# State Feedback and Observers

Real-Time Systems, Lecture 9

Anton Cervin

11 February 2016

Lund University, Department of Automatic Control

# Lecture 9

[IFAC PB Chapter 8]

- State feedback
- Observers
- Integral action and disturbance estimation

# Control Design

Many factors to consider, including:

- Attenuation of load disturbances
- Reduction of the effect of measurement noise
- Command signal following
- Variations and uncertainties in process behavior

# Two Classes of Control Problems

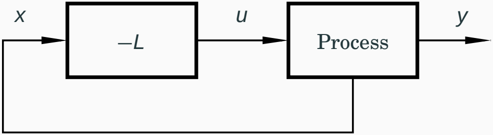
**Regulation problems:** compromise between rejection of load disturbances and injection of measurement noise

- Feedback
- Lecture 9

**Servo problems:** make the output respond to command signals in the desired way

- Feedforward
- Lecture 10

# State Feedback: Problem Formulation




- Discrete-time process model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

- Linear feedback from all state variables

$$u(k) = -Lx(k)$$

- Disturbances modelled by nonzero initial state $x(0) = x_0$
- Goal: Control the state to the origin, using a reasonable control signal

# Closed-Loop System

The state equation

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

with the control law

$$u(k) = -Lx(k)$$

gives the closed-loop system

$$x(k+1) = (\Phi - \Gamma L)\, x(k)$$

Pole placement design: Choose $L$ to obtain the desired characteristic equation

$$\det(zI - \Phi + \Gamma L) = 0$$

(Matlab: `place` or `acker`)

# Example – Double Integrator

$$x(k+1) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(k)$$

Linear state-feedback controller

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

The closed-loop system becomes

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L) x(k) \\ &= \begin{pmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{pmatrix} x(k) \end{aligned}$$

Characteristic equation

$$z^2 + \left( \frac{l_1 h^2}{2} + l_2 h - 2 \right) z + \left( \frac{l_1 h^2}{2} - l_2 h + 1 \right) = 0$$

# Example Cont'd

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Assume desired characteristic equation $z^2 + a_1 z + a_2 = 0$.

Linear equations for $l_1$ and $l_2$

$$\frac{l_1 h^2}{2} + l_2 h - 2 = a_1 \qquad \frac{l_1 h^2}{2} - l_2 h + 1 = a_2$$

Solution:

$$l_1 = \frac{1}{h^2}(1 + a_1 + a_2)$$
$$l_2 = \frac{1}{2h}(3 + a_1 - a_2)$$

- $L$ depends on $h$

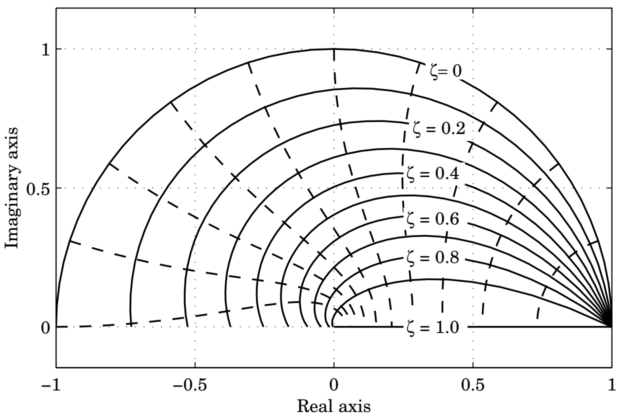
# Where to Place the Poles?

Recall from Lecture 7:

Loci of constant $\zeta$ (solid) and $\omega h$ (dashed) when

$$\frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

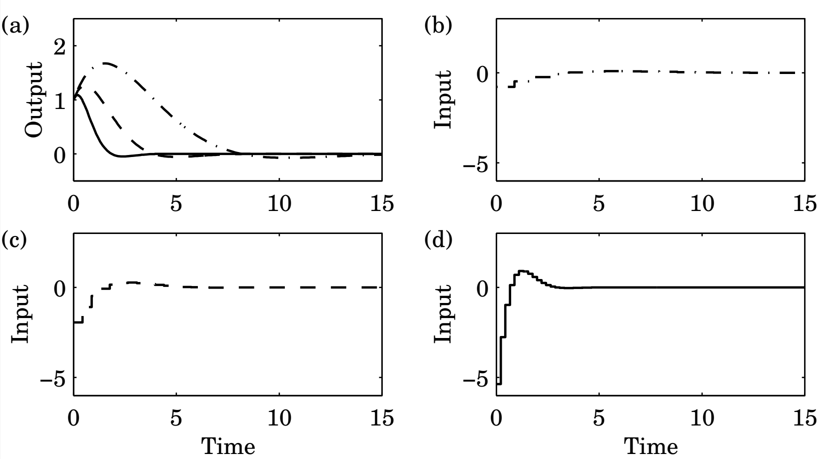
is sampled using ZOH ($z_i = e^{s_i h}$)



