LUND INSTITUTE
OF TECHNOLOGY
Lund University

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRT 075)

Exam - March 8, 2005 at 2 pm – 7 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Most subproblems can be solved independently of each other.

Preliminary grades:

3: 12 – 16 points

4: 16.5 – 20.5 points

5: 21 – 25 points

Accepted aid

All course material, except for exercises and solutions to old exams, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Results

The exam results will be posted within a week after the day of the exam on the notice-board at the Department. Contact the lecturer Anders Robertsson for checking your corrected exam.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

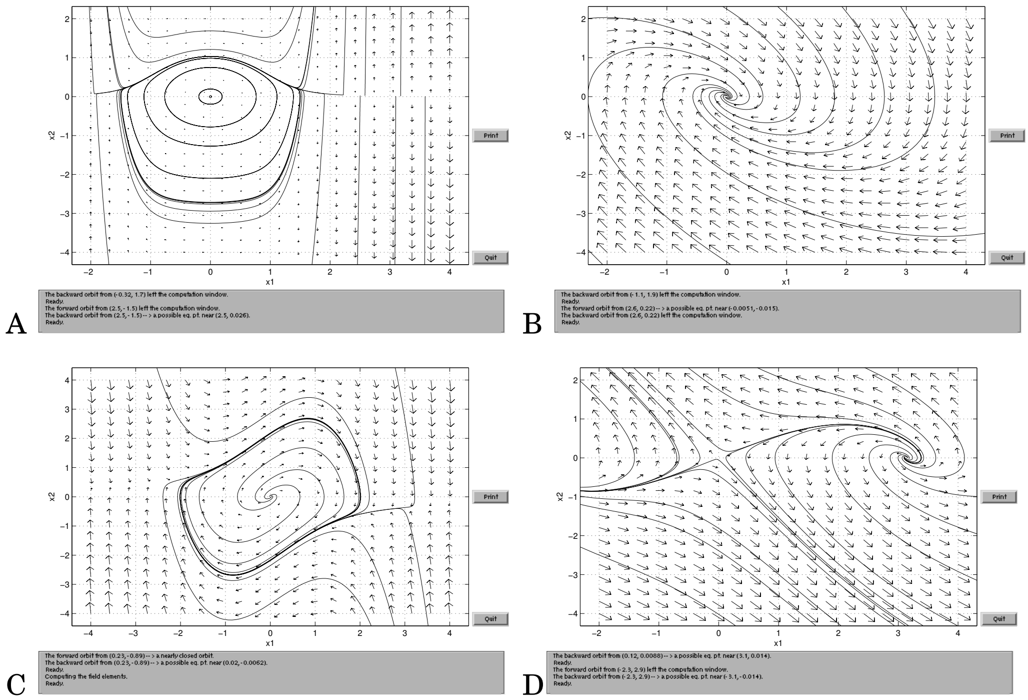

Figure 1 The phase portraits in Problem 1

0. Do you want an e-mail with your result? If so, please confirm this and write your e-mail address where you want us to send it.

Solution

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1.

a. Which phase portrait in Figure 1 belongs to what system? Briefly motivate your answers (no extensive calculations needed).

(i) $$\dot{x}_1 = -x_2$$ $$\dot{x}_2 = x_2 - \sin(x_1)$$

(ii) $$\dot{x}_1 = x_2$$ $$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2$$

(iii) $$\dot{x}_1 = x_2$$ $$\dot{x}_2 = -x_1 + x_2 x_1^5$$

(iv) $$\dot{x}_1 = x_2$$ $$\dot{x}_2 = -x_1 - x_2$$

(2 p)

Solution

a. (i)-D, (ii)-C, (iii)-A, (iv)-B

2. Consider the system

$$\dot{x}_1 = -x_1 - x_2 + sign(-x_1 - 2x_2)$$
$$\dot{x}_2 = x_1$$

Find the sliding set and determine the sliding dynamics on/along the sliding set. (Hint: Use equivalent control and state in which region u_{eq} is valid!)
(2 p)

