LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

# Nonlinear Control and Servo Systems (FRT 075)

**Exam - April 15, 2004 at 8 am –1 pm**

### Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

*Preliminary* grades:

3: 12 – 16 points

4: 16.5 – 20.5 points

5: 21 – 25 points

### Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/”Collection of Formulae”. Pocket calculator.

### Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact the lecturer Anders Robertsson for checking your corrected exam.

The result will also be published on the course homepage
http://www.control .lth.se/~kursolin/
for those who have accepted this.

*Note!*
In many cases the sub-problems can be solved independently of each other.

**Good Luck!**

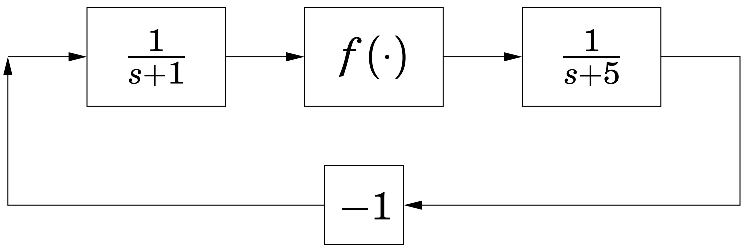


**Figure 1** System in Problem 1.

**0.** Do you accept web-publication of your result?

*Solution*

—

**1.** Consider the control system in Fig 1. The function $f(\cdot)$ is a sector bounded nonlinearity.

**a.** Suppose $f(\cdot)$ belongs to the sector $[-k, k]$ Find the largest value of $k$ for which you can guarantee that the closed loop system is stable. (1 p)

**b.** Suppose $f(\cdot)$ belongs to the sector $[0, \beta]$ Find the largest value of $\beta$ for which you can guarantee that the closed loop system is stable. (1.5 p)

**c.** Introduce states and write the system in state space form. (1 p)

*Solution*

The system consists of a negative feedback connection of the nonlinearity $f(\cdot)$ and the linear system $G(s) = \dfrac{1}{(s+1)(s+5)}$ (linear systems commute).

**a.** Small gain: Max gain of $G(s) = \dfrac{1}{(s+1)(s+5)}$ is 0.2 $\Longrightarrow k = 5$.

**b.** Sketch the Nyquist diagram for the stable system $G(s) = \dfrac{1}{(s+1)(s+5)}$, see Fig 2. According to the Circle criterion the Nyquist plot for the stable system $G(i\omega)$ should be to the right of the line $-1/\beta = -0.016$, see Fig. 2. This means that $\beta < 62$. An alternative way to find $\beta$ is to write the expression for the real part of $G(i\omega)$ and minimize (analytically or numerically) w.r.t $\omega$.

**c.** Introduce state $x_1$ at the output of block $\frac{1}{s+1}$ and state $x_2$ at the output of block $\frac{1}{s+5}$.

$$
\begin{aligned}
\dot{x}_1 &= -x_1 - x_2 \\
\dot{x}_2 &= -5x_2 + f(x_1)
\end{aligned}
\tag{1}
$$

**2.** Consider the system

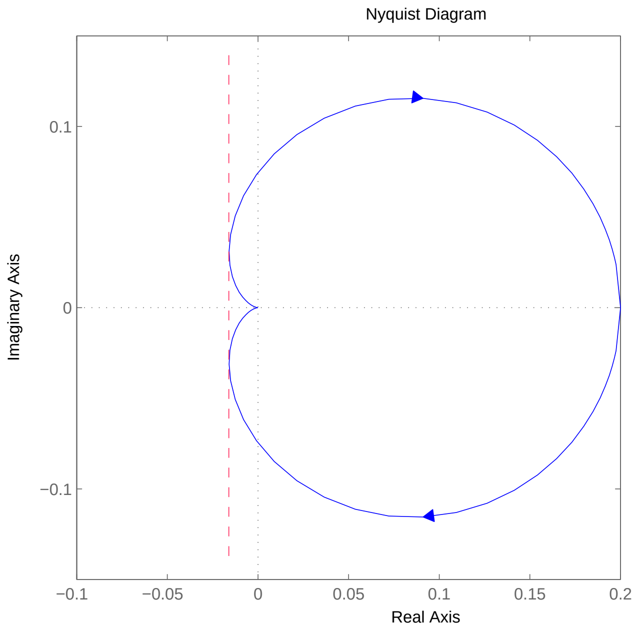


**Figure 2** Nyquist plot for $\frac{1}{(s+1)(s+5)}$ in problem 1.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_2^2 - x_1 - u^2 + t \\
y &= x_1
\end{aligned}
\tag{2}
$$

