LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
**AUTOMATIC CONTROL**

# Nonlinear Control and Servo Systems (FRTN05)

**Exam - June 1, 2009 at 14-19**

### Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: $12 - 16$ points

4: $16.5 - 20.5$ points

5: $21 - 25$ points

### Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

### Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact the lecturer Anders Robertsson for checking your corrected exam.

*Note!*

In many cases the sub-problems can be solved independently of each other.

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
June, 2009.

**1.**

**a.** Rewrite the following differential equation into state space form.

$$\ddot{y} + (\dot{y} - 1)\dot{y} - y(1 - y^2) = 0 \tag{1}$$

(1 p)

**b.** Draw a block diagram of the system in (1) (for instance showing how you would simulate the system in Simulink). (1 p)

*Solution*

**a.** Introduce $\{x_1 = y, x_2 = \dot{y}\}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1(1 - x_1^2) + x_2 - x_2^2 \end{aligned}$$

**b.** Either we can construct it from the answer in **(a)** or as follows: We have a second order system and thus need two integrators with input/outputs corresponding to $\{\frac{dy}{dt}, y\}$ and $\{\frac{d^2y}{dt^2}, \frac{dy}{dt}\}$, where the matching $\frac{dy}{dt}$ is connected. Rewrite the equation as $\ddot{y} = -(\dot{y} - 1)\dot{y} + y(1 - y^2)$ where we see what should be the inputs connected to the integrator with input $\ddot{y}$, see Figure 1.

**2.** Imagine an isolated island with no population, i.e., no animals or other living creatures. At a particular time, $t_0$, two species manage to arrive to the island at the same time. One of the species is a plant eater, but the other is a meat–eater, and the first species therefore serves as foot source for the other species. Now, denote the population size (the number of creatures) of the plant–eaters by $x_1$ and the population size of the meat–eater by $x_2$. A simple model of the evaluation of the species will then follow the *Lotka–Volterra* model, given by (a simplified version)

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1) \quad (2) \\ \dot{x}_2 &= x_2(x_2 - x_1) \quad (3) \end{aligned}$$

**a.** Find and classify all equilibria and determine if they are stable or asymptotically stable if possible. Discuss if the stability results are global or local. (2 p)

**b.** Consider the results of the stability analysis. Which of the following scenarios will appear on the island after many years (according to the model), when the initial populations were relative small (Motivate your answer!):

1. The two species will both have survived and now live in a balance, where the meat–eaters will only reproduce itself in order to compensate for what the meat–eaters eat.

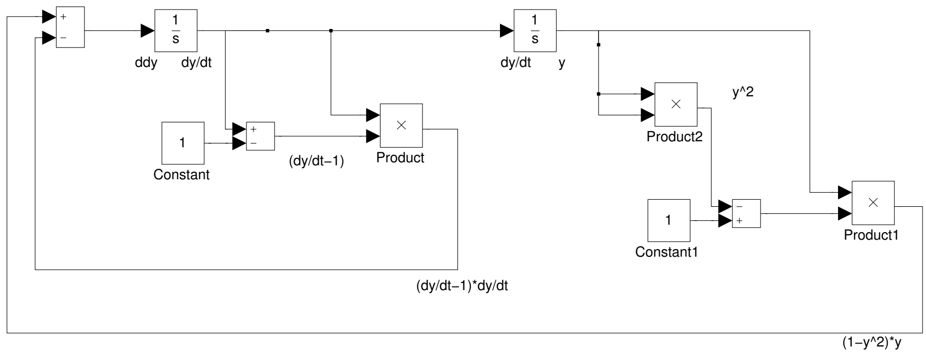


**Figure 1** Suggested solution of block diagram in Problem 1.

2. Only the plant–eaters will have survived, while the meat–eaters will all have died.
3. Only the meat–eaters will have survived, while the plant–eaters will all have died.
4. None of the species have survived.

