LUND INSTITUTE OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - August 26, 2008 at 2-7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other. *Preliminary* grades:

3: $12 - 16$ points

4: $16.5 - 20.5$ points

5: $21 - 25$ points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik". Pocket calculator.

Results

The exam results will be posted within two weeks after the day of the exam on the notice-board at the Department. Contact the lecturer Anders Robertsson for checking your corrected exam.

Note!

In many cases the sub-problems can be solved independently of each other.

Solutions to the exam in **Nonlinear Control and Servo Systems** (FRTN05)
August 26, 2008.

1.

a. The symbol of the Olympic games in Fig. 1 reminds of a phase plane plot. Can the "Olympic rings symbol" be generated by a second-order time-invariant nonlinear system by picking 5 initial conditions, one for each ring, and by simulating the system for each of these 5 initial conditions so that the 5 rings appear (one ring for each simulation)? Motivate your answer. (1 p)

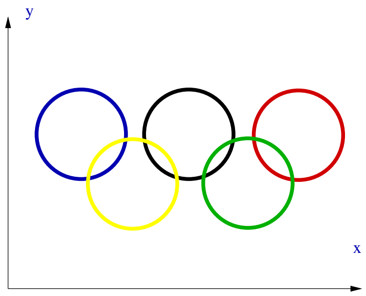

Figure 1 Olympic rings in Problem 1(a).

b. Determine which of the systems (I-IV) which generated "the rings" in Fig. 2. Also mark the direction of the solutions for two of the rings. Motivate your answer. (1.5 p)

$$(I)\quad \dot{x} = -sign(y), \quad \dot{y} = -x \qquad (II)\quad \dot{x} = y, \quad \dot{y} = -x$$

$$(III)\quad \dot{x} = cos(y), \quad \dot{y} = sin(x) \qquad (IV)\quad \dot{x} = cos(x), \quad \dot{y} = -sin(y)$$

Solution

a. If the system is time-invariant (does not change with time) we can not have trajectories crossing each others as the vector field in each point then is uniquely determined.

b. System (III) corresponds to the phase plot. There are many ways to determine this and rule out the other sysems, including drawing the phase plots, determining location and number of equilibrium points. The vector field is shown Fig. 3

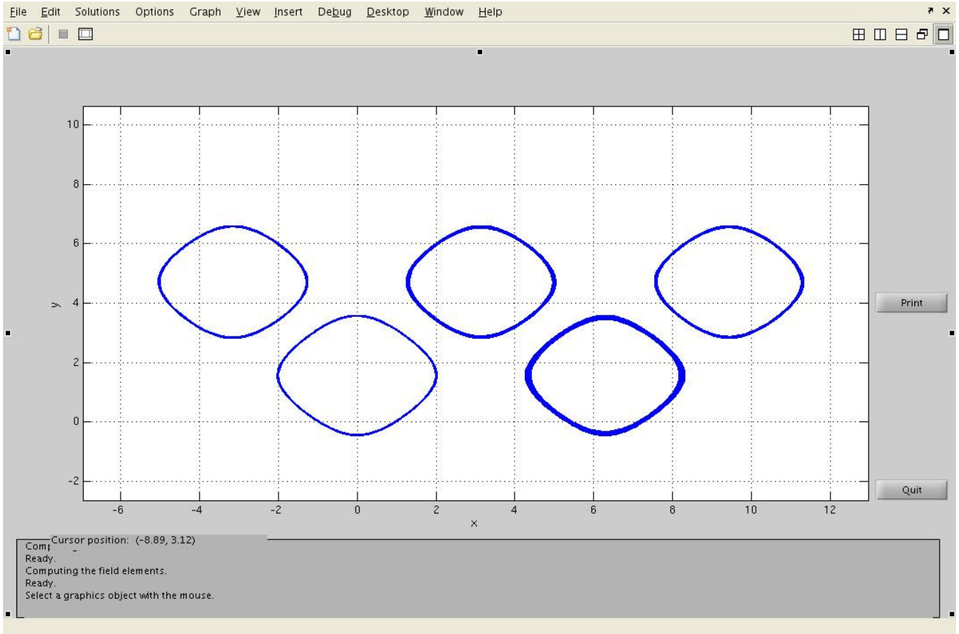

Figure 2 Phase plane plot in Problem 1(b).

