LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRT 075)

Exam - March 7, 2006 at 2 pm – 7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: $12 - 16$ points

4: $16.5 - 20.5$ points

5: $21 - 25$ points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department of Automatic Control.
Date and location for showing the corrected exams will be posted on the course home page `http://www.control.lth.se/~kursolin/`

Note!
In many cases the sub-problems can be solved independently of each other.

Good Luck!

0. Do you want an e-mail with your result? If so, please confirm this and write your e-mail address where you want us to send it.

Solution

—

1. Consider the system in Figure 1.

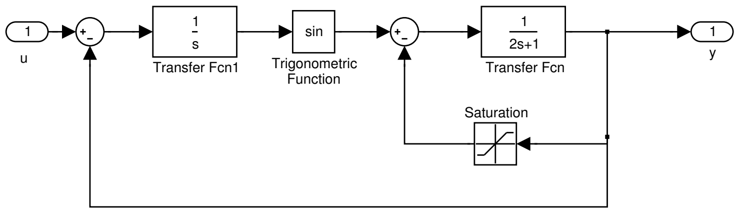

Figure 1 System in Problem 1

a. Introduce states and write the system dynamics on state space form, with input u and output y. (1 p)

b. Determine the equilibria for the system in **(a)** [1] and their local stability properties for $u = 0$. (2 p)

Solution

a. Introduce the states x_1 and x_2 at the outputs of the linear blocks.

$$
\begin{aligned}
\dot{x}_1 &= -x_2 + u \\
2\dot{x}_2 + x_2 &= \sin(x_1) - \mathrm{sat}(x_2) \\
&\Longrightarrow \\
\dot{x}_1 &= -x_2 + u \\
\dot{x}_2 &= \frac{1}{2}\left(-x_2 + \sin(x_1) - \mathrm{sat}(x_2)\right) \\
y &= x_2
\end{aligned}
$$

b. Find equilibrium points

$$
\begin{aligned}
&0 = \dot{x}_1 = -x_2 \Longrightarrow x_2 = \mathbf{0} \\
&0 = \dot{x}_1 = \frac{1}{2}\left(-\mathbf{0} + \sin(x_1) - \mathrm{sat}(\mathbf{0})\right) \Longrightarrow x_1 = k \cdot \pi,\ k \in Z
\end{aligned}
$$

Linear approximations around the equilibrium points $(x_1, x_2) = (k \cdot \pi, 0)$;

$$
\begin{bmatrix} 0 & -1 \\ 0.5\cos(k\pi) & -1 \end{bmatrix}
$$

[1] If you have not done **(a)** and can't continue, you can contact the examiner to get a hint which will cost you the point from (a).

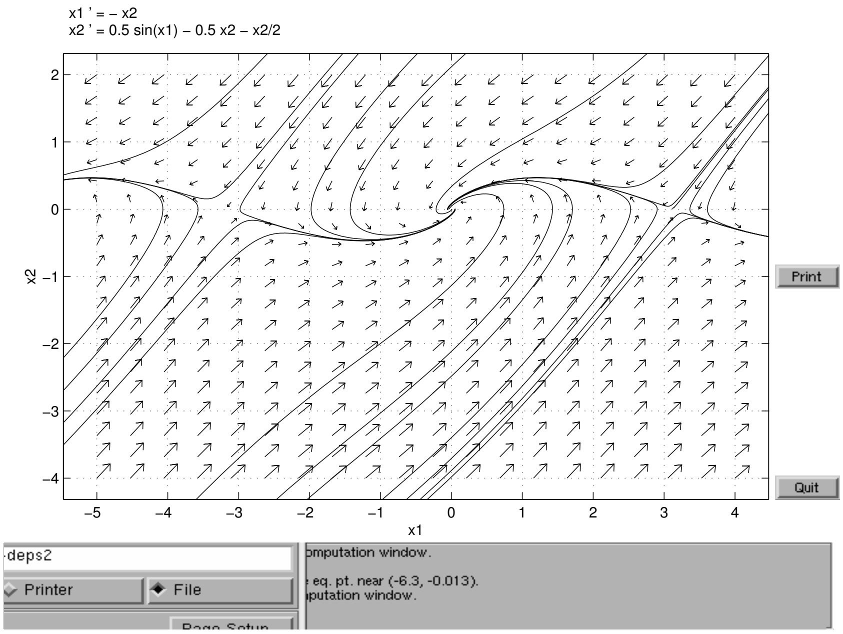

Figure 2 The phase plot in Problem 1

which has the characteristic polynomials $s^2 + s + 0.5cos(k\pi)$ and the corresponding eigenvalues $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ (saddle points for k odd) or $s = -\frac{1}{2} \pm i\frac{1}{2}$ (stable foci for k even).