# Example – Choice of Design Parameters

Double integrator, $x_0^T = [\,1 \quad 1\,]$, $\omega h = 0.44$, $\zeta = 0.707$

(b) $\omega = 0.5$ (dash-dotted), (c) $\omega = 1$ (dashed), (d) $\omega = 2$ (solid)

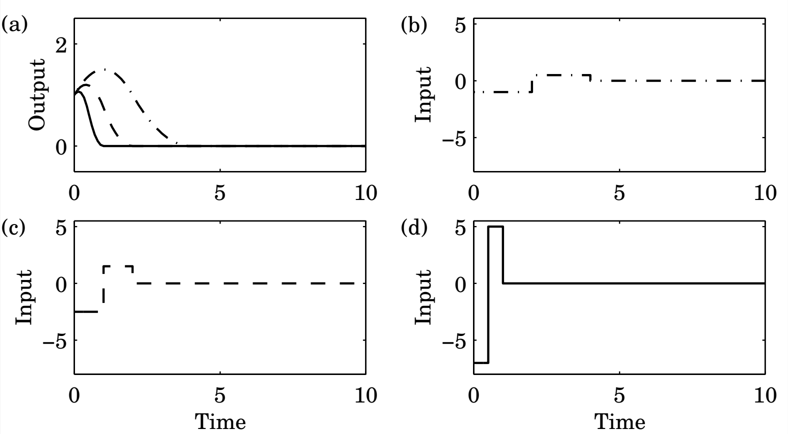


# Deadbeat Control — Only in Discrete Time

Choose $P(z) = z^n \Rightarrow\ h$ only remaining design parameter

Drives all states to zero in at most $n$ steps after an impulse disturbance in the states (can be very aggressive for small $h$!)

Example: Double integrator, $x_0^T = [\,1 \quad 1\,]$



# Controllability

The eigenvalues of $\Phi - \Gamma L$ can be placed arbitrarily if and only if the system is *controllable*, i.e. if the controllability matrix

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma & \dots & \Phi^{n-1}\Gamma \end{pmatrix}$$

has full rank.

In practice, moving some eigenvalues could require high gain and lead to bad controllers.

# State Feedback in Controllable Form

Convert the system to controllable canonical form:

$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

In this case, application of the state feedback

$$u = -l_1 x_1 - \cdots - l_n x_n$$

changes the coefficients $a_1, \dots, a_n$ to $a_1 + l_1, \dots, a_n + l_n$, so the characteristic polynomial changes to

$$z^n + (a_1 + l_1) z^{n-1} + \cdots + (a_{n-1} + l_{n-1}) z + a_n + l_n$$

Design method: Transform to controllable canonical form, apply state feedback, transform the controller back again – Ackermann's formula (see IFAC PB)

# State Feedback with Integral Action

Integral action can be introduced by augmenting the plant model with an extra state variable, $x_i$, that integrates the plant output:

$$x_i(k+1) = x_i(k) + y(k) = x_i(k) + Cx(k)$$

The augmented open-loop system becomes

$$\begin{pmatrix} x(k+1) \\ x_i(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ C & I \end{pmatrix} \begin{pmatrix} x(k) \\ x_i(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k)$$

We can then design a state feedback controller

$$u(k) = -\begin{pmatrix} L & L_i \end{pmatrix} \begin{pmatrix} x(k) \\ x_i(k) \end{pmatrix}$$

using the same techniques as before

(Integral action can also be introduced using a disturbance observer, as we will see later)

# Reconstruction

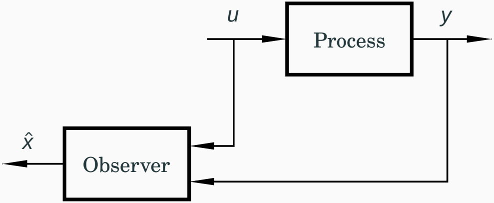
What to do if we cannot measure the full state vector or if we have noisy measurements?

# Reconstruction Through Direct Calculations

Basic idea: Reconstruct the state vector $x(k)$ through direct calculations using the output and input sequences $y(k)$, $y(k-1)$, $\dots$, $u(k-1)$, $u(k-2)$, $\dots$ together with the state-space model of the plant.

See IFAB PB p. 57 for details.

