

Lecture 9: State Feedback and Observers

[IFAC PB Ch 9]

- State Feedback
- Observers
- Disturbance Estimation

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Control Design

Many factors to consider, for example:

- Attenuation of load disturbances
- Reduction of the effect of measurement noise
- Command signal following
- Variations and uncertainties in process behavior

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Two Classes of Control Problems

Regulation problems: compromise between rejection of load disturbances and injection of measurement noise

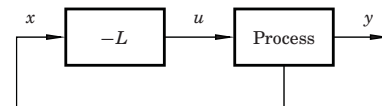
- Lecture 9

Servo problems: make the output respond to command signals in the desired way

- Lecture 10

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State Feedback: Problem Formulation



- Discrete-time process model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

- Linear feedback from all state variables

$$u(k) = -Lx(k)$$

- Disturbances modelled by nonzero initial state $x(0) = x_0$
- Goal: Control the state to the origin, using a reasonable control signal

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Closed-Loop System

The state equation

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

with the control law

$$u(k) = -Lx(k)$$

gives the closed-loop system

$$x(k+1) = (\Phi - \Gamma L)x(k)$$

Pole placement design: Choose L to obtain the desired characteristic equation

$$\det(zI - \Phi + \Gamma L) = 0$$

(Matlab: place or acker)

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Example – Double Integrator

$$x(k+1) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(k)$$

Linear state-feedback controller

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

The closed-loop system becomes

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)x(k) \\ &= \begin{pmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{pmatrix} x(k) \end{aligned}$$

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2 \right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1 \right) = 0$$

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Example Cont'd

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right)z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Assume desired characteristic equation $z^2 + a_1 z + a_2 = 0$.

Linear equations for l_1 and l_2

$$\frac{l_1 h^2}{2} + l_2 h - 2 = a_1 \quad \frac{l_1 h^2}{2} - l_2 h + 1 = a_2$$

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Example Cont'd

Solution:

$$l_1 = \frac{1}{h^2}(1 + a_1 + a_2)$$

$$l_2 = \frac{1}{2h}(3 + a_1 - a_2)$$

- L depends on h

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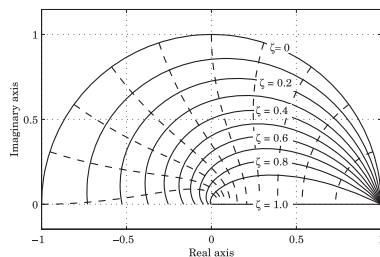
Where to Place the Poles?

Recall from Lecture 7:

Loci of constant ζ (solid) and ωh (dashed) when

$$\frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

is sampled:

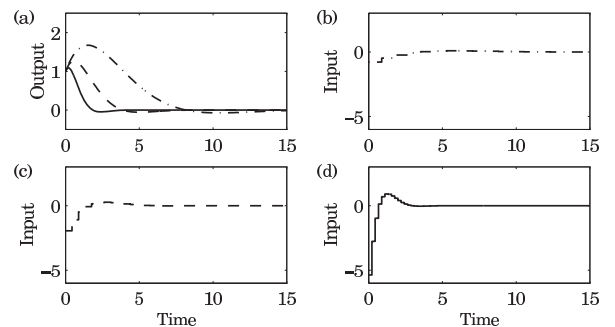


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Example – Choice of Design Parameters

Double integrator, $x_0^T = [1 \ 1]$, $\omega h = 0.44$, $\zeta = 0.707$

(b) $\omega = 0.5$ (dash-dotted), (c) $\omega = 1$ (dashed), (d) $\omega = 2$ (solid)



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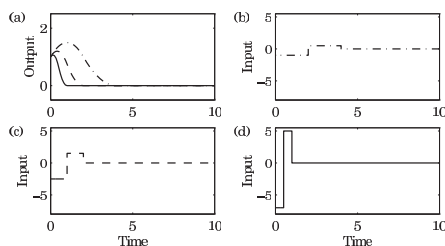
Deadbeat Control — Only in Discrete Time

Choose $P(z) = z^n \Rightarrow h$ only remaining design parameter

Drives all states to zero in at most n steps after an impulse disturbance in the states (can be very aggressive for small h !)

Finite time as opposed to infinite time in continuous time.

Example: Double integrator, $x_0^T = [1 \ 1]$



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Controllability

The eigenvalues of $\Phi - \Gamma L$ can be assigned to arbitrary positions if and only if the system is *controllable*, i.e. if the controllability matrix

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \dots & \Phi^{n-1}\Gamma \end{bmatrix}$$

has full rank.

In practice, moving some eigenvalues could require high gain and lead to bad controllers.