The vector field points towards smaller x_1-value when $x_2 > 0$ and vice versa. For large values of $|x_2|$ the vector fields points towards lower values of $|x_2|$.

2.

a. Consider the linear system

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} x.$$

Can you find a Lyapunov function for this system? If yes, suggest one!

(1 p)

b. Consider the linear system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x.$$

Can you find a Lyapunov function for this system? If yes, suggest one!

(1 p)

Solution

a. The system is unstable (one pole in $\lambda = 1$ (i.e., in the RHP), and hence, no Lyapunov function exists.

b. The system is asymptotically stable (with eigenvalues $\lambda = \{-1, -2\}$) and hence there is always a Lyapunov function $V(x) = x^T P x$ such that $P > 0$ and P solves the linear system of equations

$$A^T P + PA = -Q,$$

for any $Q > 0$. For instance, you could choose $Q = I \Longrightarrow$

$$\begin{aligned} -Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -2p_{11} & -3p_{12} + p_{11} \\ -3p_{12} + p_{11} & 2p_{12} - 4p_{22} \end{bmatrix} \Longrightarrow \end{aligned}$$

$$p_{11} = \frac{1}{2}, \qquad p_{12} = \frac{1}{6}, \qquad p_{22} = \frac{1}{3}$$

One can check that $V = x^T P x$ is a positive definite function with $V(0,0) = 0$, but this follows directly from the theory.

3.

a. Use the Lyapunov function candidate $V(x) = x_1^2 + x_2^2$ to show that the origin of the following system is GAS.

$$\begin{aligned}\dot{x}_1 &= -x_1^3 - 2x_2 \\ \dot{x}_2 &= 2x_1 - x_2\end{aligned}$$

(2 p)

b. Use the Lyapunov function candidate $V(x) = x_1^2 + x_2^2$ to show that the origin of the following system is AS and give an estimate of the region of attraction.

$$\begin{aligned}\dot{x}_1 &= -1.5x_1 + 3x_1x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

(2 p)

Solution

a.

$$\begin{aligned}\dot{V}(x) &= -2x_1^4 - 4x_1x_2 + 4x_1x_2 - 2x_2^2 \\ &= -2(x_1^4 + x_2^2) < 0\end{aligned}$$

The set $D = \mathbb{R}^2$ is open and invariant, on which $V(x) > 0$ if $x \neq 0$, and $\dot{V}(x) < 0$. Furthermore, $V(0) = 0$ and $V(x)$ is radially unbounded. Hence the origin is GAS.

b.

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -3x_1^2 + 6x_1^2x_2 - 2x_2^2 + 2x_1x_2 \\ &= -2x_1^2 - x_2^2 + 6x_1^2x_2 - (x_1 - x_2)^2 \\ &\leq -2x_1^2 - x_2^2 + 6x_1^2x_2 \\ &= -2x_1^2(1 - 3x_2) - x_2^2 \\ &\leq 0 \text{ if } 3x_2 < 1.\end{aligned}$$

The set $D = \{||x||^2 = V(x) < \frac{1}{9}\}$ is open and invariant, on which $V(x) > 0$ if $x \neq 0$, $V(0) = 0$ and $\dot{V}(x) < 0$. Hence the origin is locally AS but we can not claim that it is GAS.