If you have not been able to solve subproblem **(a)**, you can assume that the following equilibria exist: $(x_1, x_2) = (2, 3)$ with poles/eigenvalues in $(-1, -4)$, and $(x_1, x_2) = (0, 4)$ with poles/eigenvalues in $(0, -1)$,
**Note**: This is not necessarily the correct answer to subproblem (**a**)!!! (1 p)

*Solution*

**a.** The equilibria of the system is found by solving

$$0 = x_1(1 - x_1) \qquad (4)$$
$$0 = x_2(x_2 - x_1) \qquad (5)$$

which gives the points $(x_1, x_2) = (0, 0), (1, 0), (1, 1)$
Linearization of the system gives

$$\Delta\dot{x}_1 = (1 - 2x_1^0)\Delta x_1 \qquad (6)$$
$$\Delta\dot{x}_2 = (2x_2^0 - x_1^0)\Delta x_2 - x_2^0\Delta x_1 \qquad (7)$$

which gives the stability matrix

$$A = \begin{bmatrix} (1-2x_1^0) & 0 \\ -x_2^0 & (2x_2^0 - x_1^0) \end{bmatrix} \tag{8}$$

For the three equlibria we have:

$(x_1, x_2) = (0,0)$ : $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ eigenvalues of $(0,1) \Rightarrow$ unstable node

$(x_1, x_2) = (1,0)$ : $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ eigenvalues of $(-1,-1) \Rightarrow$ stable node

$(x_1, x_2) = (1,0)$ : $A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$ eigenvalues of $(-1,1) \Rightarrow$ saddle point

All stability results found from linearization are local, since the linearized model is a local approximation of the nonlinear model.

**b.** Since we have an equilibrium in (1,0) which is locally asymptotically stable, we are likely to converge to this point. This means that the meat–eaters will eventually die ($x_2 \rightarrow 0$), and the plant–eaters will converge to a steady population of 1. Therefore, the answer is 2). The equilibrium in the origin is not stable, so any deviation (like some small population landing on the island, will move away from the origin.

Actually, the system behaves rather strange if $x_2$ is initiated too high (above 1), since the population of the meat–eaters will go towards infinity, even if the amount of foot (plant–eaters) are limited. This would mean that 1) is also correct, but that can not be seen from the stability results. By stating that the initial conditions is close to the origin, this situation is not relevant.

If a) was not solved correctly, and the given equlibria–information is used, the correct answer is 1), since this is the only stable equilibrium. The equilibrium in (0,4) (which is unstable) does not really give any meaning, since it means that meat–eaters can survive without any meat (the plant–eaters).

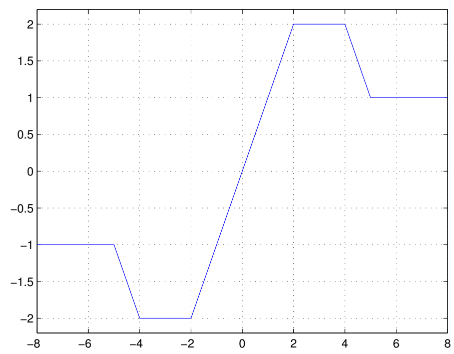


**Figure 2** The non-linear function $f(x)$ in Problem 3.

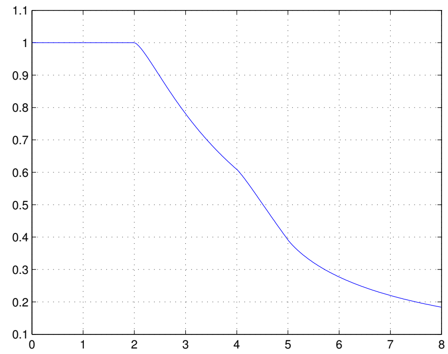


**Figure 3** Describing function 1 in Problem 3.

**3.** In Figure 2 a non-linear function $f(x)$ is shown. In Figures 3 – 5, three describing functions of non-linearities are drawn.

**a.** Which one of the three describing functions (Figs. 3 – 5) corresponds to $f(x)$? Motivate your answer. (1 p)

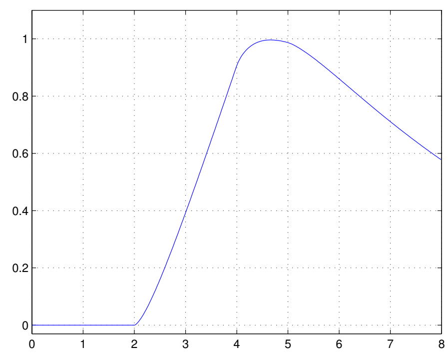


**Figure 4** Describing function 2 in Problem 3.

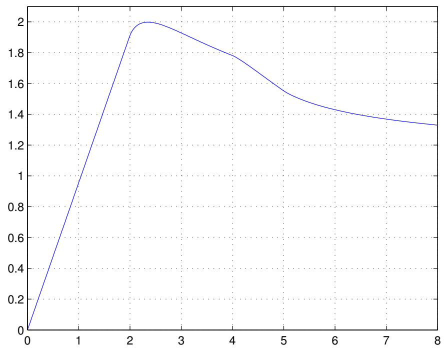


**Figure 5** Describing function 3 in Problem 3.

**b.** In Figure 6 we find the Nyquist and Bode curves of a linear system $G$. Assume that the non-linearity that gives rise to the describing function in Figure 3 is used in a negative feedback connection with $G$. Do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?

