \leq -(\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P))\|x\|^2 < 0$$

LaSalle's Theorem for Global Asymptotic Stability

Theorem: Let $\dot{x} = f(x)$ and $f(0) = 0$. If there exists a $\mathbb{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(0) = 0$

(2) $V(x) > 0$ for all $x \neq 0$

(3) $\dot{V}(x) \leq 0$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

(5) The only solution of $\dot{x} = f(x)$, $\dot{V}(x) = 0$ is $x(t) = 0$ for all t

then $x = 0$ is globally asymptotically stable.

A Motivating Example (cont'd)

$$\begin{aligned} m\ddot{x} &= -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3 \\ V(x) &= (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0 \\ \dot{V}(x) &= -b|\dot{x}|^3 \end{aligned}$$

Assume that there is a trajectory with $\dot{x}(t) = 0$, $x(t) \neq 0$. Then

$$\frac{d}{dt}\dot{x}(t) = -\frac{k_0}{m}x(t) - \frac{k_1}{m}x^3(t) \neq 0,$$

which means that $\dot{x}(t)$ can not stay constant.

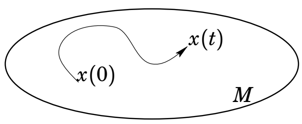
Hence, $x(t) = 0$ is the only possible trajectory for which $\dot{V}(x) = 0$, and the LaSalle theorem gives global asymptotic stability.

Invariant Sets

Definition: A set M is called **invariant** if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M$ for all $t \geq 0$.

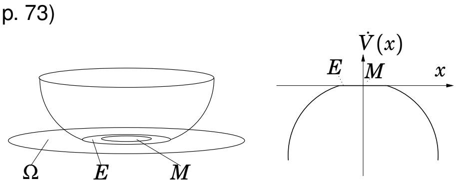


Invariant Set Theorem

Theorem Let $\Omega \in \mathbf{R}^n$ be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a radially unbounded $\mathbb{C}^1$ function such that $\dot{V}(x) \leq 0$ for $x \in \Omega$. Let E be the set of points in Ω where $\dot{V}(x) = 0$. If M is the largest invariant set in E, then every solution with $x(0) \in \Omega$ approaches M as $t \to \infty$ (proof on p. 73)



Note that V must **not** be a positive definite function in this case.

Example—Stable Limit Cycle

Show that $M = \{x : \|x\| = 1\}$ is a asymptotically stable limit cycle for (almost globally, except for starting at x=0):

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

Let $V(x) = (x_1^2 + x_2^2 - 1)^2$.

$$\begin{aligned} \frac{dV}{dt} &= 2(x_1^2 + x_2^2 - 1)\frac{d}{dt}(x_1^2 + x_2^2 - 1) \\ &= -2(x_1^2 + x_2^2 - 1)^2(x_1^2 + x_2^2) \leq 0 \quad \text{for } x \in \Omega \end{aligned}$$

$\Omega = \{0 < \|x\| \leq R\}$ is invariant for $R = 1$.

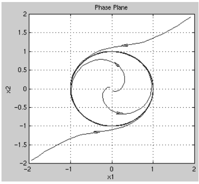
Example— Stable Limit Cycle

$$E = \{x \in \Omega : \dot{V}(x) = 0\} = \{x : \|x\| = 1\}$$

$M = E$ is an invariant set, because

$$\frac{d}{dt}V = -2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2) = 0 \quad \text{for } x \in M$$

We have shown that M is a asymtotically stable limit cycle (globally stable in $R - \{0\}$)



A Motivating Example (revisited)

$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3$$

$$V(x, \dot{x}) = (2m\dot{x}^2 + 2k_0 x^2 + k_1 x^4)/4 > 0, \qquad V(0,0) = 0$$

$\dot{V}(x, \dot{x}) = -b|\dot{x}|^3$ gives $E = \{(x, \dot{x}) : \dot{x} = 0\}$.

