LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

Nonlinear Control and Servo Systems (FRTN05)

Exam - May 28, 2010 at 08.00–13.00

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: $12 - 16.5$ points

4: $17 - 21.5$ points

5: $22 - 25$ points

Accepted aid

All course material, except for the exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

Results

The result of the exam will be posted on the notice-board at the Department. Contact the lecturer Anders Robertsson to see the corrected exam. The result will also be published on the course homepage `http://www.control.lth.se/course/FRTN05/`.

Note!

In many cases the sub-problems can be solved independently of each other.

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
2010–05–28

1. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \sin x_1 - x_1 x_2 + u$$

Let $u = 0$. Determine all equilibrium points of the system, and determine their phase plane characteristics. (3 p)

Solution

From the first state equation, we have $x_2 = 0$. Plugging this into the second state equation we have $\sin x_1 = 0 \;\Rightarrow x_1 = k\pi,\; k = \dots, -1, 0, 1, \dots$. The stationary points are:

$$(x_1^0, x_2^0) = (k\pi, 0), \quad k = \dots, -1, 0, 1, \dots$$

To check the phase plane characteristics we linearize the system.

$$A = \begin{pmatrix} 0 & 1 \\ \cos x_1^0 - x_2^0 & -x_1^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \cos k\pi & -k\pi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^k & -k\pi \end{pmatrix}$$

The characteristic polynomial becomes

$$s^2 + k\pi s + (-1)^{k+1}$$

For $k = 0$ we have

$$s^2 - 1 = 0 \quad \Rightarrow \text{saddle point}$$

For $k \neq 0$ the poles are located at:

$$s = -\frac{k\pi}{2} \pm \sqrt{\left(\frac{k\pi}{2}\right)^2 + (-1)^k}$$

Since

$$\left(\frac{k\pi}{2}\right)^2 + (-1)^k > 0 \quad \forall k \neq 0$$

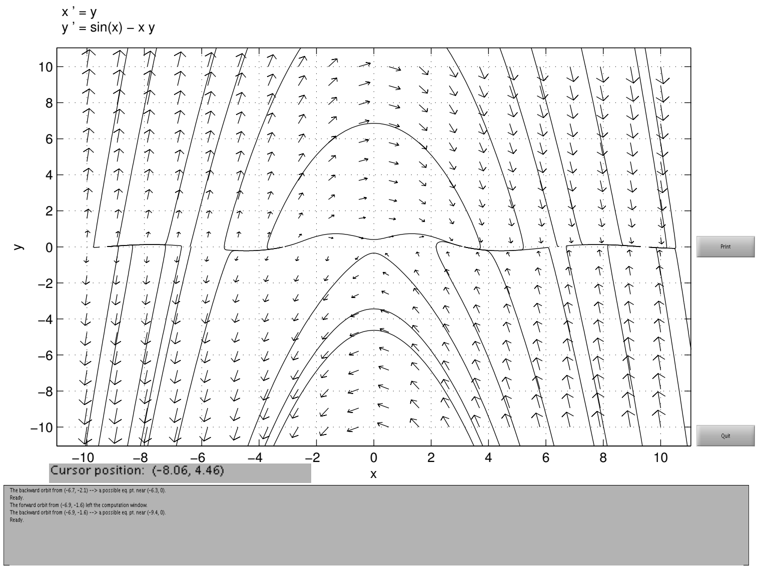
the poles will always be real. Also

$$\left|\frac{k\pi}{2}\right| < \sqrt{\left(\frac{k\pi}{2}\right)^2 + (-1)^k} \quad \text{if } k \text{ even}$$
$$\left|\frac{k\pi}{2}\right| > \sqrt{\left(\frac{k\pi}{2}\right)^2 + (-1)^k} \quad \text{if } k \text{ uneven}$$

which shows that the poles will have different signs if k is uneven, which implies a saddle point, and the same sign if k is even, which implies a node. We conclude:

$$
\begin{aligned}
&k = 0 \quad \Rightarrow \text{saddle point}\\
&k > 0 \text{ and } k \text{ uneven} \quad \Rightarrow \text{stable node}\\
&k < 0 \text{ and } k \text{ uneven} \quad \Rightarrow \text{unstable node}\\
&k \neq 0 \text{ and } k \text{ even} \quad \Rightarrow \text{saddle point}
\end{aligned}
$$

The phase plane of the system is shown in Figure 1



Figur 1 Phase plane of the system in 4

2. Show that the system

$$
\begin{aligned}
\dot{x} &= -x^3 + y^2\\
\dot{y} &= -2xy - y
\end{aligned}
$$

is globally asymptotically stable. (2 p)