Do the same for the non-linearities corresponding to the describing functions in Figures 4 and 5. (1.5 p)

**c.** What would the corresponding frequency of the limit cycles in **(b)** be? Use Figure 6. (1 p)

*Solution*

**a.** We see that the non-linearity $f(x)$ starts of with a constant slope of 1. Hence the describing function should be constant at 1 in the beginning. The only alternative is then the one in Figure 3. (Of course we could relate all the changes in the slope of $f(x)$ to the appearance of the describing function).

**b.** If we should expect a limit cycle, then $-\frac{1}{N(A)}$ should intersect the Nyquist curve. For the first and second describing function we have that $-\frac{1}{N(A)} \leq -1$ and hence these should not give rise to limit cycles.

Since the third describing function fulfills that $-\frac{1}{N(2)} = -\frac{1}{2}$ and $G(i\omega_o) \approx -0.6$, we understand that we have two intersections. The first intersection occurs when $A \approx 1.8$ and the second intersection occurs when $A \approx 4.5$. Examining the describing function around the first intersection, we see that $-\frac{1}{N(A)}$ goes from the outside of $G(i\omega)$ to the inside, with increasing $A$. Hence, we conclude that the possible limit cycle at $A \approx 1.8$ is unstable. By similar argument, we understand that the possible limit cycle at $A \approx 4.5$ is stable.

**c.** The frequency of all possible limit cycles is approximately 2.5 rad/s. To understand this, we see in the Bode plot that for $\omega \approx 2.5$ we have that $\arg(G(i\omega)) \approx -180$.

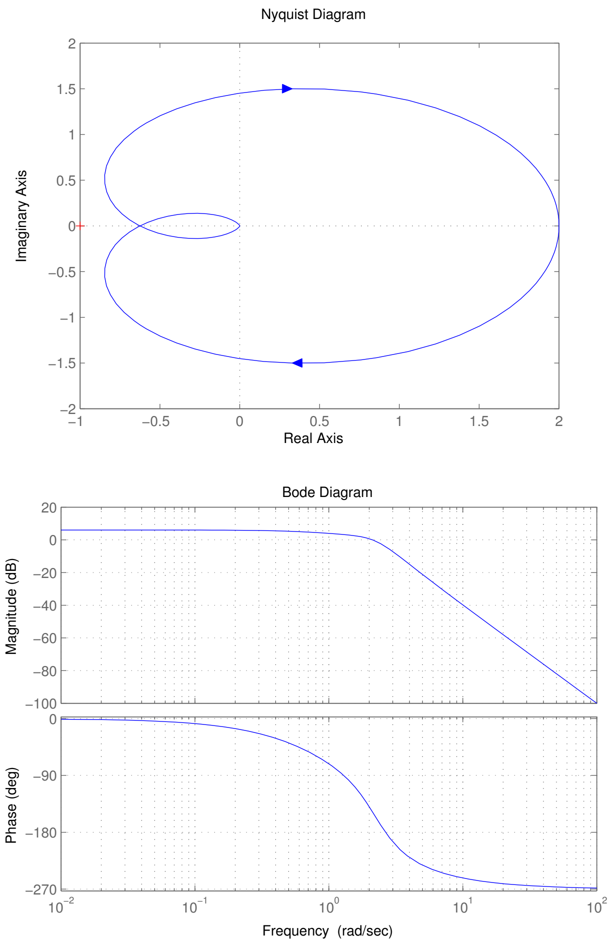


**Figure 6** Nyquist and Bode curves of the linear system $G$ considered in Problem 3.

**4.** Consider an uncontrolled second–order time–invariant system, expressed in the states $x_1(t)$ and $x_2(t)$. Assume that a limit–cycle exists, as illustrated in the phase–plane diagram in Figure 7.