Solution

Rewriting the system as

$$\dot{x}_1 = -x_1 - x_2 - u$$
$$\dot{x}_2 = x_1$$
$$u = -sgn(\sigma) \qquad \sigma = -x_1 - 2x_2$$

The switch curve is $\sigma = -x_1 - 2x_2 = 0$. Calculation of u_{eq} gives the sliding set.

$$\sigma = -x_1 - 2x_2 = 0 \rightarrow x_1 = -2x_2$$

$$\dot{\sigma} = -\dot{x}_1 - 2\dot{x}_2 = -x_1 + x_2 + u_{eq} = 0 \rightarrow u_{eq} = x_1 - x_2 = -3x_2$$

$u_{eq} \in [-1, 1]$ so can only satisfy $u_{eq} = -3x_2$ on the interval $\{x_1 = -2x_2, x_2 \in [-1/3, 1/3]\}$

The sliding dynamics with the calculated u_{eq} inserted in the system

$$\dot{x}_1 = -x_1 - x_2 - (x_1 - x_2) = -2x_1$$
$$\dot{x}_2 = x_1 = -2x_2$$

Thus the system is stable along the sliding set.

3. Consider the design model for an anti-lock braking system in Figure 2.

In case of *ABS control* the quantities in the figure have the following physical interpretation:

- λ - tire slip,
- y - torque caused by the friction between the tire and the road,
- u - commanded braking torque.

The value of the friction torque y can not increase above a maximum value as a function of the tire slip λ (maximum achieved at point B). By choosing a constant value for the control signal u we can achieve one or two different equilibrium points of the system in Figure 2.

How will the stability properties for the different equilibrium points A and C differ? Motivate your answer. (1 p)

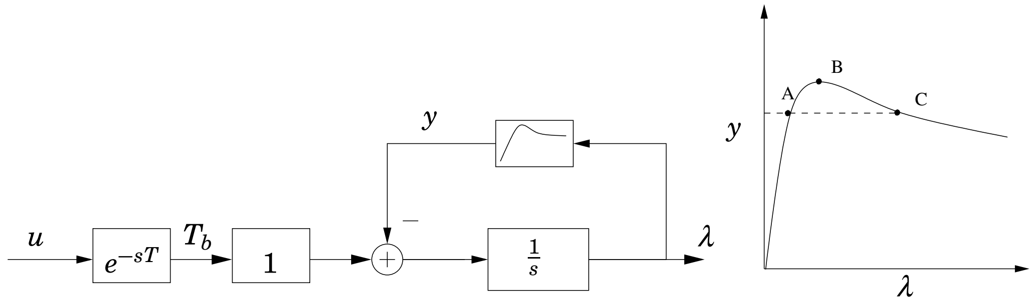

Figure 2 *(Left)* Design model for an anti-lock braking system (ABS). *(Right)* Break torque y as a function of the tire slip λ. Note that the system can have two equilibrium points, represented by the points **A** and **C** in the rightmost figure.

Solution

The system consists of a time delay in series with a negative feedback system. The time delay will not affect the stability as there is no feedback loop around it so we can concentrate on the feedback loop. For some level of u which balances y the difference will be zero and the integrator will keep its value (equilibrium points either in A and C). However if we look at the slopes of the friction curve we see that we have positive slope in the point A and negative slope in point C. This means that for a small disturbance from point A we will have in total negative feedback and we will come back to A, while in the case of a small disturbance from C we will continue move further away (unstable equilibrium point).