**a.** Show that system. (2) satisfies the solution

$$y(t) = -\sin(t) + t, \;\; u(t) = 1 - \cos(t)$$

(1 p)

**b.** Linearize the system around the solution in **(a)**. (1.5 p)

*Solution*

**a.** Direct calculations show that

$$
\begin{aligned}
y = x_1 &= -\sin(t) + t \\
x_2 = \dot{x}_1 &= -\cos(t) + 1 \\
\dot{x}_2 &= \sin(t)
\end{aligned}
\tag{3}
$$

and as

$$x_2^2 - x_1 - u^2 + t = (1 - \cos(t))^2 - (-\sin(t) + t) - (1 - cos(t))^2 + t = \sin(t)$$

we have shown that the solution satisfies the system dynamics.

**b.** The linearized system is thus

$$\begin{aligned}\delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 2x_2 \end{bmatrix}_{(x_1^o, x_2^o, u_o)} \delta x + \begin{bmatrix} 0 \\ -2u \end{bmatrix}_{(x_1^o, x_2^o, u_o)} \delta u \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 2(-\cos(t)+1) \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -2(1-cos(t)) \end{bmatrix} \delta u \end{aligned} \tag{4}$$

where

$$\delta x = \begin{bmatrix} x_1(t) - x_{1o}(t) \\ x_2(t) - x_{2o}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - (-\sin(t)+t) \\ x_2(t) - (-\cos(t)+1) \end{bmatrix}, \quad \delta u = u(t) - u_o(t) = u - (1-\cos(t))$$

**3.** Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned} \tag{5}$$

We have two suggested control laws to examine.

$$\begin{aligned}u = u_I &= -x_1 - x_2 - x_2^3 \\ u = u_{II} &= -x_1 - x_2 x_1^2\end{aligned}$$

Do any or both of the controllers above render the origin GAS?
Hint: You may consider the function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$$

(3 p)

*Solution*

$$\begin{aligned} V(x) &= \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 > 0 \text{ for } x \neq 0, \text{ radially unbounded} \\ \dot{V} &= \dot{x}_1 x_1 + \dot{x}_2 x_2^3 = -x_2^3(x_1 + u) \\ \dot{V}_I &= -x_2^4 - x_2^6 \leq 0, \qquad (\text{applying } u_I) \\ \dot{V}_{II} &= -x_2^4 x_1^2 \leq 0, \qquad (\text{applying } u_{II}) \end{aligned} \tag{6}$$

The resulting dynamics for the two cases are
case I:

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2 - x_2^3\end{aligned} \tag{7}$$

$\dot{V}_I = 0$ if $x_2 = 0$. The only possibility to stay on $x_2 = 0$ is if $x_1 = 0$. According to the Invariant Set Theorem (LaSalle) the origin is GAS.
case II:

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2 x_1^2\end{aligned} \tag{8}$$

$\dot{V}_{II} = 0$ if $x_1 = 0$ or $x_2 = 0$. To stay on $x_1 = 0$ we need $x_2 = 0$ and to stay on $x_2 = 0$ we need $x_1 = 0$, thus also this controller renders the origin GAS.

**4.** Consider the dynamic system

$$\ddot{y} + \dot{y} + 2y - sat(y) = \dot{u} - u, \quad y(0) = y_0$$

Is the system passive? (Hint: What can you say about the system when it operates in the linear region?) (1 p)

*Solution* Pick $u$ such that $|y| \leq 1$ for some amount of time $T$. (After the time $T$, you can set the input $u(t) = 0$) Then, for $t \in [0, T]$, the system behaves like the linear time invariant system

$$G(s) = \frac{s-1}{s^2+s+1},$$

which is not passive (static gain <0 and thus real part of Nyquist plot not strictly >0). Hence the nonlinear system is not passive.