**a.** Assume that the limit cycle is stable, and that the system is initiated in the point $A$ $((x_1(0), x_2(0)) = A)$. Is it possible for the system to pass through the point $B$? Motivate your answer. (0.5 p)

**b.** Assume that the limit cycle is unstable, and that the system is initiated in the point $A$ $((x_1(0), x_2(0)) = A)$. Is it possible for the system to pass through the point $B$? Motivate your answer. (0.5 p)

*Solution*

**a.** In order for the system to reach point $B$, it will have to pass the limit cycle. For a stable limit cycle, the system will remain on the limit cycle, and will thus never move toward point $B$. The answer is therefore "No".

**b.** Again, the system will have to pass the limit cycle in order to go to point $B$. When the system is exactly on the limit cycle, it will remain there, even though it is unstable. The answer is therefore again "No".

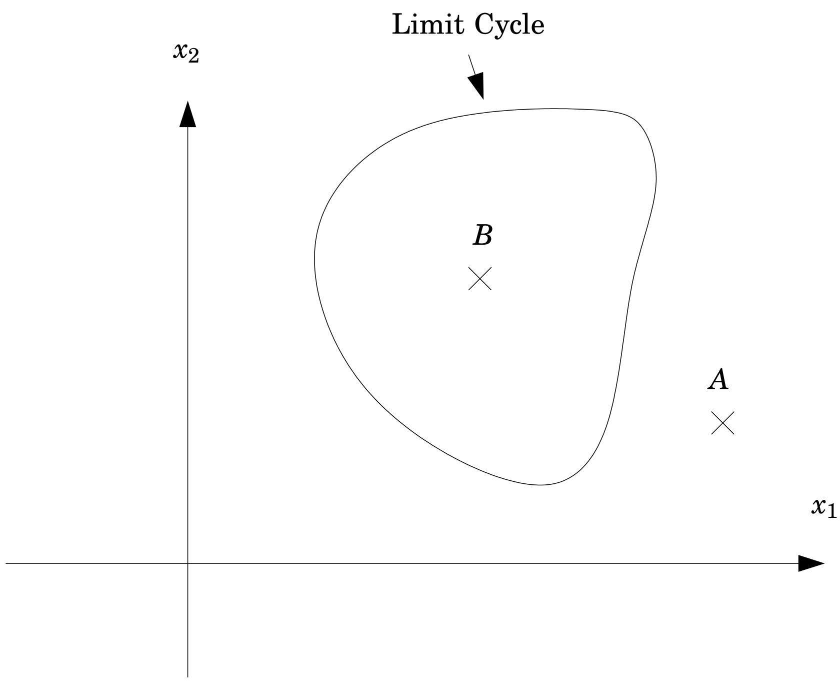


**Figure 7** Phase–plane diagram for problem 4

**5.** Consider the system

$$\dot{x}_1 = 5\,x_1 x_2$$
$$\dot{x}_2 = 2\,x_1^5 + 3\,u$$

Use Lyapunov based methods to find a feedback control law using, $u$, such that the origin can be shown to be globally asymptotically stable.

Hint: You may start with

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

as a Lyapunov function *candidate*. (3 p)

*Solution*

Consider the Lyapunov function *candidate*

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$V(0,0) = 0$
$V(x_1, x_2) > 0$ when $(x_1, x_2) \neq (0,0)$
$V(x_1, x_2) \to \infty$ when $|x| \to \infty$.

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= (5x_1x_2)x_1 + (2\,x_1^5 + s\,u)x_2 \\ &= x_2\,(5\,x_1^2 + 2\,x_1^5 + 3\,u)\end{aligned}$$

If we choose

$$u = -\frac{2}{3}x_1^5 - \frac{5}{3}x_1^2 - x_2$$

we get

$$\dot{V}(x) = -3\,x_2^2 \leq 0$$

which implies stability, but not asymptotic stability. $\dot{V} = 0$ along the line $x_2 = 0$ but the only *invariant subset* of this line is the origin. This can be seen by inserting the suggested control law.

$$\dot{x}_2 = 2\,x_1^5 + 3\,u = -5\,x_1^2 - 3\,x_2 = \{x_2 = 0 \text{ on the line}\} = -5\,x_1^2$$

Thus $\dot{x}_2 \neq 0$ whenever $x_1 \neq 0$ and the solution curves can not stay on the line $x_2 = 0$ except for at the origin. This shows global asymptotic stability of the origin.