Solution

Use the function $V = ax^2 + by^2$, where a, $b > 0$ as a candidate function. The function fulfills the requirements $V(0) = 0$, $V > 0$, and $V \to \infty$, $|x| \to \infty$. Then we have that

$$
\begin{aligned}
\dot{V} &= 2ax(-x^3 + y^2) + 2by(-2xy - y) =\\
&= -2ax^4 + (2a - 4b)xy^2 - 2by^2.
\end{aligned}
$$

By choosing , for example, $a = 2$ and $b = 1$ one finds that $\dot{V} = -4x^4 - 2y^2 < 0$ when $(x, y) \neq 0$, which proves that the system is GAS.

3. Consider the following system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2 + x_1^2 + u$$

a. Verify that the origin of the system is not globally asymptotically stable for $u = 0$. (1 p)

b. Your goal is to design a sliding mode controller that makes the origin globally asymptotically stable. Which of the sliding sets, $\sigma_1(x) = 0$, or $\sigma_2(x) = 0$, guarantee(s) asymptotic convergence of the states to the origin. *Motivate your answer by analysis of the dynamics on the set* $\{x|\sigma(x) = 0\}$.

$$\sigma_1(x) = x_1 + x_2$$
$$\sigma_2(x) = x_1 - x_2$$

(1 p)

c. Design a control law that drives the states of the system to the sliding set you chose above. (2 p)

Solution

a. Linearization around the origin yields:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$$

The linearized system has an eigenvalue in 1, which shows that the origin is unstable.

b. A sliding mode is an invariant set, i.e. $\dot{\sigma} = 0$. We have

$$0 = \dot{\sigma}_1 = \dot{x}_1 + \dot{x}_2 = x_2 + \dot{x}_2 \Rightarrow \dot{x}_2 = -x_2$$

which shows that $x_2(t) \to 0,\ t \to \infty$ along $\sigma_1(x) = 0$. Since $x_1(t) = -x_2(t)$ on the sliding set, also $x_1(t) \to 0,\ t \to \infty$.
For $\sigma_2(x)$ we have:

$$0 = \dot{\sigma}_2 = \dot{x}_1 - \dot{x}_2 = x_2 - \dot{x}_2 \Rightarrow \dot{x}_2 = x_2$$

which means $x_2(t) \to \pm\infty$, along $\sigma_2(x) = 0$.
The answer is: $\sigma_1(x) = 0$ guarantees asymptotic convergence to the origin.

c. Let

$$V(\sigma) = \frac{\sigma^2}{2}$$

We have that:

$$\dot{V} = \frac{dV}{d\sigma}\frac{d\sigma}{dx}\frac{dx}{dt} = \sigma(\dot{x}_1 + \dot{x}_2) = \sigma(2x_2 + x_1^2 + u)$$

By choosing

$$u = -2x_2 - x_1^2 - k\sigma = -(2+k)x_2 - x_1(x_1 + k)$$

with $k > 0$ we have

$$\dot{V} = -k\sigma^2$$

which means that $\sigma \to 0$ as $t \to \infty$ Another choice is

$$u = -2x_2 - x_1^2 - \text{sign}(\sigma)$$

4. Consider two systems, S_1 and S_2, where S_2 has an input nonlinearity Ψ:

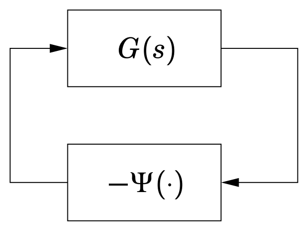
$$S_1 : \quad \begin{cases} \dot{x}_1 = -x_1 + ku_1 \\ y_1 = x_1 \end{cases}$$

$$S_2 : \quad \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_1 - z_2 - \Psi(u_2) \\ y_2 = z_1 \end{cases}$$

The systems are connected according to

$$\begin{aligned} u_1 &= y_2 \\ u_2 &= y_1 \end{aligned}$$

a. The interconnected system can be described by the system in Figure 2. Compute the transfer function G. (2 p)



