Nonlinear Control (FRTN05)

Model Predictive Control (MPC) Exercise

Last updated: Spring of 2009

1. Consider the state-space system:

$$\hat{x}(k+1|k) = 2\hat{x}(k|k-1) + u(k)$$
$$\hat{y}(k|k-1) = 3\hat{x}(k|k-1)$$

with the output constraint:

$$-1 \le y(k) \le 2, \quad \forall k$$

If the current state estimate $\hat{x}(k|k-1) = 3$ and the previous control input $u(k-1) = -1$, show that the corresponding constraint on the control signal change $\Delta u(k)$ is given by:

$$-\frac{16}{3} \le \Delta u(k) \le -\frac{13}{3}$$

2. Consider the same system as in exercise 1. Assume that the prediction horizon $N_p = 3$, the control horizon $N_u = 2$ and the cost function

$$V = \sum_{k=1}^{N_p} |\hat{y}(k|k-1)|^2$$

Formulate the optimization problem (from an initial point $\hat{x}(0)$), that is considered in MPC. Use first $u(k)$ as decision variables, then use $\Delta u(k)$. Assume that $u(k) = u(N_u - 1)$ for $k \ge N_u$ and that $u(-1)$ is given if needed.

Solutions

1. Consider the state-space system: The output constraint must be fulfilled at all time instances k, hence we have

$$-1 \leq y(k) \leq 2, \quad \forall k$$

We also know that

$$\hat{y}(k+1|k) = 3\hat{x}(k+1|k) = 6\hat{x}(k|k-1) + 3u(k)$$

Using that $u(k) = u(k-1) + \Delta u(k)$, we get

$$\hat{y}(k+1|k) = 6\hat{x}(k|k-1) + 3u(k-1) + 3\Delta u(k)$$

Substituting this relation into the output constraint gives

$$-1 \leq 6\hat{x}(k|k-1) + 3u(k-1) + 3\Delta u(k) \leq 2$$

and using the given values of $\hat{x}(k|k-1) = 3$ and $u(k-1) = -1$, we get the correct constraint on $\Delta u(k)$:

$$-\frac{16}{3} \leq \Delta u(k) \leq -\frac{13}{3}$$

2. First, instead of writing $\hat{x}(k|0)$ we write $\hat{x}_k$. For any system

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k \\ \hat{y}_k &= C\hat{x}_k \end{aligned}$$

the states' and the outputs' evolution in time is

$$\begin{aligned} \hat{x}_k &= A^k\hat{x}_0 + \sum_{i=0}^{k-1} A^{k-1-i}Bu_i \\ \hat{y}_k &= CA^k\hat{x}_0 + \sum_{i=0}^{k-1} CA^{k-1-i}Bu_i \end{aligned}$$

Using the system equation, we have that

$$\begin{aligned} \hat{x}_1 &= 2\hat{x}_0 + u_0 \\ \hat{x}_2 &= 4\hat{x}_0 + 2u_0 + u_1 \\ \hat{x}_3 &= 8\hat{x}_0 + 4u_0 + 2u_1 + u_2 \end{aligned}$$

Since $N_u = 2$ we have that $u_2 = u_1$, and the relations for $\hat{y}_k$ becomes

$$\begin{aligned} \hat{y}_1 &= 6\hat{x}_0 + 3u_0 \\ \hat{y}_2 &= 12\hat{x}_0 + 6u_0 + 3u_1 \\ \hat{y}_3 &= 24\hat{x}_0 + 12u_0 + 9u_1 \end{aligned}$$

Hence, the optimization problem we need to solve is

$$\begin{aligned}
\text{minimize} \quad & |6\hat{x}_0 + 3u_0|^2 + |12\hat{x}_0 + 6u_0 + 3u_1|^2 + |24\hat{x}_0 + 12u_0 + 9u_1|^2 \\
\text{subject to :} \quad & -1 - 6\hat{x}_0 \leq 3u_0 \leq 2 - 6\hat{x}_0 \\
& -1 - 12\hat{x}_0 \leq 6u_0 + 3u_1 \leq 2 - 12\hat{x}_0 \\
& -1 - 24\hat{x}_0 \leq 12u_0 + 9u_1 \leq 2 - 24\hat{x}_0
\end{aligned}$$

If we instead want to use Δu_k as decision variables we have that

$$\begin{aligned}
u_0 &= u_{-1} + \Delta u_0 \\
u_1 &= u_{-1} + \Delta u_0 + \Delta u_1 \\
u_2 &= u_{-1} + \Delta u_0 + \Delta u_1 + \Delta u_2
\end{aligned}$$

Since the control horizon $N_u = 2$, $\Delta u_2 = 0$. We now immediately get the optimization problem

$$\begin{aligned}
\text{minimize} \quad & |6\hat{x}_0 + 3u_{-1} + 3\Delta u_0|^2 + |12\hat{x}_0 + 9u_{-1} + 9\Delta u_0 + 3\Delta u_1|^2 + \\
& + |24\hat{x}_0 + 21u_{-1} + 21\Delta u_0 + 9\Delta u_1|^2 \\
\text{subject to :} \quad & -1 - 6x_0 - 3u_{-1} \leq 3\Delta u_0 \leq 2 - 6x_0 - 3u_{-1} \\
& -1 - 12\hat{x}_0 - 9u_{-1} \leq 9\Delta u_0 + 3\Delta u_1 \leq 2 - 12\hat{x}_0 - 9u_{-1} \\
& -1 - 24\hat{x}_0 - 21u_{-1} \leq 21\Delta u_0 + 9\Delta u_1 \leq 2 - 24\hat{x}_0 - 21u_{-1}
\end{aligned}$$

(The reason to use Δu_k instead of u_k is that we usually want to penalize changes in the control input, i.e. penalize Δu_k. This can obviously still be done if we use u_k as well, but it involves a bit larger expressions.)