**5.** Sketch the describing function corresponding to the nonlinearity in Fig. 3. (1 p)

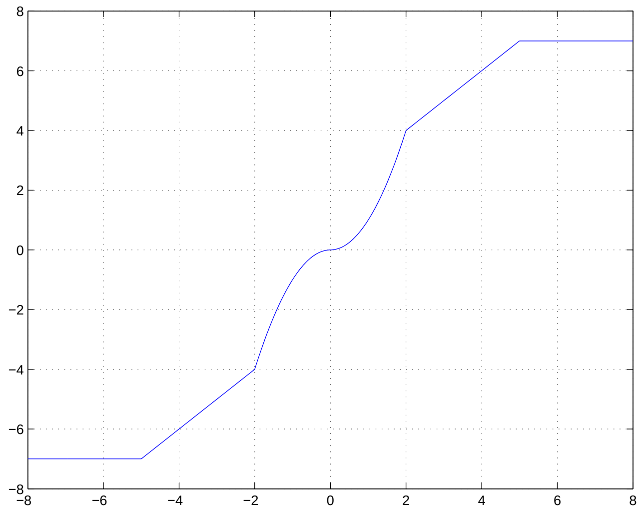


**Figure 3** Nonlinear function in problem 5.

*Solution*
Increasing growth in region $A \in [0, 2]$, (slow) decay towards 1 in region $A \in [2, 5]$, (slow) decay towards 0 in region $A > 5$, see Figure 4.

**6.** Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= sign(1 - x_1 - x_2) - x_2 - x_1 \end{aligned} \tag{9}$$

**a.** Find the equilibrium of system 9, (0.5 p)
(Note: In this subproblem you may consider the sign-function as $sign(x) = -1$ if $x < 0$ and $sign(x) = +1$ for $x \geq 0$).

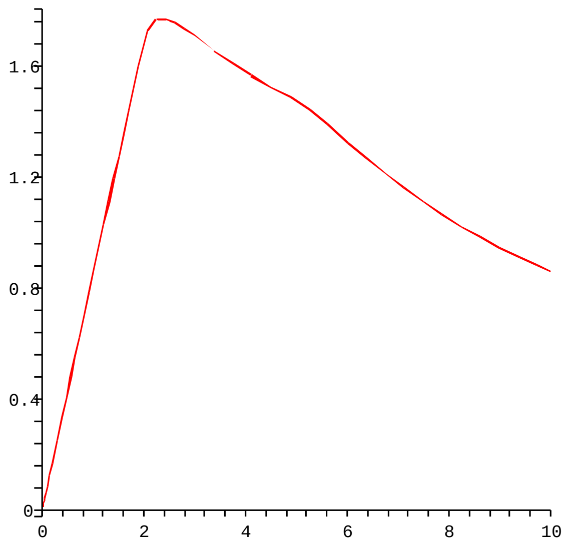


**Figure 4** Describing function in problem 5.

**b.** Find the sliding set and determine the behavior (the sliding dynamics) on/along the sliding set. Is the equilibrium in **(a)** asymptotically stable? (2 p)

*Solution*

**a.** $\dot{x}_1 = \dot{x}_2 = 0 \Longrightarrow (x_1^o, x_2^o) = (1, 0)$

**b.** Rewriting the system as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - 1 \cdot u \\ u &= -sgn(\sigma) \qquad \sigma = 1 - x_1 - x_2 \end{aligned}$$

The switch curve is $\sigma = 1 - x_1 - x_2 = 0$. Note that the sliding region is not the whole line but only for $-1 < x_1 < 1$, where the vector field on both side of the switch slide, $f_+$ and $f_-$, have opposite normal directions. "Equivalent control" gives

$$\dot{\sigma} = 0 - \dot{x}_1 - \dot{x}_2 = x_2 - x_1 - x_2 - u_{eq} = 0 \Longrightarrow u_{eq} = -x_1$$

Inserting this in the dynamics above on the line $1 - x_1 - x_2 = 0$ gives

$$\begin{aligned} \dot{x}_1 &= x_2 = -x_1 + 1 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

Decoupled dynamics for $x_1$ and $x_2$ with asymptotically stable poles in $s = -1$, shows us that the solution along the switch line $\sigma$ will slide towards the point (1,0).