# Reconstruction Using An Observer




Simulated process model:

$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k)$$
$$\hat{y}(k) = C\hat{x}(k)$$

Introduce "feedback" from measured $y(k)$

$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + K\left(y(k) - C\hat{x}(k)\right)$$

# Reconstruction Using An Observer

Form the estimation error $\widetilde{x} = x - \hat{x}$

$$\begin{aligned} \widetilde{x}(k+1) &= \Phi\widetilde{x}(k) - KC\widetilde{x}(k) \\ &= [\Phi - KC]\widetilde{x}(k) \end{aligned}$$

- Any observer pole placement possible, provided the observability matrix

$$W_o = \begin{pmatrix} C \\ \vdots \\ C\Phi^{n-1} \end{pmatrix}$$

  has full rank
- Choose $K$ to get good convergence but not too much amplification of measurement noise

# Deadbeat Observer

A *deadbeat observer* is obtained if the observer gain $K$ is chosen so that the matrix $\Phi - KC$ has all eigenvalues zero.

The observer error goes to zero in finite time (in at most $n$ steps, where $n$ is the order of the system)

Noise sensitive (fast observer dynamics)

Equivalent to reconstruction using direct calculations.

# Observer for the Double Integrator

$$\Phi - KC = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 - k_1 & h \\ -k_2 & 1 \end{pmatrix}$$

Characteristic equation

$$z^2 - (2 - k_1)z + 1 - k_1 + k_2 h = 0$$

Desired characteristic equation:

$$z^2 + p_1 z + p_2 = 0$$

Gives:

$$\begin{aligned} 2 - k_1 &= -p_1 \\ 1 - k_1 + k_2 h &= p_2 \end{aligned}$$

# Observer for the Double Integrator cont'd

Solution:

$$k_1 = 2 + p_1$$

$$k_2 = (1 + p_1 + p_2)/h$$

Assume deadbeat observer ($p_1 = p_2 = 0$)

$$k_1 = 2$$

$$k_2 = 1/h$$

Resulting observer (assuming $u = 0$)

$$\hat{x}_1(k+1) = \hat{x}_1(k) + h\hat{x}_2(k) + 2\Big(y(k) - \hat{x}_1(k)\Big)$$

$$\hat{x}_2(k+1) = \hat{x}_2(k) + \frac{1}{h}\Big(y(k) - \hat{x}_1(k)\Big)$$

# An Alternative Observer

The observer presented so far has a one sample delay: $\hat{x}(k \mid k-1)$ depends only on measurements up to time $k-1$

Alternative observer with direct term:

$$\begin{aligned}\hat{x}(k \mid k) &= \Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1) \\ &\quad + K\Big[y(k) - C\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big)\Big] \\ &= (I - KC)\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big) + Ky(k)\end{aligned}$$

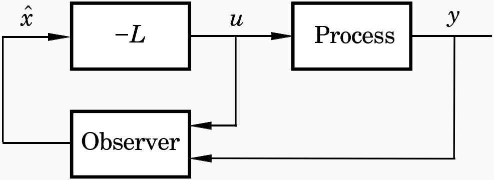
Reconstruction error:

$$\tilde{x}(k \mid k) = x(k) - \hat{x}(k \mid k) = (\Phi - KC\Phi)\tilde{x}(k-1 \mid k-1)$$

- $\Phi - KC\Phi$ can be given arbitrary eigenvalues if $\Phi - KC$ can
- $K$ may be chosen so that some of the states will be observed directly through $y \Rightarrow$ the order of the observer can be reduced
  - Reduced order observer or *Luenberger observer*

# Output Feedback

State feedback from observed state:




Controller:

$$
\begin{aligned}
\hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K\left(y(k) - C\hat{x}(k)\right) \\
u(k) &= -L\hat{x}(k)
\end{aligned}
$$

Controller transfer function (from $y$ to $u$):

$$H_c(z) = -L(zI - \Phi + \Gamma L + KC)^{-1}K$$

# Analysis of the Closed-Loop System

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
\widetilde{x}(k+1) &= (\Phi - KC)\widetilde{x}(k) \\
u(k) &= -L\hat{x}(k) = -L(x(k) - \widetilde{x}(k))
\end{aligned}
$$

Eliminate $u(k)$

$$
\begin{pmatrix} x(k+1) \\ \widetilde{x}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ \widetilde{x}(k) \end{pmatrix}
$$

Separation

Control poles: $A_c(z) = \det(zI - \Phi + \Gamma L)$

Observer poles: $A_o(z) = \det(zI - \Phi + KC)$

# Disturbance Estimation

How to handle disturbances that can not be modeled as impulse disturbances in the process state?