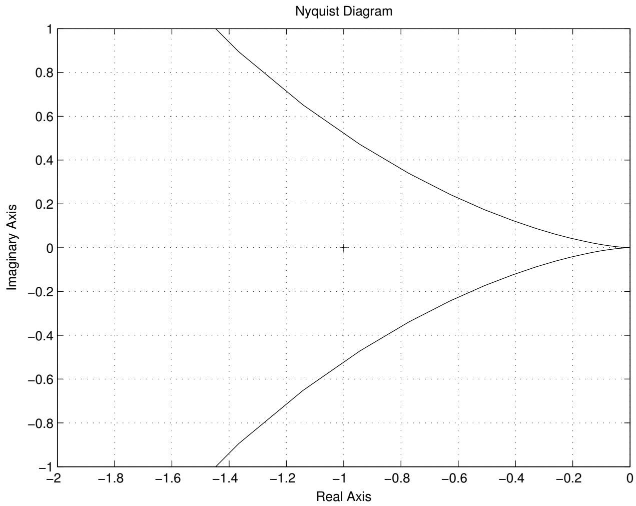


**Figure 8** Nyquist diagram for Problem 6.

**6.** Parts of the Nyquist plot of the open-loop transfer function

$$G(s) = \frac{20}{(s+1)^2}$$

is shown i Figure 8. The signals in the system are defined by $y = G(s)u$. The system is then closed by the uncertain feedback

$$u = -(1+\Delta)y.$$

with $\Delta(y)$ belonging to a symmetric cone $[-k, k]$. For how large values of $k$ can you guarantee stability? (2 p)

*Solution*

We should find the maximum uncertainty sector $[1-k, 1+k]$ for which we can guarantee stability. We will use the circle criterion. Therefore try different values of $k$ and draw the approximate circle in the Nyquist plot. The choice $k = 0.45$ gives the plot in Figure 9. Notice the circle is not centered around -1.

An alternative solution is to consider the system $\dfrac{G}{1+G}$ in negative feedback connection with the uncertainty $\Delta$ and apply the Small gain theorem. $\sup\left|\dfrac{G}{1+G}\right| \approx 2.18 \Rightarrow k = \sup|\Delta| \approx 0.45$. To realize that $\sup\left|\dfrac{G}{1+G}\right| \approx 2.24$ we have that

$$\left|\frac{G}{1+G}\right|^2 = \frac{20^2}{(21-\omega^2)^2 + 4\omega^2} = \frac{20^2}{21^2 - 38\omega^2 + \omega^4}$$

This expression has its maximum when $\omega^2 = 38/2 = 19$ and we get

$$\sup\left|\frac{G}{1+G}\right|^2 = \frac{20^2}{21^2 - 19^2}$$

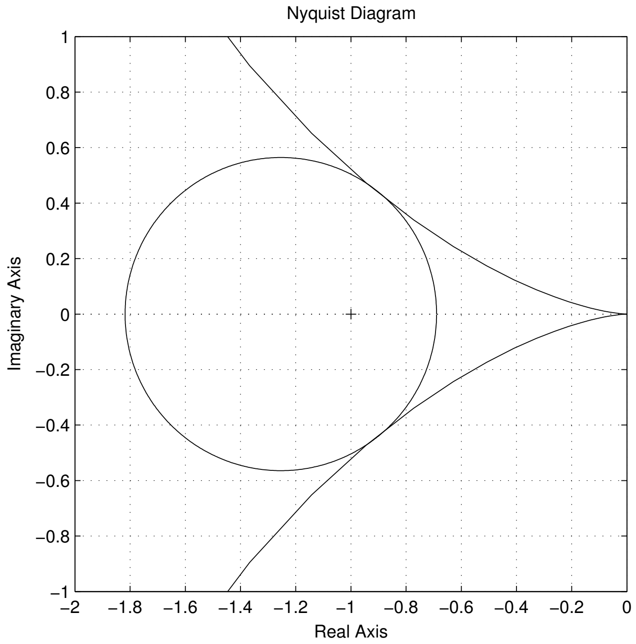


**Figure 9** Nyquist diagram for Problem 6 with circle.

**7.** Consider a linear system which is represented by the transfer function

$$Y(s) = \frac{2(s+a)}{(s+5)(s+15)}\,U(s)$$

where you can choose the *positive* parameter $a$. You want the mapping from $u$ to $y$ to be passive. Which values of $a$ can you choose? (1 p)