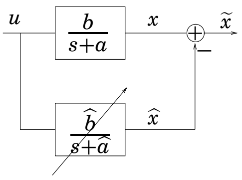
Assume there exists $(\bar{x}, \dot{\bar{x}}) \in M$ such that $\bar{x}(t_0) \neq 0$. Then

$$m\ddot{\bar{x}}(t_0) = -k_0\bar{x}(t_0) - k_1\bar{x}^3(t_0) \neq 0$$

so $\dot{\bar{x}}(t_0+) \neq 0$ so the trajectory will immediately leave M. A contradiction to that M is invariant.

Hence, $M = \{(0,0)\}$ so the origin is asymptotically stable.

Adaptive Noise Cancellation by Lyapunov Design



$$\dot{x} + ax = bu$$
$$\dot{\hat{x}} + \hat{a}\hat{x} = \hat{b}u$$

Introduce $\tilde{x} = x - \hat{x}$, $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$.

Want to design adaptation law so that $\tilde{x} \to 0$

Let us try the Lyapunov function

$$V = \frac{1}{2}(\tilde{x}^2 + \gamma_a\tilde{a}^2 + \gamma_b\tilde{b}^2)$$
$$\dot{V} = \tilde{x}\dot{\tilde{x}} + \gamma_a\tilde{a}\dot{\tilde{a}} + \gamma_b\tilde{b}\dot{\tilde{b}} =$$
$$= \tilde{x}(-a\tilde{x} - \tilde{a}\hat{x} + \tilde{b}u) + \gamma_a\tilde{a}\dot{\tilde{a}} + \gamma_b\tilde{b}\dot{\tilde{b}} = -a\tilde{x}^2$$

where the last equality follows if we choose

$$\dot{\tilde{a}} = -\dot{\hat{a}} = \frac{1}{\gamma_a}\tilde{x}\hat{x} \qquad \dot{\tilde{b}} = -\dot{\hat{b}} = -\frac{1}{\gamma_b}\tilde{x}u$$

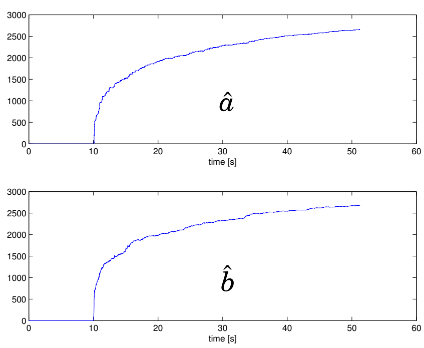
Invariant set: $\tilde{x} = 0$.

This proves that $\tilde{x} \to 0$.

(The parameters $\tilde{a}$ and $\tilde{b}$ do not necessarily converge: $u \equiv 0$.)

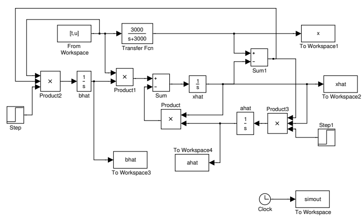
Demonstration if time permits

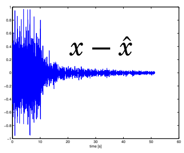
Results



Estimation of parameters starts at t=10 s.

Results



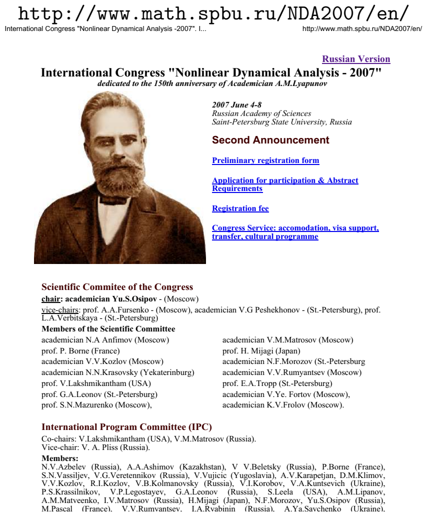


Estimation of parameters starts at t=10 s.

Next Lecture

- Stability analysis using input-output (frequency) methods

http://www.math.spbu.ru/NDA2007/en/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Russian Version

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