Assume that the process is described by

$$\frac{dx}{dt} = Ax + Bu + v$$
$$y = Cx$$

where $v$ is a disturbance modeled as

$$\frac{dw}{dt} = A_w w$$
$$v = C_w w$$

Since disturbances typically have most of their energy at low frequencies, $A_w$ often has eigenvalues in the origin (constant disturbance) or on the imaginary axis (sinusoidal disturbance)

# Disturbance Estimation

Augment the state vector: $\begin{pmatrix} x \\ w \end{pmatrix}$

Gives the augmented system

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A & C_w \\ 0 & A_w \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

Sample this using ZOH:

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix}\begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k)$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x(k) \\ w(k) \end{pmatrix}$$

# Augmented Observer and State Feedback

Augmented observer:

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix} \begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} K \\ K_w \end{pmatrix} \epsilon(k)$$

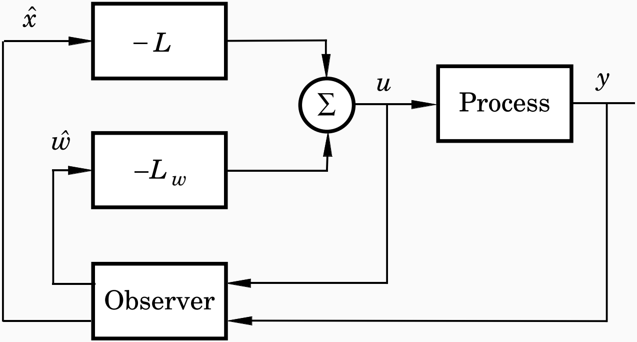
with $\epsilon(k) = y(k) - C\hat{x}(k)$

Augmented state feedback control law:

$$u(k) = -L\hat{x}(k) - L_w\hat{w}(k)$$

If possible, select $L_w$ such that $\Phi_{xw} - \Gamma L_w = 0$

# Disturbance Estimation: Block Diagram



# Disturbance Estimation: Closed-Loop System

The closed-loop system can be written

$$
\begin{aligned}
x(k+1) &= (\Phi - \Gamma L)x(k) + (\Phi_{xw} - \Gamma L_w)w + \Gamma L\widetilde{x}(k) + \Gamma L_w\widetilde{w} \\
w(k+1) &= \Phi_w w(k) \\
\widetilde{x}(k+1) &= (\Phi - KC)\widetilde{x}(k) + \Phi_{xw}\widetilde{w}(k) \\
\widetilde{w}(k+1) &= \Phi_w\widetilde{w}(k) - K_w C\widetilde{x}(k)
\end{aligned}
$$

- $L$ ensures that $x$ goes to zero at the desired rate after a disturbance.
- The gain $L_w$ reduces the effect of the disturbance $v$ on the system by feedforward from the estimated disturbance $\hat{w}$.
- $K$ and $K_w$ influence the rate at which the estimation errors go to zero.

# Special Case: Constant Input Disturbance

Assume constant disturbance acting on the plant input:

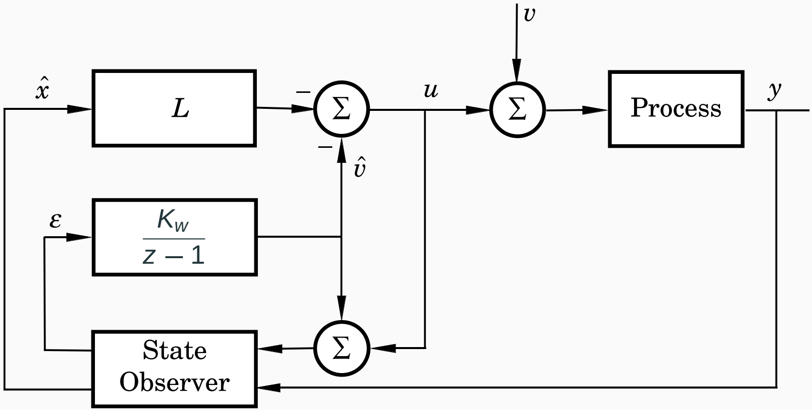
- $v = w$
- $\Phi_w = 1$
- $\Phi_{xw} = \Gamma$

If we choose $L_w = 1$ we will have perfect cancellation of the load disturbance

New controller + estimator

$$
\begin{aligned}
u(k) &= -L\hat{x}(k) - \hat{v}(k) \\
\hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma\Big(\hat{v}(k) + u(k)\Big) + K\epsilon(k) \\
\hat{v}(k+1) &= \hat{v}(k) + K_w\epsilon(k) \qquad \text{(integrator)} \\
\epsilon(k) &= y(k) - C\hat{x}(k)
\end{aligned}
$$