4. A nonlinear system is given below.

$$\dot{x}_1 = -x_1^3 + x_2 + u$$
$$\dot{x}_2 = -x_1 + a x_2^2 x_1$$

a. Find all equilibrium points to the nonlinear system if $u = 0$ and $a = 1$. Determine their local stability properties. (2 p)

b. Let $u = 0$ and $a = 0$, prove that the origin is globally asymptotically stable using the Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

(2 p)

c. When $a = 1$ the uncontrolled system is not stable. Design a control law $u = f(x)$ such that the origin becomes stable (asymptotic stability not required). (2 p)

Solution

a. All singular points are given by $\{\dot{x}_1 = \dot{x}_2 = 0\}$:

$$0 = x_1(x_2^2 - 1) \Rightarrow x_1^0 = 0 \text{ and } x_2^0 = \pm 1$$

$$0 = -x_1^3 + x_2 \Rightarrow x_1^0 = \pm 1 \text{ and } x_2^0 = 0 \Rightarrow$$
$$x^0 = \{0, 0\} \text{ and } x^0 = \{1, 1\} \text{ and } x^0 = \{-1, -1\}$$

Now let $\dot{x} = f(x)$ and then

$$\frac{d}{dt}(x - x^0) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x^0} (x - x^0)$$

$$A(x_1, x_2) = \frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ -1 + x_2^2 & 2x_2x_1 \end{bmatrix}$$
$$\text{eig } A(0,0) = \pm i \text{ (no conclusion)}$$
$$\text{eig } A(1,1) = \{-3, 2\} \text{ (saddle point)}$$
$$\text{eig } A(-1,-1) = \{-3, 2\} \text{ (saddle point)}$$

Because the linearization around the origin gives eigenvalues on the imaginary axis no conclusion can be drawn.

b.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1^3 + x_2) + x_2(-x_1) = -x_1^4 \le 0$$

Because $\dot{V}$ is only negative semi definite, we can only conclude that the origin is stable. To prove asymptotic stability we use the Invariant Set Theorem. The set E for which $\dot{V} = 0$ in this case is $x_1 = 0$, but when $x_1 = 0$ then $\dot{x}_1 = x_2 \neq 0$ unless $x_2 = 0$. Therefore the origin is the largest invariant set in E and all solutions will eventually end up there.

c.

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1^3 + x_2 + u) + x_2(-x_1 + x_2^2x_1) = -x_1^4 + ux_1 + x_2^3x_1$$

Choosing $u = -x_2^3$ gives

$$\dot{V} = -x_1^4 \le 0$$

and thus the system is stable.

5. A linear time-invariant system $G(s)$ is feedback interconnected with the nonlinear function $h(y)$ and a static gain k, see Figure 3.

$$G(s) = 0.8\frac{(s+2)(s+1.9)}{s^3 - 1}$$

The Nyquist plot of $G(s)$ can be seen in Figure 4. Find a k which makes the system globally asymptotically stable for the nonlinearity in Figure 5.

(2 p)