4. Let Figure (3) be the phase plane plot for the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

Assume that the solution to the system is unique. Is the phase plane in Figure 3 possible? What if the system is time-varying, that is

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, t) \\ \dot{x}_2 &= f_2(x_1, x_2, t)\end{aligned}$$

Motivate your answer. (1 p)

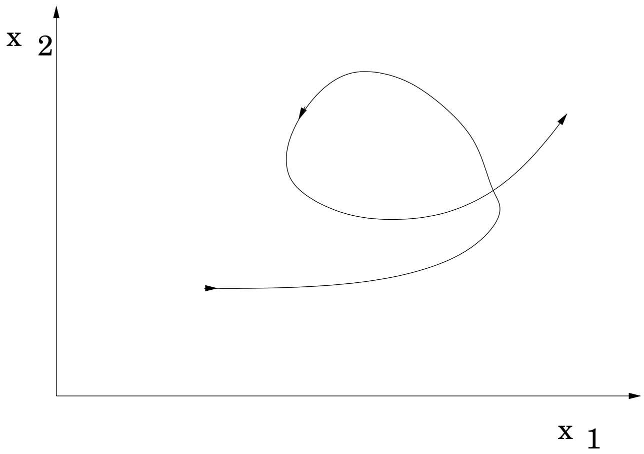

Figure 3 The phase plane plot for problem 4

Solution

If the system is time-invariant, then the phase plot is impossible, since the curve crosses itself, which means that the solution is not unique. On the other hand, it could be possible if the system is time-varying, since the system parameters could be different even though that state value is the same.

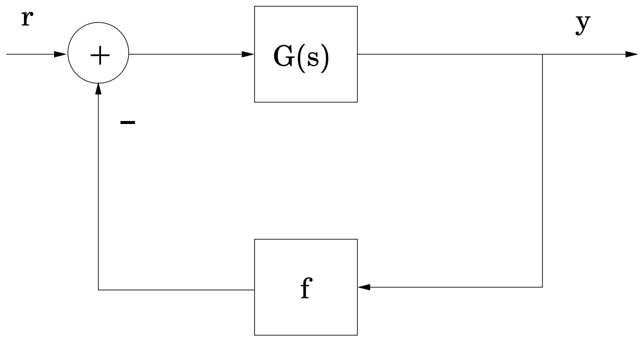

Figure 4 The feedback system for problem 5

5. Consider the feedback system given by Figure 4 with

$$G(s) = \frac{\Delta}{s(s+1)},$$

and

$$f(y) = K \cdot \arctan(y).$$

a. For what values of the uncertain (but constant) parameters $\Delta > 0$ and $K > 0$ does BIBO stability for the feedback system follow from the Circle Criterion? (2 p)

b. Is it possible to use the Small Gain Theorem to derive stability conditions for this system? (1 p)

Solution

a. We include Δ in the nonlinearity, and thus consider $\hat{G}(s) = \frac{1}{s(s+1)}$ in feedback with $\hat{f}(y) = \Delta K \arctan(y)$. Note that $\text{Re}(\hat{G}(i\omega)) = -1/(1+\omega^2) > -1$ for all $\omega \neq 0$, and $\hat{f}(\cdot)$ is bounded by the sector $[0, \Delta K]$. The Circle Criterion gives BIBO stability for $\Delta K < 1$.

b. Small Gain Theorem is not applicable since the gain of $G(s)$ is infinite.

6.

a. Use describing function analysis to predict the amplitude of a possible limit cycle for the system shown in Figure 5. The non-linearity, $\Psi(y)$, at the plant input is given by $\Psi(y) = -|y| \cdot y$. Figure 6 shows Nyquist plots of some closed-loop transfer functions involving the linear plant, $P(s)$, and the linear controller, $C(s)$. (3 p)

b. Is the limit cycle predicted by the analysis stable? If you have not solved sub-problem **a.**, you may use $N(A) = \frac{6A}{5\pi}$ in this problem. (*Note:* This is not the solution in **a.**) (1 p)

Solution

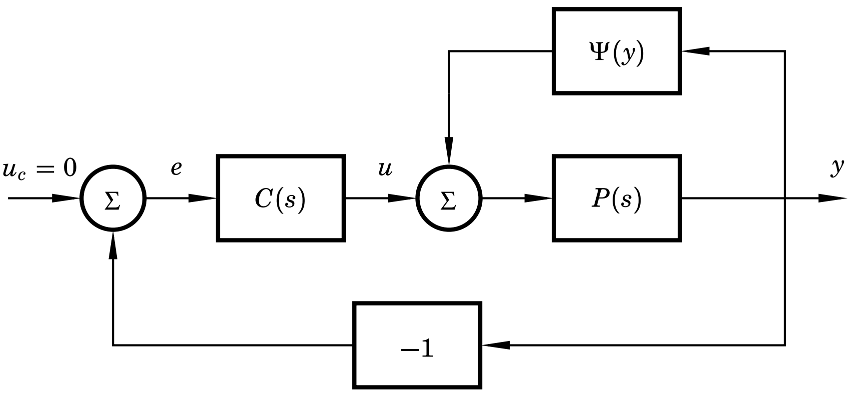

Figure 5 The feedback control system of Problem 6.