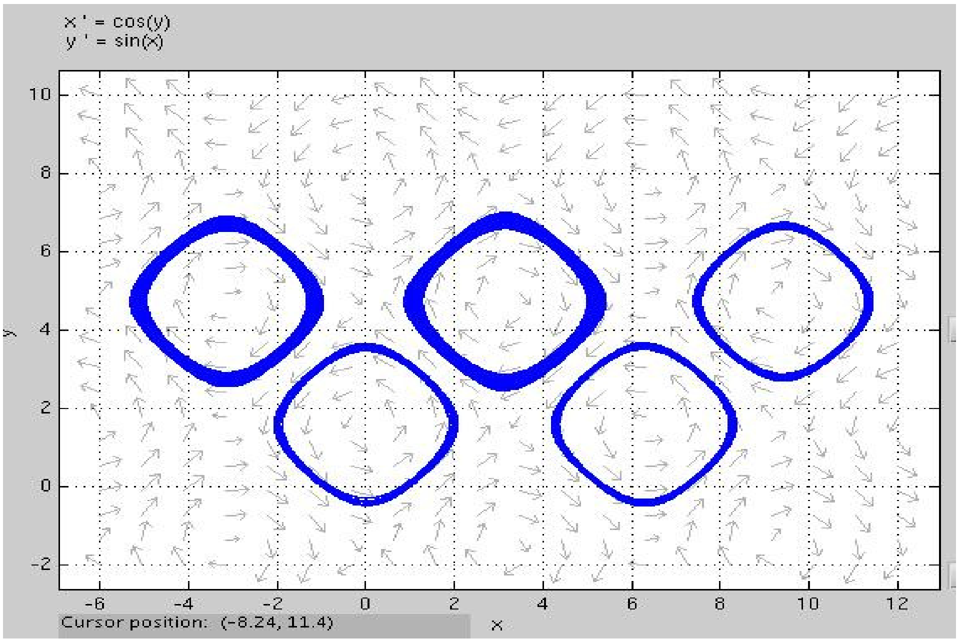

Figure 3 Phase plane plot with vector field in Problem 1(b). Note that there are many other limit cycles than the 5 "Olympic rings".

2. Consider the feedback loop in Figure 4. The linear system

$$G(s) = \frac{1}{(s+1)^4}$$

is connected with the static nonlinearity $u(t) = f(e(t))$, where $f(\cdot)$ has the describing function

$$N(A) = A + 3A^2.$$

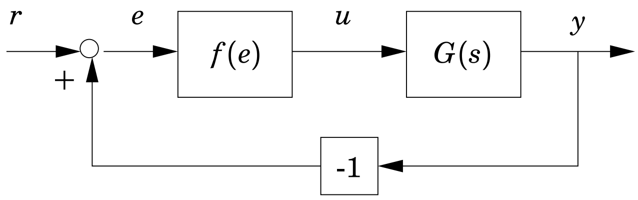

Figure 4 Block diagram for Problem 2.

What is the amplitude and frequency of a possible limit cycle? Will it be a stable limit cycle? (2 p)

Solution

Possible crossing points between $G(i\omega)$ and $1/N(A)$ must be on the negative real axis. Thus we need the frequency ω' where

$$\arg G(i\omega') = -\pi.$$

Now $\arg G(i\omega) = -4 \arctan \omega$ which leads to the solution $\omega' = 1$. Also $|G(i\omega')| = 1/4$. This gives us the equation

$$1/N(A') = 1/4$$

with the solution $A' = 1$ (and $A' = -4/3$).

The limit cycle will be unstable, as G(s) has stable poles and for increased amplitude A>A' the Nyquist curve for $G(s)$ will encircle the point -1/N(A) (i.e. $|G(j\omega_{\{-180\,\mathrm{deg}\}})| \cdot |N(A)| > 1$).

3. Consider the second order system

$$\ddot{x} + \alpha \dot{x} - x + x^3 = 0 \tag{1}$$

where α is a constant parameter.

a. Rewrite the system in state-space form. (0.5 p)

b. Determine all equilibrium points for the system and determine their local stability properties for positive values of the parameter α. (2.5 p)

c. Consider the case $\alpha = 0$. Determine the time-derivative of the function

$$H = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}(\dot{x})^2$$

along the solutions of the system and comment on the properties of the system when $\alpha = 0$.