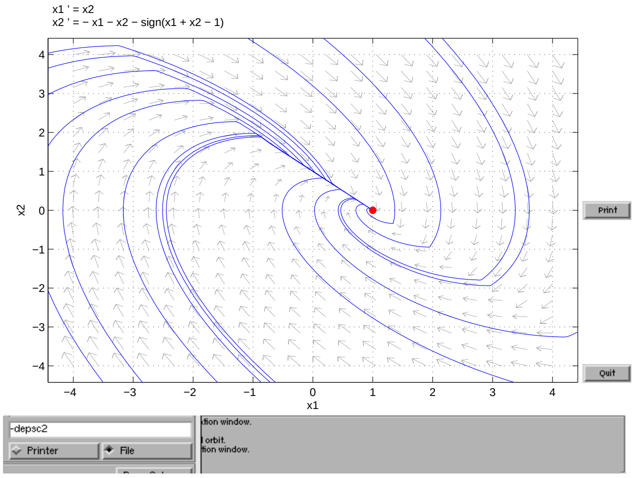


**Figure 5**

Remark: It might be helpful to use

$$
f_+ = \begin{bmatrix} x_2 \\ 1 - x_1 - x_2 \end{bmatrix}
$$

$$
f_- = \begin{bmatrix} x_2 \\ -1 - x_1 - x_2 \end{bmatrix}
$$

to find the sliding region. The normal to the switching line is $[1 \quad 1]$ The components of $f_+$ and $f_-$ perpendicular to the switching line are thus $[1 \quad 1]\begin{bmatrix} x_2 \\ 1 - x_1 - x_2 \end{bmatrix} = 1 - x_1$ and $[1 \quad 1]\begin{bmatrix} x_2 \\ -1 - x_1 - x_2 \end{bmatrix} = -1 - x_1$ respectively. These expressions have different signs (here pointing towards the switching line) only in the interval $x_1 \in (-1..1)$

**7.** Consider the following dynamical system

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 + x_2 \\
\dot{x}_2 &= u
\end{aligned} \tag{10}
$$

**a.** Use a linear feedback law and show that the origin is locally asymptotically stable (2 p)

**b.** Is it possible by using a linear feedback law to make the origin globally asymptotically stable? (1 p)

**c.** Design a nonlinear feedback law for (10), such that the origin of the closed loop system is globally exponentially stable.
Hint: Change the variables of system (10) by the transformation

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_1^2 + x_2
\end{aligned} \tag{11}
$$

Then design a stabilizing nonlinear feedback law for the new system. (2 p)

*Solution*

**a.** Use $u = -k_1 x_1 - k_2 x_2$ and Lyapunov's linearization method. From the system equations we conclude that $x = 0$ is an equilibrium point. Linearization of system (10) give

$$A = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

The poles lie in the left half plane (Re $\lambda_{1,2} < 0$), then $x = 0$ is locally asymptotically stable.

**b.** No. Choose for example $x_1 >> 0$ and $|x_1| >> |x_2|$ ($x_1^2$ dominates) then $x_1$ will increase and $x_2$ decrease.

**c.** The new system is

$$\begin{aligned} \dot{z}_1 &= x_1^2 + x_2 = z_2 \\ \dot{z}_2 &= 2x_1\dot{x}_1 + \dot{x}_2 = 2x_1(x_1^2 + x_2) + u \end{aligned}$$

The state equation in the new coordinates$(z_1, z_2)$ can be found by inverting the transformation (11)

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= -z_1^2 + z_2 \end{aligned}$$

The transformed state equation is the given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 2z_1 z_2 + u \end{aligned}$$

Then the nonlinearities can be can canceled by the feedback

$$u = -k_1 z_1 - k_2 z_2 - 2z_1 z_2$$

With the feedback the system is given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -k_1 z_1 - k_2 z_2 \end{aligned}$$

As the resulting system is linear and time invariant with poles in the left half plane it is globally asymptotically and even globally exponentially stable.

**8.** [Tuning at Tuna]

Former economy student CIA is now working as controller at ICA Tuna. She notices that there are some problems with the new conveyor belts. To save time at the cashier the new conveyor belts were set up to be much faster than the old ones, but instead of increasing the flow there are now problems with the milk containers which are tipping over [1] and this slows down the procedure at the cashier.

Her old engineering friend H A Milton heard about the problem and told her about a "beautiful principle" which could be used for solving these kinds of optimization problems.

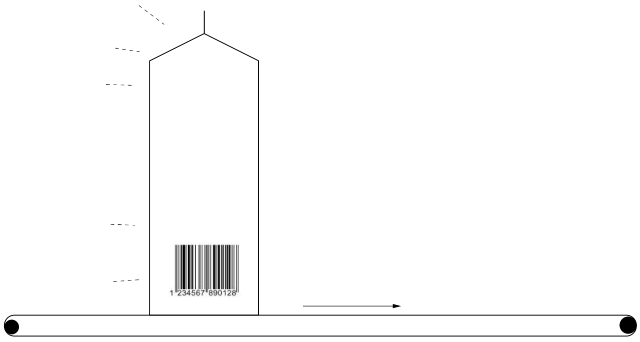

**Figure 6** Milk container on a conveyor belt

The new laser scanners can detect the EAN code (so called "streckkod") if the container passes at speeds below 0.2 m/s.