*Solution*

If the real part of $G(i\omega) \geq 0$ for all $\omega$, the system is passive.

$$\begin{aligned} G(i\omega) &= \frac{2(i\omega+a)}{(i\omega+5)(i\omega+15)} \\ &= \frac{(40\,-2\,a)\omega^2+150\,a}{w^4+5625+250\,w^2} + \frac{-2\,\omega^3+150\,\omega-40\,a\,\omega}{\omega^4+5625+250\,\omega^2}\,i \end{aligned}$$

As $\Re e\{G(i\omega)\}$ should be $\geq 0$ for all $\omega$, we see that $0 \leq a \leq 20$.

**8.** Consider the following non-linear system:

$$\begin{aligned} \dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= x_1 - x_2^3 + u \end{aligned}$$

which is controlled by the nonlinear control law

$$u = -\text{sign}(x_2)$$

**a.** This switching control law suggests that we might get a sliding mode. Let

$$\sigma(x) = x_2$$

On which part of the $x_1$ axis (where $\sigma(x) = x_2 = 0$) will there be a sliding mode? (Any method is OK to use). (1 p)

**b.** Will the system converge to the origin when it has reached the sliding mode? Use the *equivalent–control method* to describe the dynamics in the sliding mode. (1 p)

*Solution*

**a.** The requirement for a sliding mode is that

$$\begin{cases} \nabla\sigma(x) f^+(x) < 0 \\ \nabla\sigma(x) f^-(x) > 0 \end{cases}$$

where $f^+(x)$ is the system dynamics when $\sigma(x) > 0$ and vice verse. Thus

$$f^+(x) = \begin{pmatrix} x_2^2 \\ x_1 - x_2^3 - 1 \end{pmatrix}, \quad f^-(x) = \begin{pmatrix} x_2^2 \\ x_1 - x_2^3 + 1 \end{pmatrix}.$$

and

$$\nabla\sigma(x) = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

There will be sliding mode at $\sigma(x) = x_2 = 0$ when

$$\nabla\sigma(x) f^+(x) = x_1 - x_2^3 + 1 = x_1 - 1 < 0 \Rightarrow x_1 < 1$$

and

$$\nabla\sigma(x) f^-(x) = x_1 - x_2^3 + 1 = x_1 + 1 > 0 \Rightarrow x_1 > -1.$$

$\Rightarrow$ Sliding mode when $|x_1| < 1$ and $x_2 = 0$.

**b.** Equivalent control method: Find $u_{eq}$ which makes

$$\sigma(x) = \dot\sigma(x) = 0$$

on the sliding surface.

$$\dot\sigma(x) = \dot x_2 = x_1 - x_2^3 + u_{eq} = x_1 + u_{eq} = 0$$

$$\Rightarrow u_{eq} = -x_1$$

$$\Rightarrow \begin{pmatrix} \dot x_1 \\ \dot x_2 \end{pmatrix} = \begin{pmatrix} x_2^2 \\ x_1 - x_2^3 + u_{eq} \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 - x_1 \end{pmatrix} = 0$$

The system will stop as soon as it reaches the sliding surface, and will thus NOT go to the origin.

**9.** Consider the system

$$\begin{aligned} \dot x_1 &= (x_1 - x_2)^2 + (x_1 - x_2)x_2 + u \qquad (= x_1^2 - x_1 x_2 + u) \\ \dot x_2 &= -x_2 + u \end{aligned} \tag{9}$$

**a.** Show that the open system of (9) is not stable, (open: $u = 0$). (1 p)

**b.** We would like to stabilize the system (9) using backstepping. Explain the problem of directly applying backstepping to stabilize the system. (0.5 p)

**c.** To get the system in strict feedback form we could introduce an appropriate state transformation. Find this transformation by suggesting a simple change of coordinates (Hint: see (9)) and use backstepping to make the origin globally asymptotically stable. (2 p)

*Solution*

**a.** Since $\dot{x}_2 = -x_2$, after a while $x_2 = 0$. Then $\dot{x}_1 = x_1^2$ and if $x_1 \geq r > 0$ then $\dot{x}_1 \geq r^2 > 0$ for all times, meaning that $x_1 \to \infty$.