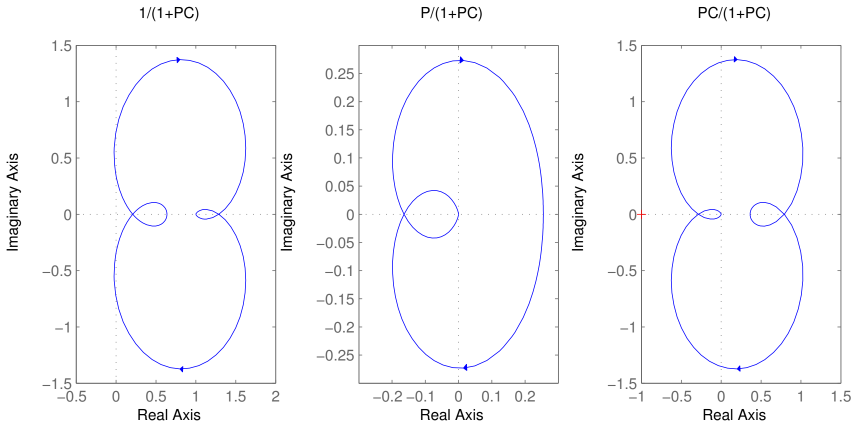

Figure 6 Nyquist curves for Problem 6.

a. The system can be transfered to standard feedback form according to Figure 7, where $G(s) = \frac{P(s)}{1+P(s)C(s)}$. The predicted amplitude is given by the intersection $-\frac{1}{N(A)} = G(i\omega)$, which assumes negative feedback. Thus, we should take out the minus sign from the nonlinearity when we compute the describing function.

We now assume that the input to the nonlinearity is given by $y = A \cdot \sin \phi$. Since the nonlinearity is static and odd, we have

$$N(A) = \frac{b_1}{A},$$

where

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \cdot \sin \phi \; d\phi$$

From the characteristics of the nonlinearity, we have

$$u(\phi) = \begin{cases} A^2 \sin^2 \phi, & \phi \in (0, \pi) \\ -A^2 \sin^2 \phi, & \phi \in (\pi, 2\pi) \end{cases}$$

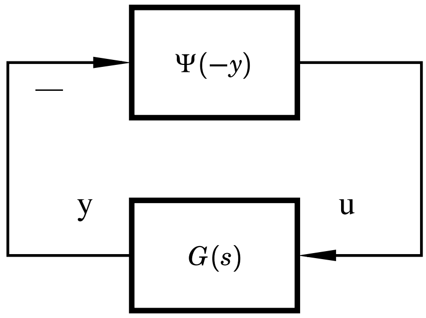

Figure 7 The feedback control system of Problem 6.

which gives

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} A^2 \cdot \sin^3\phi \, d\phi = \frac{2A^2}{\pi}\int_0^{\pi} \sin\phi(1-\cos^2\phi)d\phi \\
&= \frac{2A^2}{\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} = \frac{8A^2}{3\pi}
\end{aligned}
$$

The describing function if thus $N(A) = \frac{8A}{3\pi}$, and we conclude that the intersection between $-\frac{1}{N(A)}$ and $G(i\omega)$ will be at the negative real axis. From the Nyquist plot of $\frac{P(s)}{1+P(s)C(s)}$ we get

$$-\frac{3\pi}{8A} \approx -0.15 \Rightarrow A \approx 7.9$$

b. The function $-\frac{1}{N(A)}$ goes from $-\infty$ for $A = 0$ to 0 when $A \to \infty$. The limit cycle predicted above is not stable, since points on $-\frac{1}{N(A)}$ corresponding to amplitudes larger than $A = 7.9$ are encircled by the Nyquist curve, whereas points with decreasing amplitude are not.

7. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 - sign(x_1 + x_2)
\end{aligned}
$$

Determine the sliding set and the dynamics on the sliding set. (2 p)

Solution

Alt 1. Introduce and determine the equivalent control signal $u_{eq} \in [-1, 1]$ on the sliding set.

$$\dot{x}_2 = -x_1 - 2x_2 - sign\,\underbrace{(x_1 + x_2)}_{\sigma(x)} \,\widehat{=}\, -x_1 - 2x_2 + u_{eq}, \qquad u_{eq} = -sign(x_1 + x_2)$$

To stay on the set, put $\dot{\sigma}(x) = 0$:

$$\dot{\sigma} = \dot{x}_1 + \dot{x}_2 = x_2 - x_1 - 2x_2 + u_{eq} = 0 \Rightarrow u_{eq} = x_1 + x_2 = 0(!!)$$

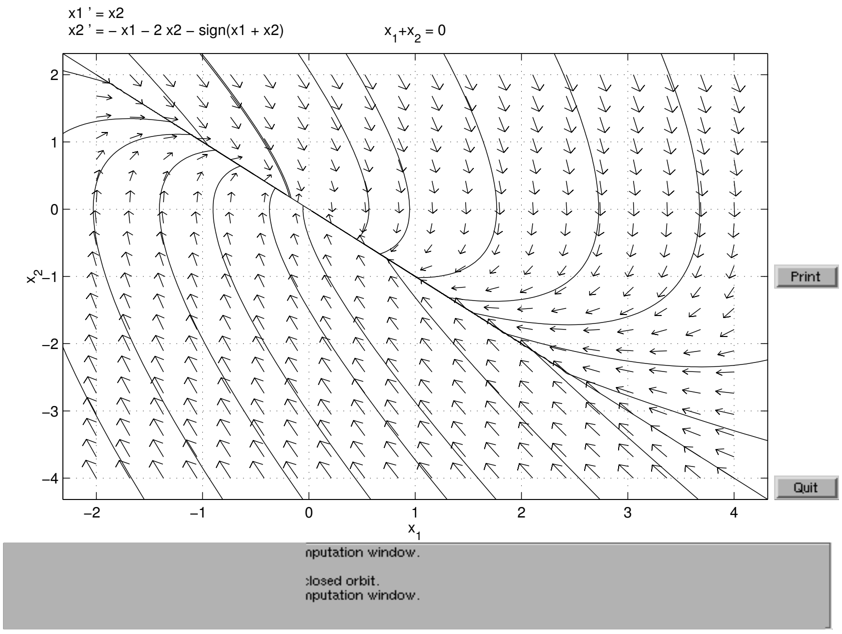

Figure 8 The phase plot in Problem 7

In general, we would express one of the states in the other from the relation $\sigma(x) = 0$ which is valid on the switch curve, and impose the constraint that $u_{eq} \in [-1, 1]$ to find the region for the sliding set. As we in this case have that $u_{eq} = 0$ this can be fulfilled for all values of $x_1 + x_2 = 0$, we have that the whole line belongs to the sliding set.

On the line $x_1 + x_2 = 0$ (or equivalently $x_1 = -x_2$) we have the sliding dynamics

$$
\begin{aligned}
\dot{x}_1 &= x_2 && = -x_1 \\
\dot{x}_2 &= -x_1 - 2x_2 + \underbrace{u_{eq}}_{=0} && = -x_2
\end{aligned}
$$

Thus the system will be asymptotically stable and slide to the origin, see Figure 8.

8. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2ax_1x_2^2 + u \\ \dot{x}_2 &= -ax_2^3 - (2a-1)x_2,\end{aligned} \tag{1}$$

where a is an unknown parameter.

a. Determine a bound γ, such that the origin of (1) for $u = 0$ is **locally** asymptotically stable for all $a > \gamma$. (1 p)

b. Show that the nonlinear feedback control law, $u = -x_1^3$, makes the origin globally asymptotically stable in the case $a = 1$. (2 p)

Solution

a. Close to the origin, the linear terms dominate over the higher-order terms and the dynamics are governed by

$$\begin{aligned}\dot{x}_1 &= -3x_1 \\ \dot{x}_2 &= -(2a-1)x_2\end{aligned}$$

(This is the linearized system around the origin.)
Lyapunov's linearization method now gives LAS around the origin if the eigenvalues of