The conveyor belt is torque controlled, so it can be accelerated directly by the control signal $u$ according to

$$\ddot{x} = u$$

where $x$ is the position.

A crude approximation gives that the control signal should be bounded by $|u| < 5$ for the container not to tip over.

**a.** Help CIA to formulate the optimization problem of moving a container from $x(0) = \dot{x}(0) = 0$ to $x(T) = 0.5\,m$, $\dot{x}(T) = 0.2\,m/s$ as fast as possible, without tipping over the container. (1 p)

**b.** Heuristic reasoning to get bounds on the optimal time: What would the control signal look like if we were to move the container 0.5 m as fast as possible? How could it look like if you were to move it from "rest to rest" (zero velocity at both start and end of motion)? (1 p)

**c.** Solve for $u$ which optimizes the criterion in **(a)**. For full point you should characterize how $u$ depend on time. The optimal time should of course be within the bounds you found in **(b)**, but you do not need to explicitly solve for $T$ here. (2 p)

*Solution*

**a.** We take the state vector to be $(x_1, x_2) = (x, \dot{x})$ then

$$\min_u \int_0^T 1dt$$

$$\text{subject to} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x_1(0) = x_2(0) = 0,\ |u| < 5$$

and $x_1(T) = 0.5$, $x_2(T) = 0.2$,
i.e. $\phi(x) = 0$, $\psi_1(x) = x_1 - 0.5$, $\psi_2(x) = x_2 - 0.2$

**b.** To move the distance 0.5 m in as short time as possible, we should of course use the maximal acceleration all the time. $\ddot{x} = 5$, $x(0) = 0$, $\dot{x}(0) = 0 \Longrightarrow x(t) = 5t^2/2$. $x(T) = 0.5 \Longrightarrow 5T^2/2 = 0.5 \Longrightarrow T \approx 0.45 sec$.

[1](which in some rare cases indeed increases the flow)

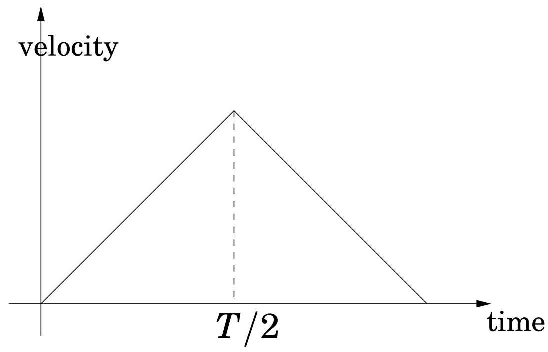


**Figure 7** Velocity profile in **(b)** for max acceleration, followed by max retardation.

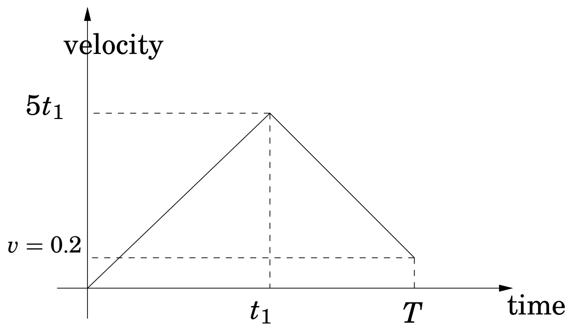


**Figure 8** Velocity profile in **(c)** for max acceleration, followed by max retardation.

To instead get to rest at the end, we could first use max acceleration for half the time and then max retardation. The velocity profile would then look like Fig. 7, where the area under the triangle ($= u_{max}T^2/4$ ) corresponds to the distance. The slope of the velocity is the max acceleration $u_{max} = 5$. It will thus take $T = \sqrt{2/5} \approx 0.63$ seconds.

**c.** We first consider the normal case, i.e. $n_0 = 1$. The Hamiltonian is given by

$$H(y,u,\lambda) = L + \lambda^T f(x) = 1 + \lambda_1 x_2 + \lambda_2 u$$

At optimality we should minimize H with respect to $u$ within its bounds. We see that if $\lambda_2 < 0$ we should choose $u = +5$ and if $\lambda_2 > 0$ we should choose $u = -5$.