**b.** The control signal enters both state equation and hence, the system is not in strict feedback form. This means that the requirements for applying backstepping are not fulfilled.

**c.** If we subtract the first state equation by the second state equation we would get rid of the control signal. That is, try the transformation $z_1 = x_1 - x_2$ (and leave $x_2$ as it is). This gives

$$\begin{aligned}\dot{z}_1 &= z_1^2 + (z_1 + 1)x_2 \\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

Now, start with the equation

$$\dot{z}_1 = z_1^2 + (z_1 + 1)\phi_1(z_1)$$

Choosing $\phi_1(z_1) = -z_1$ and the Lyapunov function $V_1(z_1) = \frac{z_1^2}{2}$ we stabilize the system. To backstep, introduce

$$z_2 = x_2 - \phi_1(z_1) = z_1 + x_2$$

We get the system

$$\begin{aligned}\dot{z}_1 &= -z_1 + (z_1 + 1)z_2 \\ \dot{z}_2 &= z_1 z_2 + u\end{aligned}$$

Let $V_2(z) = V_1(z_1) + \frac{z_2^2}{2}$. Then

$$\begin{aligned}\dot{V}_2(z) &= z_1\dot{z}_1 + z_2\dot{z}_2 = z_1(-z_1 + (z_1 + 1)z_2) + z_2(z_1 z_2 + u) = \\ &= -z_1^2 + z_2(z_1(z_1 + 1) + z_1 z_2 + u) = [u = -z_1(z_1 + 1) - z_1 z_2 - z_2] = \\ &= -z_1^2 - z_2^2\end{aligned}$$

Hence $u = -2x_1^2 + 3x_1x_2 - x_2^2 - 2x_1 + x_2$ stabilizes the system.

# New formulation of the Optimal Control problem

**10.** You should have a meeting in the conference room in 24 hours. The temperature, $T$, is 15°$C$. You want to increase $T$ to 20°$C$ before the meeting but do not want to use too much energy and have decided to minimize $\int u^2 dt$ over the time interval. The temperature dynamics can be modeled with the following first order model in this region.

$$\dot{T} = -\frac{1}{48}T + u \tag{10}$$

where $u$ is your control signal. The units are [C] for T and [C/h] for $\dot{T}$.

**a.** State the above described optimal control problem. (1 p)

**b.** Solve the optimal control problem (2 p)

**c.** Your heater has a maximum capacity of $u_{max} = 0.5$. Is the optimal control problem feasible then, i.e., will the conference room be heated on time? Motivate your answer. (1 p)

*Solution*

**a.** The optimal control problem becomes

$\min \int_0^{24} u^2$ dt
subject to: $\dot{T} = -T/48 + u$
$T(0) = 15$
$T(24) = 20$

**b.** The Hamiltonian is $H = u^2 + \lambda(-T/48 + u)$. The Maximum principle says that the following should hold

$$\frac{\partial H}{\partial u} = 2u + \lambda = 0$$

and

$$\dot{\lambda} = -\frac{\partial H}{\partial T} = \frac{1}{48}\lambda$$

$\lambda(24)$ is free. This means that $\lambda(t) = c_1 e^{t/48}$ which means that $u(t) = -c_1 e^{t/48}/2$. Insert this into the system dynamics to decide $c_1$.

$$\begin{aligned} T(24) = 20 &= 15e^{-24/48} - \frac{c_1}{2}\int_0^{24} e^{-(24-t)/48} e^{t/48} dt \\ &= 15e^{-1/2} - \frac{c_1}{2}\left[24e^{-(24-2t)/48}\right]_0^{24} \end{aligned}$$

Rearranging the terms gives

$$c_1 = \frac{15e^{-0.5} - 20}{12(e^{0.5} - e^{-0.5})}$$

**c.** Apply maximum control signal

$$
\begin{aligned}
T(24) &= 15e^{-1/2} + 0.5\int_0^{24} e^{-(24-t)/48}dt \\
&= 15e^{-1/2} + 0.5\left[48e^{-(24-t)/48}\right]_0^{24} \\
&= 15e^{-1/2} + 24(1-e^{-1/2}) = 24 - 9e^{-1/2} = 18.54 < 20
\end{aligned}
$$

If we have this constraint the optimization problem is not feasible and we cannot reach the desired temperature.