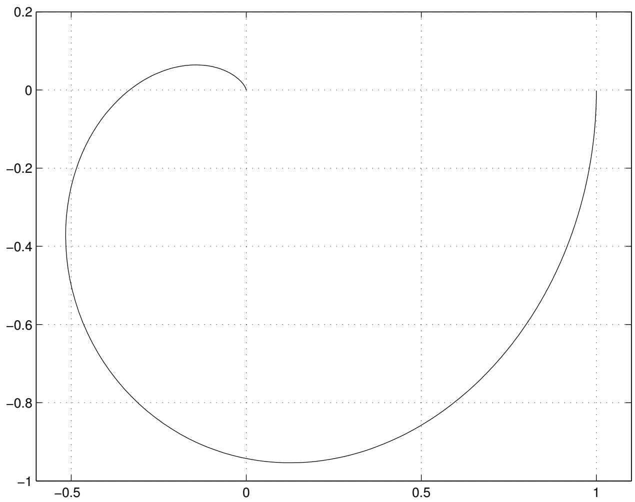
Figur 2 Figure for Problem 4

b. The Nyquist plot of G in Figure 2 is displayed in Figure 3, for $k = 1$. The nonlinearity Ψ is shown in Figure 4. For what values of k does the describing function analysis predict a limit cycle. Will the limit cycle be stable? *Note: You do not have to compute the describing function for Ψ explicitly.* (3 p)

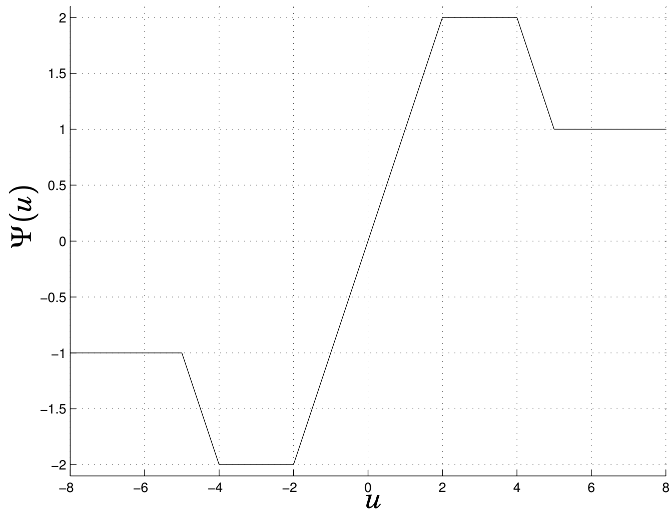
Solution

The systems are interconnected as shown in Figure 5, where G_1 and G_2 are the transfer functions of S_1 and S_2, respectively. G_1 is given by:

$$G_1 = \frac{k}{s+1}$$



Figur 3 The Nyquist plot of G in Problem 4, with $k = 1$



Figur 4 The nonlinearity in Problem 4

To obtain G_2 let $v = \Psi(u)$. G_2 can then be obtained by transforming S_2 into state space form and computing the transfer function through:

$$G_2(s) = C(sI - A)^{-1}B$$

Another (much simpler) way is to use, $\dot{z}_2 = \ddot{z}_1 = \ddot{y}_2$, in the second state equation to obtain:

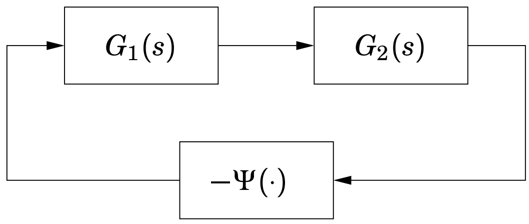
$$\ddot{y}_2 + \dot{y}_2 + y_2 = v$$

Taking the Laplace transforms of both sides results in:

$$G_2(s) = \frac{1}{s^2 + s + 1}$$

G is then given by:

$$G(s) = G_1(s)G_2(s) = \frac{k}{(s+1)(s^2+s+1)}$$



Figur 5 Figure for Problem 4

Since Ψ is odd, its describing function is given by:

$$N(A) = \frac{2}{\pi A} \int_0^{\pi} \Psi(A \sin \phi) \sin \phi d\phi$$

Understanding of describing functions is enough, we do not have to compute $N(A)$. We only need to derive upper and lower bounds, and show that these bounds are tight.

Upper bound: We know that if $A < 1$, $\Psi(A \sin \phi) = A \sin \phi$, $\forall \phi$, which means that Ψ functions as unit gain. This means that

$$N(A) = 1, \quad \text{if } A \leq 1$$

Now take $A_1 > 1$. Then we have $\Psi(A_1 \sin \phi) \; < A_1 \sin \phi$ for some $\phi \in [0.\pi]$, which means that

$$N(A_1) = \frac{2}{\pi A_1} \int_0^{\pi} \Psi(A_1 \sin \phi) \sin \phi d\phi < \frac{2}{\pi A_1} \int_0^{\pi} A_1 \sin^2 \phi d\phi < 1$$

We can thus conclude that $N(A) \leq 1$ is an upper bound, that is achieved.