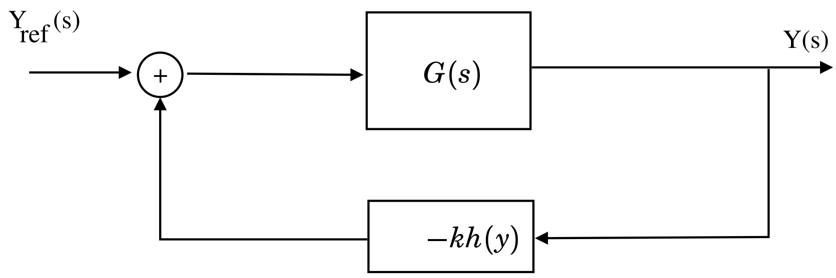

Figure 3 Block diagram in problem 5

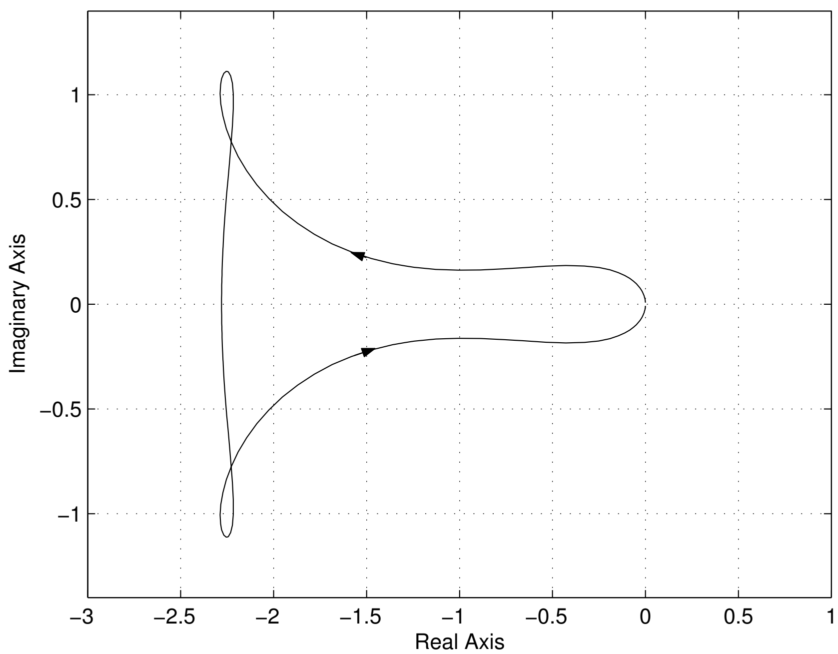

Figure 4 Nyquist plot in problem 5

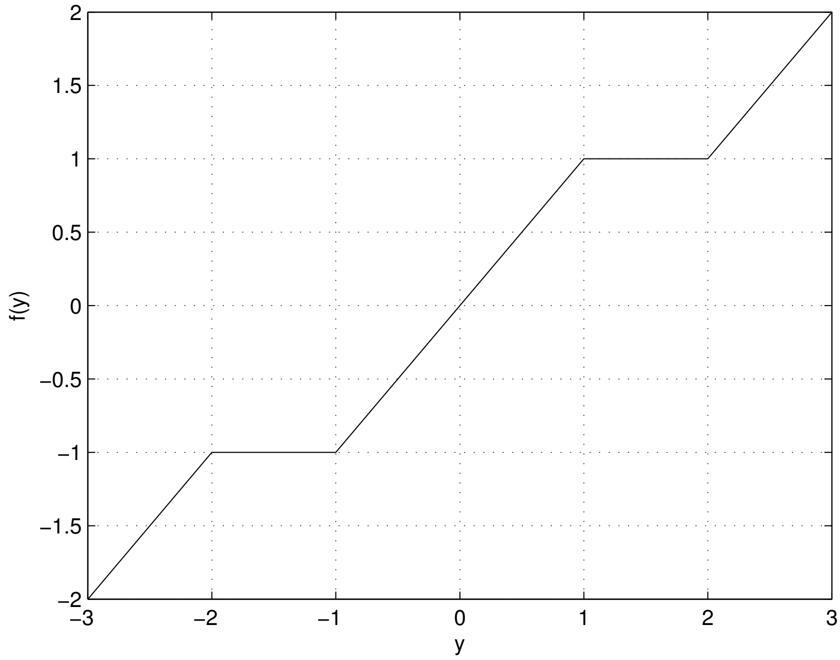

Figure 5 Nonlinearity in problem 5

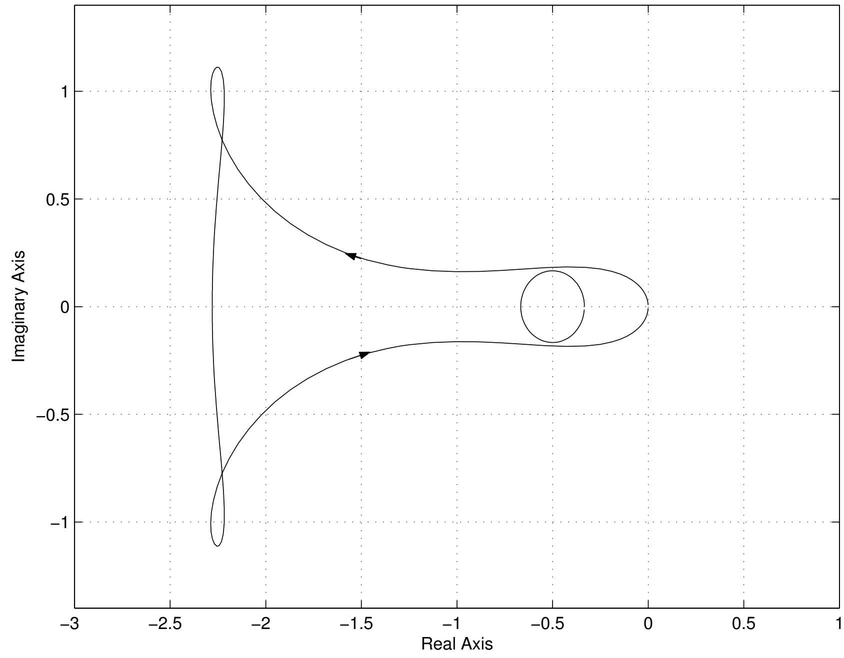

Figure 6 Circle and Nyquist plot in problem 5

Solution

By inspection of Figure 5 we conclude that the nonlinearity is contained in the sector $[0.5\ ,\ 1]$. After multiplication with k the new sector becomes $[\frac{k}{2}\ ,\ k]$.

The denominator can be written as $s^3 - 1 = (s-1)(s^2+s+1)$ and thus we conclude that the system has one pole in the right half plane. For the system to be stable the Nyquist plot of $G(s)$ must encircle the disk D($\frac{k}{2}$, k) once in the counter-clockwise direction.D($\frac{k}{2}$, k)means a circle on the real axis between $-\frac{2}{k}$ and $-\frac{1}{k}$. The circle is centered around $-\frac{3}{2k}$ and has radius $\frac{1}{2k}$. Thus making k sufficiently large will be enough. For example $k = 3$ is plotted in Figure 6.

6. Consider the nonlinear system

$$\ddot{z} + z = \epsilon(\dot{z} - \frac{1}{3}\dot{z}^3), \qquad \epsilon > 0$$

a. Show that the system can be separated into one linear and one nonlinear part as in Figure 7. Determine the transfer function $G(s)$. (2 p)

b. Calculate the describing function of $f(y) = \frac{1}{3}y^3$.
Hint:

$$\int_0^{2\pi} \sin(x)^4 dx = \frac{3\pi}{4}$$

(1 p)

c. Estimate the frequency and amplitude of possible limit cycles. (Note that your answer can be found independently of ϵ while in reality the value of ϵ affects the oscillation). (2 p)

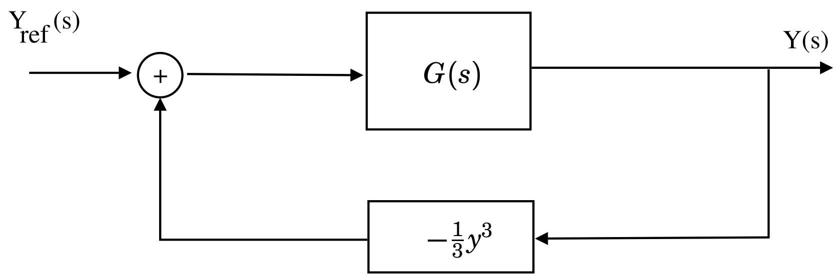

Figure 7 Block diagram in problem 6

Solution

a.