(*Note: You can choose to do this problem exercise either from the original system formulation in Eq. (1) or from your answer in* **(a)**.) (2 p)

Solution

a. Introduce two states $x_1 = x$ and $x_2 = \dot{x}$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = +x_1 - x_1^3 - \alpha x_2$$

b. The equilibrium points are given by $(\dot{x}_1, \dot{x}_2) = (0, 0)$: $\dot{x}_1 = 0 \Longrightarrow x_2 = 0$ and $\dot{x}_2 = 0 \Longrightarrow +x_1 - x_1^3 - \alpha \cdot 0 = 0 \Longrightarrow x_1 = \{0, \pm 1\}$, which gives us the three equilibria $(x_1^o, x_2^o) = \{(0, 0), (1, 0), (-1, 0)\}$ The jacobian $\dfrac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ (1-3x_1^2) & -\alpha \end{bmatrix}_{x=(x_1^o, x_2^o)} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & -\alpha \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & -\alpha \end{bmatrix} \right\}$

Equilibrium (0, 0): From solving the corresponding characteristic equation one gets the system poles to be $\frac{-\alpha}{2} \pm \sqrt{\alpha^2/4 + 1}$ (i.e., one stable and one unstable pole); The linearization has a thus saddle point and then also the nonlinear system has a saddle point in (0,0)).

Equilibria (0, ±1): At $(0, \pm 1)$ one gets the system poles for the linearized system to be $\frac{-\alpha}{2} \pm \sqrt{\alpha^2/4 - 2}$ which are in the left half-plane for $\alpha > 0$ For $\alpha > 2$ the poles will be real (stable nodes) and for $0 < \alpha < 2$ the poles will be complex conjugated (stable foci).

c. Take the time-derivative of H and insert the system dynamics (for either $\ddot{x}$ of for $\dot{x}_1$ and $\dot{x}_2$ respectively, depending on if you use the original formulation or consider your solution in subproblem **(a)**):
$\frac{dH}{dt} = \frac{dH}{dx}\frac{dx}{dt} + \frac{dH}{d\dot{x}}\frac{d\dot{x}}{dt} = -x\dot{x} + x^3\dot{x} + \dot{x}\underbrace{\cdot\, \ddot{x}}_{Eq.\ 1} = 0$. This means that the value of H is preserved constant along the system trajectories (This system property is representative for *Hamiltonian* systems).

(For this system we have that $H(x_1 = 0, x_2 = 0) = 0$, so for all points $H \neq 0$ there is no system trajectory going to the origin.)

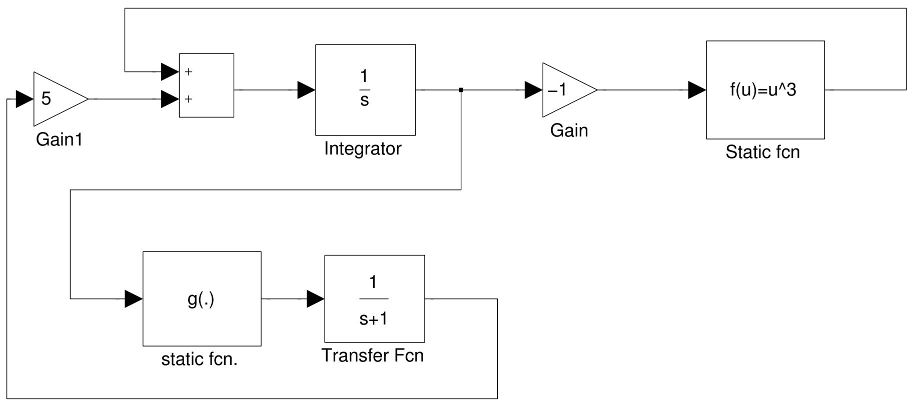

Figure 5 Block diagram for Problem 4.

4.

a. Introduce states and write the closed-loop system in Fig. 5 in a state-space form (for a general static function g). (1 p)

b. Use Lyapunov-based design to find a function $g(\cdot)$ which globally stabilizes the origin. *Hint: You may try with a quadratic Lyapunov fcn candidate.* (2 p)

Solution

a. Introduce the state x_1 at the output of the integrator and state x_2 at the output of the system $\dfrac{1}{s+1}$. This gives

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + 5x_2 \\ \dot{x}_2 &= g(x_1) - x_2\end{aligned}$$

b. Consider the Lyapunov function candidate $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ where

- $V(x_1, x_2) > 0$ if $(x_1,\, x_2) \neq (0, 0)$
- $V(0, 0) = 0$
- V is radially unbounded

$\frac{dV}{dt} = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^4 + 5x_1x_2 + x_2 g(x_1) - x_2^2$. If we e.g., choose $g(x_1) = -5x_1$ then $\dot{V} < 0$, for all $x \neq 0$ and the origin is a globally asymptotically stable equilibrium.