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -(2a-1) \end{pmatrix}$$

are both in the left half-plane. This gives the condition

$$-(2a-1) < 0 \Rightarrow a > \frac{1}{2} = \gamma$$

b. We use the standard Lyapunov function candidate

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

which is radially unbounded, $V(0,0) = 0$, and $V(x_1, x_2) > 0 \ \forall x_1, x_2 \neq 0$.
We have for $a = 1$

$$\begin{aligned}\dot{V} &= \dot{x}_1x_1 + \dot{x}_2x_2 = (-3x_1 + 2x_1x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4\end{aligned}$$

Inserting the control law, $u = -x_1^3$, we get

$$\dot{V} = -3x_1^2 - x_2^2 - x_1^4 + 2x_1^2x_2^2 - x_2^4 = -3x_1^2 - x_2^2 - \left(x_1^2 - x_2^2\right)^2 < 0 \ \forall x_1, x_2 \neq 0$$

This shows global asymptotic stability of the origin.

9. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - u \begin{bmatrix} 0 & 2\epsilon \\ \epsilon & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax + uBx \tag{2}$$

where $u \in [-1, 1]$ and $\epsilon > 0$

This is a simple second order model of a swing and the control corresponds to a child standing straight up or squatting (Swe. “sitter på huk”) to pump energy into the system (moving the center of mass and thus changing the “effective pendulum length”).

a. Show that the state cannot be moved away from the origin $(x_1, x_2) = (0, 0)$ if starting there. (0.5 p)

b. Show that for any optimal control problem with only minimal time cost, the following dynamical relation holds

$$\frac{dx}{dt} = \frac{\partial H}{\partial \lambda}$$

where H is the Hamiltonian, x the ordinary state dynamics and λ the adjoint states. (0.5 p)

c. Assume that we are starting from some initial condition $x_0 = (x_1(0), 0) \neq 0$ and would like to reach a higher (energy) level $x(t_f) = \big(x_1(t_f), 0\big) \geq (a, 0)$ as fast as possible. (You may here consider $x(t_f) = \big(x_1(t_f), 0\big) = (a, 0)$.)

Rewrite the problem as an optimal control problem and show that the optimal swing strategy is given by

$$u^*(t) = sign(2\lambda_1 x_2 + \lambda_2 x_1) \tag{3}$$

(You should write down the differential equations for the adjoint variables including the terminal conditions, but you need not solve them.) (2 p)

Solution

a. As the control signal is multiplied with the state vector we do not have any control action at $x = 0$ which is an equilibrium point and thus we can not leave it.

b.

$$H = 1 + \lambda^T f \Longrightarrow \frac{\partial H}{\partial \lambda} = f(x) = \dot{x}$$

c.

$$min_u\, t_f = min_u \int_0^{t_f} 1dt \Longrightarrow H = 1 + \lambda^T f = 1 + [\,\lambda_1 \quad \lambda_2\,]\,(Ax + uBx) \tag{4}$$

$$\begin{aligned} min_u\, H &= min_u\,(1 + \lambda_1(x_2 - 2\epsilon u x_2) + \lambda_2(-x_1 - \epsilon u x_1) = \\ &= min_u 1 + \lambda_1 x_2 - \lambda_2 x_1 - \epsilon \cdot u \cdot (2\lambda_1 x_2 + \lambda_2 x_1)) \end{aligned}$$

To minimize H with respect to $u \in [-1, 1]$:
For $(2\lambda_1 x_2 + \lambda_2 x_1) > 0$ choose $u^* = 1$ and for $(2\lambda_1 x_2 + \lambda_2 x_1) < 0$ choose $u^* = -1 \Longrightarrow$

$$u^* = sign(2\lambda_1 x_2 + \lambda_2 x_1)$$

$$\Psi = \begin{bmatrix} \Psi_1 = x_1 - a \\ \Psi_2 = x_2 - 0 \end{bmatrix} = 0$$

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = (1 + \epsilon u)\lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = (-1 + 2\epsilon u)\lambda_1 \end{aligned}$$

where

$$\begin{aligned} \lambda_1(t_f) &= n_0 \frac{\partial \phi}{\partial x_1} + \mu \frac{\partial \psi}{\partial x_1} = \mu_1 \\ \lambda_2(t_f) &= n_0 \frac{\partial \phi}{\partial x_1} + \mu \frac{\partial \psi}{\partial x_2} = \mu_2 \end{aligned}$$