# Special Case: Block Diagram



# Example – Design

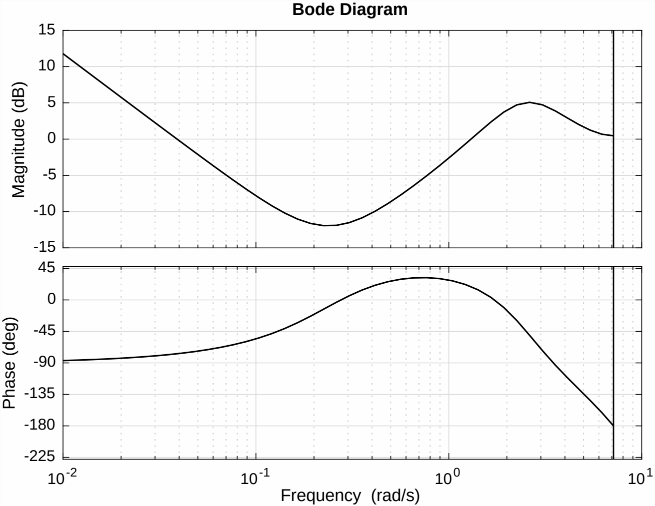
- Control of double integrator

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

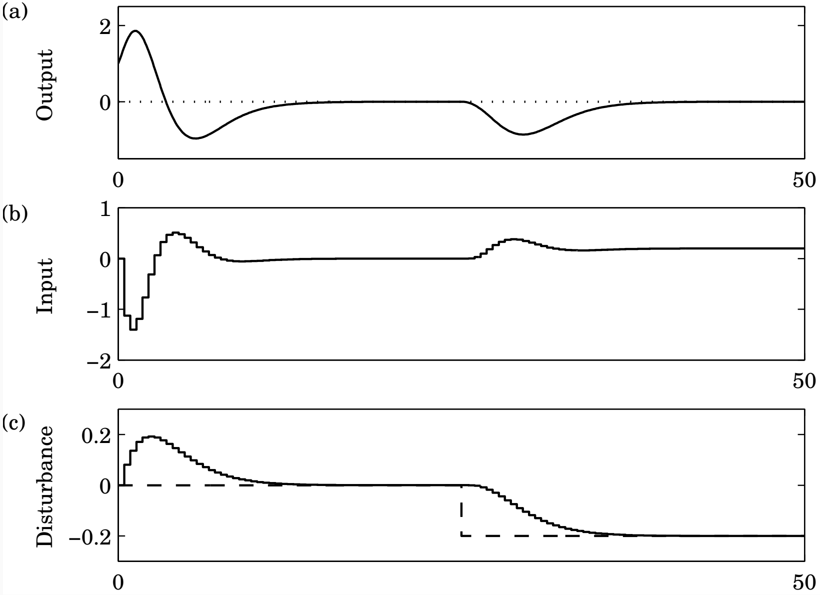
- Sample with $h = 0.44$
- Discrete state feedback designed based on continuous-time specification $\omega = 1$, $\zeta = 0.7$
    - Gives $L = [0.73 \;\; 1.21]$
- Extended observer assuming constant input disturbance to obtain integral action; all three poles placed in $z = 0.75$.

# Example – Design

Bode diagram of resulting controller:



# Example – Simulation



# Optimization-Based Design

Pole placement design used in this course:

- $L$ and $K$ derived through pole placement

In the course Multivariable Control (Flervariabel Reglering),
$L$ and $K$ are instead derived through optimization

- LQ (Linear Quadratic) and LQG (Linear Quadratic Gaussian) control
- Short overview in Chapter 11 of IFAC PB
- Not part of this course

# Example in Matlab

```
>> A = [0 1; 0 0];
>> B = [0; 1];
>> h = 0.44;
>> [Phi,Gamma] = c2d(A,B,h)
>> Hp = ss(Phi,Gamma,C,0,h);
>> % Desired poles in continuous time
>> omega = 1; zeta = 0.7;
>> pc = roots([1 2*zeta*omega omega^2])
>> % Corresponding desired discrete-time poles
>> pd = exp(pc*h)
>> % Design state feedback
>> L = place(Phi,Gamma,pd)
>> Le = [L 1];
>> % Design augmented observer
>> Phie = [Phi Gamma; zeros(1,2) 1];
>> Ce = [C 0];
>> Ke = acker(Phie',Ce',[0.75 0.75 0.75])'
>> % Form controller
>> Hc = ss(Phie-Gammae*Le-Ke*Ce,Ke,Le,0,h);
>> bode(Hc)
```