Nonlinear Control
Lecture 3

Dept. of Automatic Control
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Today's Goal

You should be able to

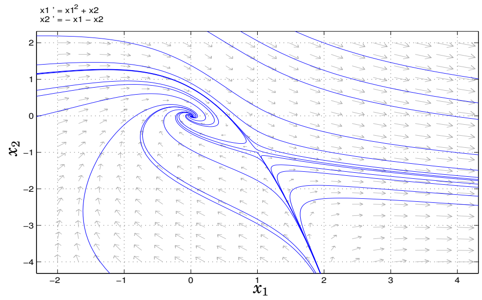
- *sketch phase portraits for two-dimensional systems*
- *classify equilibria into nodes, focus, saddle points, and center points.*
- *analyze limit cycles through Poincaré maps*

Material

- Glad and Ljung: Chapter 13
- Slotine and Li: Chapter 2 (except the isocline method and Section 2.6)
- Khalil: Chapter 2.1–2.3
- Lecture notes

First glipse of phase plane portraits: Consider the system

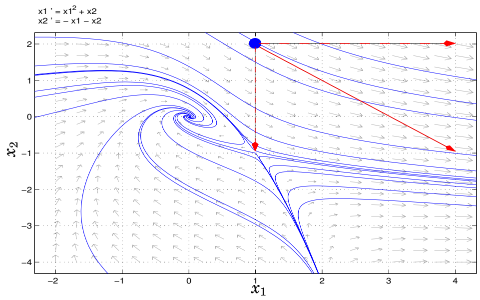
$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



Flow-interpretation: To each point $(x_1,\, x_2)$ in the plane there is an associated flow-direction $\frac{dx}{dt} = f(x_1, x_2)$

First glipse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



In the point $(x_1,\, x_2) = (1,\, 2)$ **the vector field** is pointing in the direction $(1^2 + 2, -1 - 2) = (3,\, -3)$.

Linear Systems Revival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0)$.

If A is diagonalizable, then

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A $\quad (Av_1 = \lambda_1 v_1$ etc).

Matlab:

```
>> [V,Lambda]=eig(A)
```

Linear *Time-Varying* Systems (warning)

Warning: Pointwise "Left Half-Plane eigenvalues" of $A(t)$ (*i.e., time-varying systems*) do *NOT* impose stability!!!

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are in the LHP for $0 < \alpha < 2$ (and here even constant). However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix} x(0),$$

which is an unbounded solution for $\alpha > 1$.

Example: Two real negative eigenvalues

Given the eigenvalues $\underbrace{\lambda_1}_{faster} < \underbrace{\lambda_2}_{slower} < 0$, with corresponding eigenvectors v_1 and v_2, respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t.
Moves along the fast eigenvector for small t

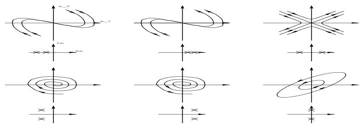
Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t.
Moves along the slow eigenvector towards $x = 0$ for large t

Phase-Plane Analysis for Linear Systems

The location of the eigenvalues $\lambda(A)$ determines the characteristics of the trajectories.

Six cases:

	stable node	unstable node	saddle point
$\text{Im}\lambda_i = 0:$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
$\text{Im}\lambda_i \neq 0:$	$\text{Re}\lambda_i < 0$	$\text{Re}\lambda_i > 0$	$\text{Re}\lambda_i = 0$
	stable focus	unstable focus	center point



Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

In polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan x_2/x_1$ $(x_1 = r\cos\theta,\ x_2 = r\sin\theta)$:

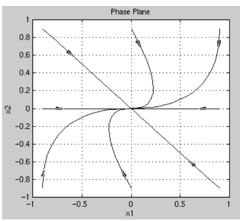
$$\dot{r} = \sigma r$$
$$\dot{\theta} = \omega$$

Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

v_1 is the slow direction and v_2 is the fast.



Phase-Plane Analysis for Nonlinear Systems

Close to equil. points "nonlinear system" ≈ "linear system".

Theorem Assume

$$\dot{x} = f(x)$$

is linearized so that

$$\dot{x} = Ax + g(x),$$

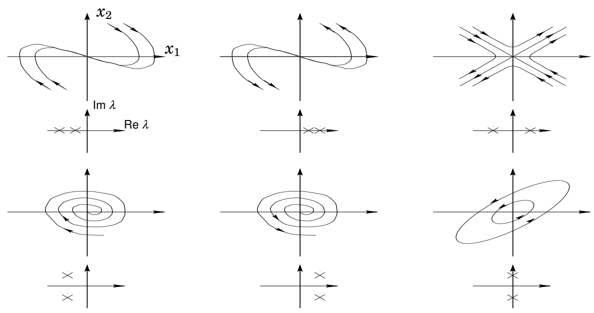
where $g \in C^1$ and $\|g(x)\| < \|x\|^{1+\epsilon}$ for some $\epsilon > 0$.