Lower bound Since the expression under the integral is nonnegative for all $\phi \in [0, \pi]$, and positive for all $\phi \in (0, \pi)$, we have the lower bound

$$N(A) > 0, \quad \forall A \geq 0$$

Since the maximum value of Ψ is 2 we have

$$N(A) = \frac{2}{\pi A} \int_0^{\pi} \underbrace{\Psi(A \sin \phi)}_{\leq 2} \underbrace{\sin \phi}_{\leq 1} d\phi \leq \frac{4}{A} \to 0, \quad A \to \infty$$

and conclude that the lower bound is approached asymptotically.

Therefore

$$N(A) \in (0, 1] \quad \Rightarrow \quad -\frac{1}{N(A)} \in (-\infty, -1)$$

Describing function analysis predicts a limit cycle if

$$G(j\omega) = -\frac{1}{N(A)}$$

which requires $G(j\omega) = \Re\{G(j\omega)\} \le -1$ for some ω. From the Nyquist diagram we see that this is achieved by choosing $k > \frac{1}{0.3} \approx 3.3$.

Note that this illustrates the interpretation of $N(A)$ as the "equivalent gain" of Ψ.

Since $\frac{1}{N(A)}$ grows as we move to the left, if we choose a smaller A it will be encircled and will grow to the value where the lmit cycle occurs. On the other hand if we choose a larger A, the amplitude will decrease to the value where the limit cycle occurs. Hence, the limit cycle is stable.

5. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2 \\ \dot{x}_2 &= x_2 + u\end{aligned}$$

a. Show that the system is on strict feedback form. (1 p)

b. Design a controller using backstepping to globally stabilize the origin. (2 p)

Solution

a. Define $f_1(x_1) = x_1^2$, $g_1(x_1) = 1$, $f_2(x_1, x_2) = x_2$, $g_2(x_1, x_2) = 1$. Then the system can be written as

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u\end{aligned}$$

which is on strict feedback form.

b. Start with the system $\dot{x}_1 = x_1^2 + \phi(x_1)$ which can be stabilized using $\phi(x_1) = -x_1^2 - x_1$, where $\phi(0) = 0$. The first Lyapunov function is $V_1 = \frac{1}{2}x_1^2$.
Then define $z_2 = x_2 - \phi$ to transfer the system to

$$\begin{aligned}\dot{x}_1 &= -x_1 + z_2 \\ \dot{z}_2 &= z_2 - x_1 - x_1^2 + (1 + 2x_1)(-x_1 + z_2) + u\end{aligned}$$

Take $V_2 = V_1 + \frac{1}{2}z_2^2$ as the second Lyapunov function and note that $V_2(0) = 0$. Then

$$\begin{aligned}\dot{V}_2 &= x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-x_1 + z_2) + z_2(z_2 - 2x_1 - 3x_1^2 + 2x_1z_2 + u) \\ &= -x_1^2 + z_2(-x_1 + 2z_2 - 3x_1^2 + 2x_1z_2 + u),\end{aligned}$$

which is made negative $\forall\,(x_1, z_2) \neq 0$ if $u = x_1 - 3z_2 + 3x_1^2 - 2x_1z_2$. Noting that V is radially unbounded, it is possible to conclude that the origin is globally asymptotically stable.

6. A mass is moving around an equilibrium governed by an external force, a linear spring and viscous friction. The motion is described by the equation

$$\ddot{z} + \phi(\dot{z}) + z = u(t)$$

Suppose that ϕ is Lipschitz and satisfies $0.1v^2 \le \phi(v)v \le v^2$ for all v.

a. Prove that ϕ is strictly passive. (1 p)

b. Show that the system can be written as a feedback interconnection between ϕ and a linear system. (1 p)

c. Show that the linear system is passive, and conclude that the map from u to $\dot{z}$ is passive. (2 p)

Hint: A transfer function G is passive if and only if it is positive real. A definition of positive real is given below:

Positive Real

A transfer function G is positive real if and only if the following 3 conditions hold:

1. The function has no poles in the right half-plane.
2. If the function has poles on the imaginary axis or at infinity, denoted $i\omega_p$, they are simple poles and $\lim_{s \to i\omega_p}(s - i\omega_p)G(s)$ is positive.
3. The real part of G is nonnegative along the real axis, that is,

$$\text{Re}(G(i\omega)) \ge 0.$$

d. Verify BIBO stability from u to $\dot{z}$. (1 p)