The adjoint equations are given by

$$\dot{\lambda} = -\frac{\partial H(x,u,\lambda)^T}{\partial x} \lambda(T) = \frac{\partial \phi}{\partial x} + \mu^T \frac{\partial \Psi(T, x(T))}{\partial x}$$

$$\begin{aligned} \dot{\lambda}_1 &= 0, \quad \lambda_1(T) = 0 + \mu_1 \cdot 1 + \mu_2 \cdot 0 \\ \dot{\lambda}_2 &= -\lambda_1, \quad \lambda_2(T) = 0 + \mu_1 \cdot 0 + \mu_2 \cdot 1 \end{aligned} \tag{12}$$

Integrating, we get

$$\lambda_1(t) = \mu_1$$

Then

$$\lambda_2(t) = -\mu_1 t + C, \quad \lambda_2(T) = \mu_1 \cdot T + C = \mu_2 \Longrightarrow \lambda_2(t) = \mu_1(T-t) + \mu_2$$

Note that $\lambda_2$ increases or decreases linearly with time and may shift sign at most once. From the minimization of H we thus can conclude that $u$ should shift sign when $\mu_1(T-t_1)+\mu_2 = 0$.

The velocity profile is showed in Fig. 8, where the switch from $u = 5$ to $u = -5$ occurs at time $t_1$. The total area under the curve should be 0.5 and the velocity $v_{end} = 0.2$

$$\begin{aligned} 0.5 &= 5t_1^2/2 + (5t_1 - v_{end})(T-t_1)/2 + v_{end}(T-t_1) \\ v_{end} = 0.2 &= 5t_1 - 5(T-t_1)5 \\ &\Longrightarrow t_1 \approx 0.32,\ T \approx 0.59 \end{aligned} \tag{13}$$

**9.** Consider a linear time-varying system

$$\text{S: } \dot{x} = A(t)x, \quad t \geq 0$$

**a.** Show that the system is globally asymptotically stable if all the eigenvalues of $A(t)+A(t)^T$ are strictly negative, i.e. show that if there exists a constant $r > 0$

$$\text{s.t. } \lambda_j(A(t)+A(t)^T) \leq -r, \forall j \text{ and } t \geq 0 \Longrightarrow \text{ S is GAS.}$$

(Hint: Use $V(x) = \frac{1}{2}x^T x$) (1 p)

**b.** Suppose that the system matrix is given by

$$A(t) = \begin{bmatrix} -1 & \exp(t/2) \\ 0 & -1 \end{bmatrix}$$

Show that the condition in **(a)** is not satisfied for all values of $t$, but that the origin of this particular system is GAS anyway. (1.5 p)

*Solution*

**a.**

$$\dot{V} = x^T\dot{x} + \dot{x}^T x = x^T A(t)x + (A(t)x)^T x = x^T(A(t)+A(t)^T)x \leq -rx^T x \leq 0$$

with equality iff. $x = 0$. Hence S is GAS since V is radially unbounded.

**b.** Note that the condition in a) is not necessary. In fact we have

$$\det(\lambda I - (A(t)+A(t)^T)) = \det\begin{bmatrix} \lambda+2 & -\exp(t/2) \\ -\exp(t/2) & \lambda+2 \end{bmatrix} = 0$$
$$\Longleftrightarrow\ (\lambda+2)^2 = \exp(t) \ \Longleftrightarrow\ \lambda = -2 \pm \exp(t/2)$$

So that whenever $t > 2\ln(2)$ one of the eigenvalues are greater then 0. But we can solve directly for $x_2$ and then $x_1$

$$\dot{x_2} = -x_2 \Longrightarrow x_2(t) = x_2(0)\exp(-t) \Longrightarrow \dot{x_1} = -x_1 + x_2(0)\exp(-t/2) \Longrightarrow$$
$$\frac{d(x_1\exp(t))}{dt} = x_2(0)\exp(t/2) \Longrightarrow x_1(t) = 2x_2(0)\exp(-t/2) + C\exp(-t)$$

with $x_1(0) = 2x_2(0) + C$, i.e.

$$x_1(t) = 2x_2(0)\exp(-t/2) + C\exp(-t) \text{ and } x_2(t) = x_2(0)\exp(-t)$$

so the system is GAS.