5. Consider the system in Figure 6, where Δ denotes some unknown nonlinear system (this is often called “multiplicative uncertainty”). The system with $\Delta = 0$ is stable. Some relevant amplitude curves are shown in Figure 7. Use the figures to find a bound γ so that the system is stable for all Δ with $\|\Delta\| < \gamma$. (3 p)

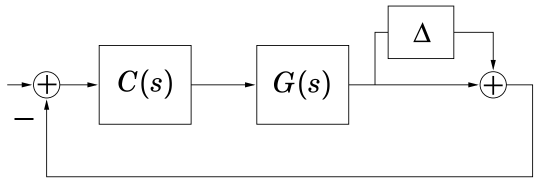

Figure 6 The system in Problem 5

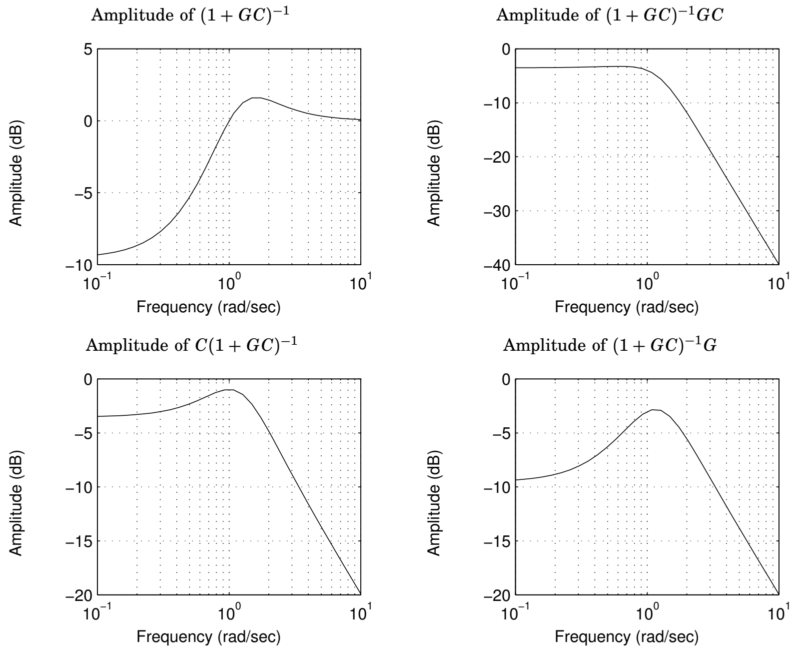

Figure 7 Amplitude curves for $(1+GC)^{-1}$, $(1+GC)^{-1}GC$, $C(1+GC)^{-1}$ and $(1+GC)^{-1}G$ in Problem 5. The y-axes are graded in $20\log_{10}(\cdot)$.

Solution

The diagram can be rewritten as as a feedback diagram with $(1+GC)^{-1}GC$ in the lower box and Δ in the upper. The small gain theorem says that the loop is stable if

$$\|\Delta\| \cdot \|(1+GC)^{-1}GC\| < 1$$

From the diagram we read $\|(1+GC)^{-1}GC\| = -3dB$ hence we have stability if $\|\Delta\| < 3dB$. Hence $\gamma = 3dB = 1.4$.

6. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 - \operatorname{sign}(x_1 + x_2)\\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

Determine which part of the switching line which belongs to the sliding surface. Also determine the dynamics on the sliding surface and its stability properties. (3 p)

Solution

The dynamics are

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 - \text{sign}(x_1 + x_2)\\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

We have thus the switch-line at $x_1 + x_2 = 0$ and will first determine on what subset of the line there may be sliding (i.e. where the vector fields on either side of the switching line points towards it). Set $\sigma(x) = x_1 + x_2$ and use e.g., equivalent control to calculate the sliding surface. Use $u_{eq} \in [-1\ 1]$

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 + u_{eq}\\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

Set $\dot{\sigma}(x) = 0$

$$\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = -x_1^2 + x_2 + u_{eq} + x_1 - x_2 = 0 \tag{2}$$

Thus $u_{eq} = x_1(x_1 - 1)$. Since $u_{eq} \in [-1\ 1]$ the sliding surface is in the region where x_1 satisfies $f(x_1) = x_1(x_1 - 1) \in [-1\ 1]$. $f(x_1)$ is convex and the minimum of $f(x_1)$ is at $x_1 = 1/2$ where $f(1/2) = -1/4$. Thus the lower bound is never reached and the region is between the roots of $x_1^2 - x_1 - 1 = 0$. The solution to this is $x_1 = 1/2 \pm \sqrt{1/4 + 1} = (1 \pm \sqrt{5})/2$, see also Fig. 8.