$$\ddot{z} + z - \epsilon\dot{z} = -\epsilon\frac{1}{3}\dot{z}^3 = \epsilon u.$$

Take the Laplace transform on both sides

$$s^2 Z + Z - \epsilon s Z = \epsilon U$$

$$Z = \frac{\epsilon}{s^2 - \epsilon s + 1} U$$

Let $y = \dot{z}$ then $u = -\frac{1}{3}y^3$ and $Y = sZ$ which gives

$$G(s) = \frac{\epsilon s}{s^2 - \epsilon s + 1}$$

b. The function is odd $\Rightarrow a_0 = a_1 = 0$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(A\sin(\phi))\sin(\phi)d\phi = \frac{A^3}{3\pi}\int_0^{2\pi}\sin(\phi)^4 d\phi = \frac{A^3}{4}$$

$$N(A) = \frac{ia_1 + b_1}{A} = \frac{A^2}{4}$$

c. Possible limit cycles occur when

$$G(iw) = -\frac{1}{N(A)}.$$

Because $N(A)$ is real we want to calculate the points were $Im(G(iw)) = 0$.

$$G(iw) = \frac{\epsilon i w}{-w^2 - i\epsilon w + 1} = \frac{\epsilon i w(1 - w^2 + i\epsilon w)}{(1 - w^2)^2 + \epsilon^2 w^2}$$

so

$$Im G(iw) = \frac{\epsilon w(1 - w^2)}{(1 - w^2)^2 + \epsilon^2 w^2} = 0$$

which gives $w = 0, \pm 1$. The only valid solution is thus $w = 1$.

$$G(i1) = -1 = -\frac{1}{N(A)} = -\frac{4}{A^2} \Rightarrow A = \pm 2$$

So finally $w = 1$ and $A = 2$.

7. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + \alpha x_1(\beta^2 - x_1^2 - x_2^2)\\ \dot{x}_2 &= -x_1 + \alpha x_2(\beta^2 - x_1^2 - x_2^2)\end{aligned}$$

where α and β are positive constants.

a. Show that $x_0(t) = [A\sin(t) \;\; A\cos(t)]^T$ is a solution if $A = \beta > 0$. (1 p)

b. Linearize the system around the periodic solution in a). (2 p)

Solution

a. Differentiate the solution and substitute into the system equations.

$$\begin{aligned}A\cos(t) &= A\cos(t) + \alpha A\sin(t)(\beta^2 - A^2\sin^2(t) - A^2\cos^2(t))\\ &= A\cos(t) + \alpha A\sin(t)(\beta^2 - A^2)\\ -A\sin(t) &= -A\sin(t) + \alpha A\cos(t)(\beta^2 - A^2\cos^2(t) - A^2\sin^2(t))\\ &= -A\sin(t) + \alpha A\cos(t)(\beta^2 - A^2)\end{aligned}$$

and thus $A = \beta$.

b. Denote $\dot{x} = f(x)$ and let

$$\frac{d}{dt}(x(t) - x_0(t)) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_0(t)} (x(t) - x_0(t))$$

$$A(x_1, x_2) = \frac{\partial f}{\partial x} = \begin{bmatrix}\alpha(\beta^2 - x_1^2 - x_2^2) - 2\alpha x_1^2 & 1 - 2\alpha x_1 x_2\\ -1 - 2\alpha x_1 x_2 & \alpha(\beta^2 - x_1^2 - x_2^2) - 2\alpha x_2^2\end{bmatrix}$$

$$A(t) = \begin{bmatrix}-2\alpha\beta^2\sin^2(t) & 1 - 2\alpha\beta^2\sin(t)\cos(t)\\ -1 - 2\alpha\beta^2\sin(t)\cos(t) & -2\alpha\beta^2\cos^2(t)\end{bmatrix}$$

8. Given the double integrator system with bounded control

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t), \quad u \in [-1, 1]\end{aligned} \tag{1}$$

we have seen that the time optimal control is of "bang-bang" type control. The task is now to show that this control can be written as

$$u(t) = -sign\{\sigma\,(x(t))\}$$

a. Draw the phase plane diagram for the case $u = 1$, *i.e.,* for the differential equation