Solution

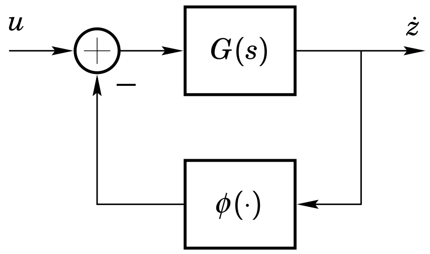
a. ϕ is obviously passive. Strict passivity follows through the definition of the nonlinearity and strict passivity in the slides with $\epsilon = 0.1/2$:

$$\int \phi v \,\mathrm{d}t \ge 0.1 \int v^2 \,\mathrm{d}t = \frac{0.1}{2} \int v^2 + v^2 \,\mathrm{d}t \ge \frac{0.1}{2} \int \phi^2 + v^2 \,\mathrm{d}t$$

b. With $x = (\dot{z}, z)$, the system can be written

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x - \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\phi([1\ 0]x - u)) = Ax - B\phi(Cx) + Bu$$

where $G(i\omega) = C(i\omega I - A)^{-1}B = i\omega/(1 - \omega^2)$ is passive.



c. G is passive, since it has no poles in $\mathrm{Re}[s]>0$, has simple poles at $\pm i$, and whose residues are $\frac{1}{2}$. Passivity of the map from u to $\dot{z}$ is a direct consequence of the fact that a feedback interconnection of two passive systems is passive.

d. G is passive and ϕ is strictly passive. Therefore, according to the passivity theorem, the system is BIBO stable.

Alternative solution: $G(i\omega) = C(i\omega I - A)^{-1}B = i\omega/(1-\omega^2)$ is purely imaginary and therefore stays outside the circle through $-1/\alpha = -10$ and $-1/\beta = -1$. Hence, by the circle criterion, the system is BIBO stable.

7. A factory has wagons moving on rails transporting goods. A simplified model could then be a mass-damper system with mass $M > 0$ and damping $d > 0$, and an external force acting u on it. The equation of motion is then given by

$$M\ddot{x} = -d\dot{x} + u,$$

where x is the position of the wagon. In order to be able to transport as much goods as possible, we would like the wagon to go from rest to high velocity in a short, precalculated time t_f. However, in order to be environmental friendly we would like to spend as low amount of energy as possible. If we further simplify the problem by setting $M = 1$, $d = 1$, and $t_f = 1$, and let $y = [y_1 \quad y_2]^T = [x \quad \dot{x}]^T$, this optimal control problem can be formulated as

$$\min_u \int_0^1 u(t)^2 \, \mathrm{d}t$$

$$\text{subject to } \dot{y} = Ay + Bu, \quad y_1(0) = y_2(0) = 0 \quad \text{and } y_2(1) = 1$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solve this optimal control problem for u. (3 p)

Solution

We first consider the normal case, i.e., $n_0 = 1$. The Hamiltonian is given by

$$H(y, u, \lambda) = u^2 + \lambda^T(Ay + Bu), \qquad \lambda \in R^2.$$

At optimality we have

$$\frac{\partial H(y, u, \lambda)}{\partial u} = 2u + \lambda^T B = 0 \iff u = -\frac{1}{2}\lambda^T B.$$

The adjoint equations are given by

$$\dot{\lambda} = -\frac{\partial H^T(y, u, \lambda)}{\partial y} = -A^T\lambda, \quad \lambda(t_f = 1) = \frac{\partial \Psi^T}{\partial y}(t_f = 1, y(t_f = 1))\mu = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mu,$$

and therefore we get that

$$\dot{\lambda}_1 = 0, \quad \lambda_1(1) = 0 \Longrightarrow \lambda_1(t) = 0.$$

Then

$$\dot{\lambda}_2 = -\lambda_1 + \lambda_2, \quad \lambda_2(1) = \mu_2 \Longrightarrow$$
$$\lambda_2(t) = \mu_2 e^{t-1}.$$

Thus

$$\lambda(t) = \begin{bmatrix} 0 \\ e^{t-1}\mu_2 \end{bmatrix}, \quad \text{and } u = -\frac{1}{2}e^{t-1}\mu_2.$$

It remains to find μ_2, which can be done by solving for the velocity

$$\dot{y}_2 + y_2 = -\frac{1}{2}e^{t-1}\mu_2 \iff \frac{d}{dt}(e^t y_2) = -\frac{1}{2}e^{2t-1}\mu_2 \iff$$
$$y_2 = -\frac{1}{4}e^{t-1}\mu_2 + Ce^{-t},$$

where C is a constant. Using the boundary condition, $y_2(0) = 0$ and $y_2(1) = 1$, we find that $\mu_2 = -\frac{4}{1-e^{-2}}$. Note that if $n_0 = 0$ H has no minimum value, hence there are no abnormal solutions.