If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

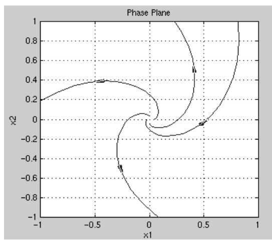
Equilibrium Points for Linear Systems

	stable node	unstable node	saddle point
$\text{Im}\lambda_i = 0:$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
$\text{Im}\lambda_i \neq 0:$	$\text{Re}\lambda_i < 0$	$\text{Re}\lambda_i > 0$	$\text{Re}\lambda_i = 0$
	stable focus	unstable focus	center point

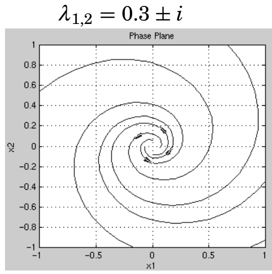


Example- unstable focus cont'd

$\lambda_{1,2} = 1 \pm i$



$\lambda_{1,2} = 0.3 \pm i$



4 minute exercise:

Hint: For $\lambda_1 = \lambda_2 = \lambda$ there are two different cases: only one linear independent eigenvector or all vectors are eigenvectors

How to Draw Phase Portraits

If done by hand then

1. Find equilibria (also called singularities)
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
4. Try to find possible limit cycles
5. Guess solutions

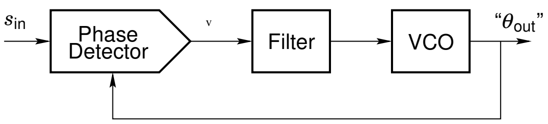
Matlab: `pptool6/pptool7, dfield6/dfield7, dee, ICTools,` etc.

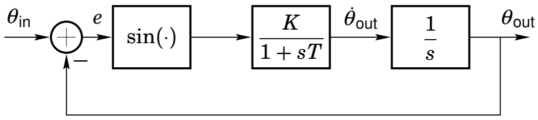
PPTool and some other tools for Matlab is available on or via

`http://www.control.lth.se/course/FRTN05`

Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A\sin[\omega t + \theta_{\text{in}}(t)]$.





Singularity Analysis of PLL

Let $x_1(t) = \theta_{\text{out}}(t)$ and $x_2(t) = \dot{\theta}_{\text{out}}(t)$.
Assume $K, T > 0$ and $\theta_{\text{in}}(t) \equiv \theta_{\text{in}}$ constant.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -T^{-1}x_2 + KT^{-1}\sin(\theta_{\text{in}} - x_1)$$

Singularities are $(\theta_{\text{in}} + n\pi, 0)$, since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Singularity Classification of Linearized System

Linearization gives the following characteristic equations:

n even:
$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

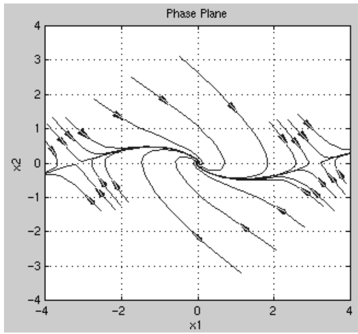
$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

n odd:
$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

$K = 1/2$, $T = 1$: Focus $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



Summary

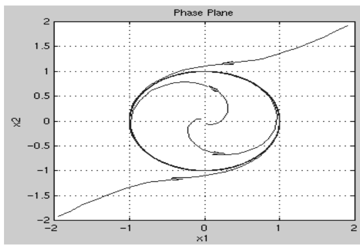
Phase-plane analysis limited to second-order systems (sometimes it is possible for higher-order systems to fix some states)

Many dynamical systems of order three and higher not fully understood (chaotic behaviors etc.)

Periodic Solutions: $x(t+T) = x(t)$

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \qquad (1)$$



Periodic solution: Polar coordinates.

Let
$$x_1 = r\cos\theta \quad \Rightarrow dx_1 = \cos\theta dr - r\sin\theta d\theta$$
$$x_2 = r\sin\theta \quad \Rightarrow dx_2 = \sin\theta dr + r\cos\theta d\theta$$

$\Rightarrow$
$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now
$$\dot{x}_1 = r(1-r^2)\cos\theta - r\sin\theta$$
$$\dot{x}_2 = r(1-r^2)\sin\theta + r\cos\theta$$

which gives
$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \quad \forall t \geq 0$$

*N*ote that a constant value for x(t) by convention not is regarded as periodic.

- When does a periodic solution exist?
- When is it locally (asymptotically) stable? When is it globally asymptotically stable?

Poincaré map (“Stroboscopic map”)

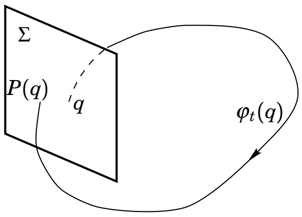
$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n$$

$\varphi_t(q)$ is the solution starting in q after time t.