Look at the decoupled dynamics on the sliding surface where $\dot{\sigma}(x) = \dot{x}_1 + \dot{x}_2 = \dot{x}_1 + x_1 - x_2 = 0$. Using that $x_1 = -x_2$ on the line we get

$$\begin{aligned}\dot{x}_1 &= -2x_1\\ \dot{x}_2 &= -2x_2\end{aligned} \tag{3}$$

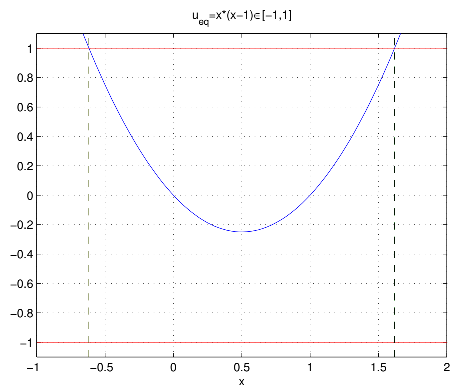

Figure 8 Condition to stay on switching line: $u_{eq} = x_1(x_1 - 1) \in [-1, 1]$.

Warning! If you don't substitute $x_1 = -x_2$ to get decoupled equations in the system above (note: only valid on on the sliding surface!!) it is easy to draw wrong conclusions! You may e.g., then get the system

$$\begin{aligned}\dot{x}_1 &= 2x_2\\ \dot{x}_2 &= 2x_1\end{aligned} \tag{4}$$

which has one stable and one unstable eigenvalue; $\lambda = \pm 2$ which may seem like a contradiction. However, the eigenvector corresponding to the stable eigenvalue is exactly along the sliding region of switching line (the only region where the “reduced dynamics” are valid), so if you start in the sliding region you will move towards the origin.

7. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2 - x_2^2)x_2$$

a. Find the only equilibrium point and show that the unit circle is a limit cycle for the system. (1 p)

b. Consider a Lyapunov function candidate

$$V(x_1, x_2) = \frac{1}{2}(1 - x_1^2 - x_2^2)^2$$

and use La Salle's principle to show that almost all trajectories converge to the limit cycle. (2.5 p)

Solution

a. Equilibrium: $\dot{x}_1 = \dot{x}_2 = 0 \Longrightarrow (x_1,\, x_2) = (0,\, 0)$ The unit circle is described by $x_1^2 + x_2^2 - 1 = 0$.

Alt 1: Check first if the unit circle is invariant: Look at the time derivative of $x_1^2+x_2^2-1$: $\dfrac{d(x_1^2 + x_2^2 - 1)}{dt} = x_1\dot{x}_1+x_2\dot{x}_2 = x_1x_2+x_2(-x_1+(1-x_1^2-x_2^2)x_2) = 0$ on the set. The motion on this set is then described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \underbrace{(1 - x_1^2 - x_2^2)}_{=0}\, x_2 = -x_1$$

reduces to a linear system with poles on the imaginary axis. The periodic solution describes exactly the unit cycle.

Alt 2. Make a change to polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and verify that $r = 1$ gives a solution.

b. $\dot{V} = (1 - x_1^2 - x_2^2)(-x_1\dot{x}_1 - x_2\dot{x}_2) = (1 - x_1^2 - x_2^2)(-x_1x_2 + x_2x_1 - x_2(1 - x_1^2 - x_2^2)x_2) = -(1 - x_1^2 - x_2^2)^2 x_2^2 \leq 0$ This means that $\dot{V} = 0$ on the unit circle $x_1^2 + x_2^2 = 1$ or when $x_2 = 0$. However, $\dot{x}_2 = x_1 - 0$, so if we do not start in the origin $(0, 0)$, $\dot{x}_2$ will be $\neq 0$. From La Salle's theorem the larges invariant set is $\{(x_1,\, x_2) | (0, 0) \cup x_1^2 + x_2^2 = 1\}$.

8. Solve the optimal control problem

$$\min_u \; x^2(T) + \int_0^T u^2(t)\,dt$$
$$\frac{d}{dt}x(t) = t\,u(t)$$
$$x(0) = 1.$$

The final time T is fixed. (3 p)

Solution

The system is normal so can put $n_0 = 1$. The Hamiltonian is

$$H = u^2 + \lambda t u$$

Minimization wrt u gives $u = -\lambda(t)t/2$. The adjoint equation is

$$\dot{\lambda} = -H_x = 0, \quad \lambda(T) = 2x(T).$$

This gives $u = -tx(T)$. If this is put into the system equation we get

$$x(T) - x(0) = \int_0^T -t^2 x(T)\, dt = -T^3/3x(T)$$

and hence $x(T) = x(0)/(1 + T^3/3)$. The optimal control signal is hence

$$u = -\frac{t}{1 + T^3/3} x(T).$$