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= 1\end{aligned} \tag{2}$$

Next, draw the phase plane diagram for the case $u = -1$, *i.e.,* for the differential equation

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -1\end{aligned} \tag{3}$$

Give an expression for and mark the two different trajectories (for $u = 1$ and $u = -1$ respectively) which go through the origin $x = 0$. (1 p)

b. From the optimal control theory we know that the time optimal control in this case switches sign at most once. Combine the phase plane diagrams from (a) to sketch the control strategy which brings the state of the double integrator to the origin by switching control at most once. (1 p)

c. **Either** use your results from (a)–(b) to show that the optimal control can be written in feedback form

$$u(t) = -sign\{\sigma\,(x(t))\}, \quad \sigma(x) = x_1 + sign\{x_2\}\frac{x_2^2}{2}$$

or
find the optimal control signal u as a function of time and μ by solving the optimization problem using Pontryagin's "Maximum/Minimum principle". (2 p)

Solution

a. Case 1 ($u = +1$):

$$x_2(t) = t + x_2(0) \Rightarrow x_1(t) = t^2/2 + x_2(0)t + x_1(0)$$

By eliminating time t from the expressions for x_1 and x_2 we get

$$x_1 - x_2^2/2 = C_1,\ where\ C_1 = x_1(0) - x_2(0)^2/2$$

which describe the parabola as seen in the leftmost picture of Figure 8.

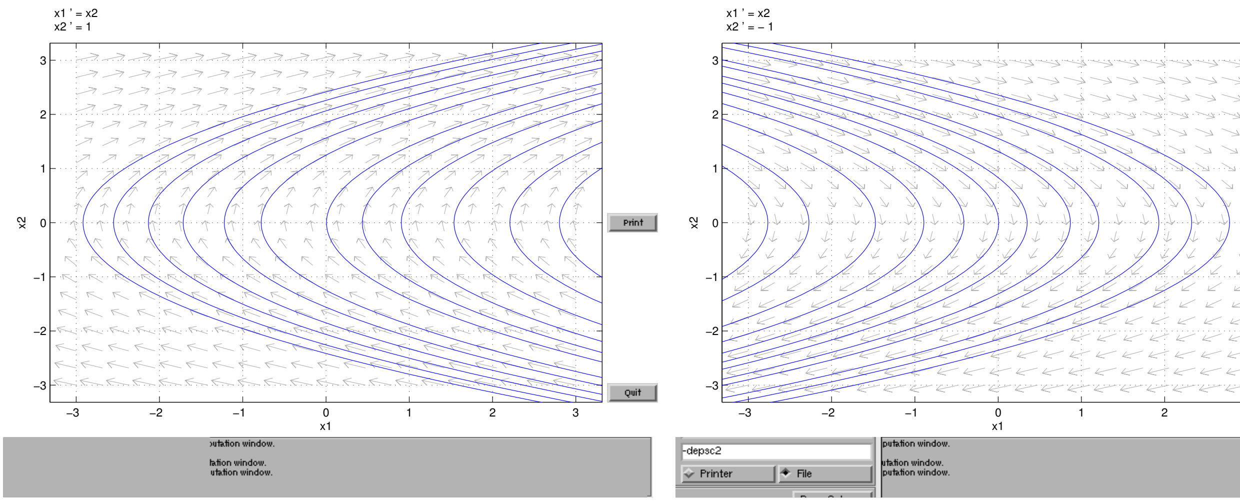

Figure 8 *(Left)* Solution curves for $u = +1$. *(Right)* Solution curves for $u = -1$.