$\Sigma \subset \mathbf{R}^{n-1}$ is a hyperplane transverse to φ_t.

The Poincaré map $P : \Sigma \to \Sigma$ is

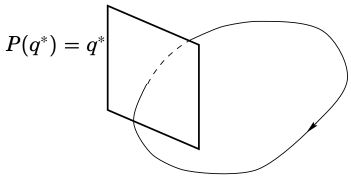
$$P(q) = \varphi_{\tau(q)}(q), \qquad \tau(q) \text{ is the first return time}$$



Limit Cycles

If a simple periodic orbit pass through q^*, then $P(q^*) = q^*$.

Such an orbit is called a *limit cycle*.
q^* is called a *fixed point* of P.



Does the iteration $q_{k+1} = P(q_k)$ converge to q^*?

Locally Stable Limit Cycles

The linearization of P around q^* gives a matrix $W = \frac{\partial P}{\partial q}\Big|_{q^*}$ so

$$(q_{k+1} - q^*) \approx W(q_k - q^*),$$

if q_k is close to q^*.

- If all $|\lambda_i(W)| < 1$, then the corresponding limit cycle is locally **asymptotically stable**.
- If $|\lambda_i(W)| > 1$, then the limit cycle is **unstable**.

Linearization Around a Periodic Solution

The linearization of

$$\dot{x}(t) = f(x(t))$$

around $x_0(t) = x_0(t+T)$ is

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$
$$A(t) = \frac{\partial f}{\partial x}(x_0(t)) = A(t+T)$$

P is the map from the solution at $t = 0$ to $t = \tau(q)$.

Example—Stable Unit Circle

Rewrite (**??**) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\varphi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is $\tau(r_0, \theta_0) = 2\pi$.

Example—Stable Unit Circle

The Poincaré map is

$$P(r_0) = [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2}$$

$r_0 = 1$ is a fixed point.

The limit cycle that corresponds to $r(t) = 1$ and $\theta(t) = t$ is locally asymptotically stable, because

$$W = \frac{dP}{dr_0}(1) = \left[e^{-4\pi}\right]$$

and

$$|W| = \left|\frac{dP}{dr_0}(1)\right| = |e^{-4\pi}| < 1$$

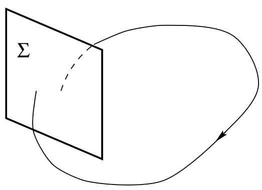
Example—The Hand Saw

Can we stabilize the inverted pendulum by vertical oscillations?

The Hand Saw—Poincaré Map

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{\ell}\left(g + a\omega^2 \sin x_3\right) \sin x_1$$
$$\dot{x}_3(t) = \omega$$

Choose $\Sigma = \{x_3 = 2\pi k\}$.



The Hand Saw–Poincaré Map

$q^* = 0$ and $T = 2\pi/\omega$. No explicit expression for P. It is, however, easy to determine W numerically. Do two (*or preferably many more*) different simulations with different, small, initial conditions $x(0) = y$ and $x(0) = z$.
Solve W through (*least squares solution of*)

$$\begin{pmatrix} x(T)\big|_{x(0)=y} & x(T)\big|_{x(0)=z} \end{pmatrix} = W \begin{pmatrix} y & z \end{pmatrix}$$

This gives for $a = 1$cm, $\ell = 17$cm, $\omega = 180$

$$W = \begin{pmatrix} 1.37 & 0.035 \\ -3.86 & 0.630 \end{pmatrix}$$

which has eigenvalues $(1.047, 0.955)$. Unstable.

W is stable for $\omega > 183$

The Hand Saw—Stability Condition

Make the assumptions that

$$\ell \gg a \qquad \text{and} \qquad a\omega^2 \gg g$$

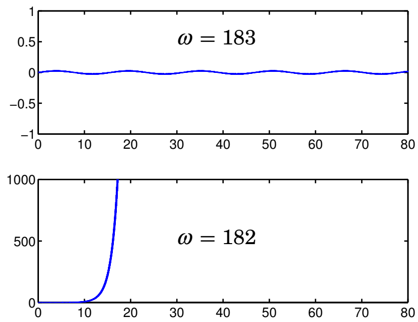
Then some calculations show that the Poincaré map is stable at $q^* = 0$ when

$$\omega > \frac{\sqrt{2g\ell}}{a}$$

$a = 1$ cm and $\ell = 17$ cm give $\omega > 182.6$ rad/s (29 Hz).

The Hand Saw—Simulation

Simulation results give good agreement

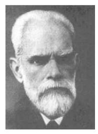


Next Lecture

- Lyapunov methods for stability analysis

Lyapunov generalized the idea of: *If the total energy is dissipated along the trajectories (i.e the solution curves), the system must be stable.*

Benefit: Might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.



Nonlinear control is a serious business... cheer up ☺

Lab 1