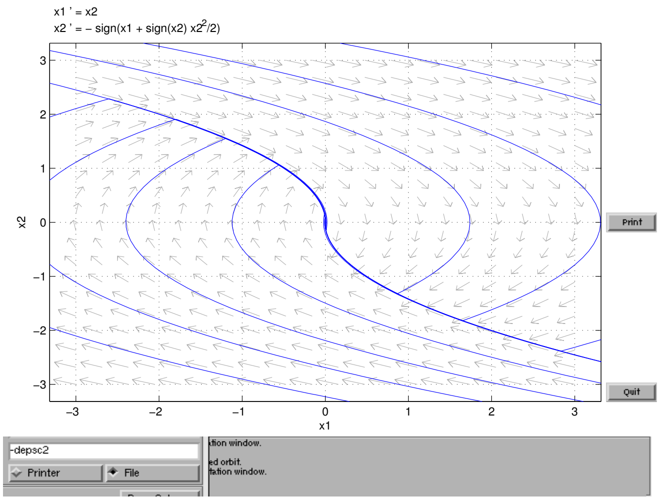

Figure 9 *(Left)* Solution curves for $u = +1$. *(Right)* Solution curves for $u = -1$.

Case 2 $(u = -1)$:

$$x_2(t) = -t + x_2(0) \Rightarrow x_1(t) = -t^2/2 + x_2(0)t + x_1(0)$$

By eliminating time t from the expressions for x_1 and x_2 we get

$$x_1 + x_2^2/2 = C_2, \textit{ where } C_2 = x_1(0) + x_2(0)^2/2$$

which describe the parabola as seen in the rightmost picture of Figure 8. The combination of the parabola and in particular the parts of those two which will have solution curves which "flows to the origin" are marked in Figure 9.

b. Follow one parabola using maximum or minimum control signal until you hit the switch line σ, switch to minimum/maximum control and then follow that parabola to the origin.

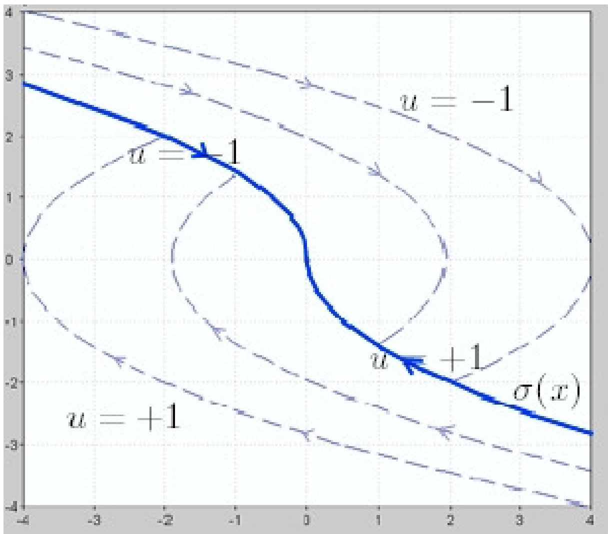

Figure 10 Combination of solution curves: Follow one parabola using maximum or minimum control signal until you hit the switch line σ, switch to minimum/maximum control and then follow that parabola to the origin.

c. The optimal control strategy requires at most one switch. We have a switch only when the initial condition is not on the "switch line" $\sigma(x)$, see Figure 10. To characterize the two areas from subproblem **b** we can see that we should choose $u = +1$ when we are to the "left" of the switch curve $\sigma(x)$ and $u = -1$ if we are to the "right" of it. The switch curve σ is a combination of the two parabola which goes through the origin and is described by the equation

$$x_1 + sign(x_2) \cdot x_2^2/2 = 0$$

Thus, the desired control law will be

$$u = -sign(\sigma(x))$$

It is also possible to state it as a standard optimal control problem, and solve for u;

$$\begin{aligned} min_u \int_0^T 1dt \\ \text{subject to} \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ u \in [-1, 1] \\ x_1(0) = x_{10},\, x_2(0) = x_{20} \\ x_1(T) = 0,\, x_2(T) = 0 \end{aligned}$$

which will give the same result but be more involved.