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State Feedback in Controllable Form

We previously derived the controllable canonical form

$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

In this case, application of the state feedback

$$u = -l_1 x_1 - \dots - l_n x_n$$

changes the coefficients a_1, \dots, a_n to $a_1 + l_1, \dots, a_n + l_n$, so the characteristic polynomial changes to

$$z^n + (a_1 + l_1)z^{n-1} + \dots + (a_{n-1} + l_{n-1})z + a_n + l_n$$

Design method: Transform to controllable canonical form, apply state feedback, transform the controller back again – Ackermann's formula (see IFAC PB)

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State Feedback with Integral Action

Integral action can be introduced by augmenting the plant model with an extra state variable, x_i , that integrates the plant output:

$$x_i(k+1) = x_i(k) + y(k) = x_i(k) + Cx(k)$$

The augmented open-loop system becomes

$$\begin{bmatrix} x(k+1) \\ x_i(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$$

We can then design a state feedback controller

$$u(k) = - \begin{bmatrix} L & L_i \end{bmatrix} \begin{bmatrix} x(k) \\ x_i(k) \end{bmatrix}$$

using the same techniques as before

(Integral action can also be introduced using a disturbance observer, as we will see later)

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Reconstruction

What should you do if you can not measure the full state vector or if you have noisy measurements?

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Reconstruction Through Direct Calculations

Basic idea: Reconstruct the state vector through direct calculations using the input and output sequences $y(k), y(k-1), \dots, u(k), u(k-l), \dots$ together with the state-space model of the plant.

Explained in detail in IFAC PB pg 61–62

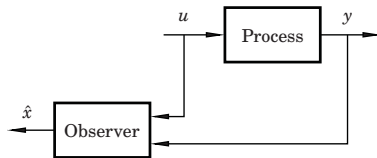
Make sure that you understand it (a lot of notation but not difficult!)

Often sensitive to disturbances.

A better alternative is to use the model information explicitly.

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Reconstruction Using An Observer



Simulated process model:

$$\begin{aligned} \hat{x}(k+1) &= \Phi \hat{x}(k) + \Gamma u(k) \\ \hat{y}(k) &= C \hat{x}(k) \end{aligned}$$

Introduce "feedback" from measured $y(k)$

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - C \hat{x}(k))$$

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Form the estimation error $\tilde{x} = x - \hat{x}$

$$\begin{aligned} \tilde{x}(k+1) &= \Phi \tilde{x}(k) - K C \tilde{x}(k) \\ &= [\Phi - K C] \tilde{x}(k) \end{aligned}$$

- Any observer poles possible, provided the observability matrix

$$W_o = \begin{bmatrix} C \\ \vdots \\ C \Phi^{n-1} \end{bmatrix}$$

has full rank

- Choose K to get good convergence
- Trade-off against measurement noise amplification

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Deadbeat Observer

A *deadbeat observer* is obtained if the observer gain K is chosen so that the matrix $\Phi - KC$ has all eigenvalues zero.

The observer error goes to zero in finite time (in at most n steps, where n is the order of the system)

Noise sensitive (fast observer dynamics)

Equivalent to reconstruction using direct calculations.

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Observer for the Double Integrator

$$\Phi - KC = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 - k_1 & h \\ -k_2 & 1 \end{pmatrix}$$

Characteristic equation

$$z^2 - (2 - k_1)z + 1 - k_1 + k_2h = 0$$

Desired characteristic equation:

$$z^2 + p_1z + p_2 = 0$$

Gives:

$$\begin{aligned} 2 - k_1 &= -p_1 \\ 1 - k_1 + k_2h &= p_2 \end{aligned}$$

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Observer for the Double Integrator cont'd

Solution:

$$\begin{aligned} k_1 &= 2 + p_1 \\ k_2 &= (1 + p_1 + p_2)/h \end{aligned}$$

Assume deadbeat observer ($p_1 = p_2 = 0$)

$$\begin{aligned} k_1 &= 2 \\ k_2 &= 1/h \end{aligned}$$

Resulting observer

$$\begin{aligned} \hat{x}_1(k+1) &= \hat{x}_1(k) + h\hat{x}_2(k) + 2(y(k) - \hat{x}_1(k)) \\ \hat{x}_2(k+1) &= \hat{x}_2(k) + \frac{1}{h}(y(k) - \hat{x}_1(k)) \end{aligned}$$

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An Alternative Observer

The observer presented so far has a one sample delay: $\hat{x}(k | k-1)$ depends only on measurements up to time $k-1$

Alternative observer with direct term:

$$\begin{aligned} \hat{x}(k | k) &= \Phi\hat{x}(k-1 | k-1) + \Gamma u(k-1) \\ &+ K[y(k) - C(\Phi\hat{x}(k-1 | k-1) + \Gamma u(k-1))] \\ &= (I - KC)(\Phi\hat{x}(k-1 | k-1) + \Gamma u(k-1)) + Ky(k) \end{aligned}$$

Reconstruction error:

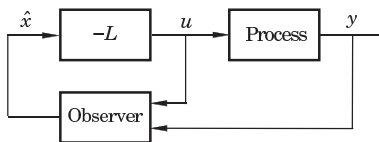
$$\tilde{x}(k | k) = x(k) - \hat{x}(k | k) = (\Phi - KC\Phi)\tilde{x}(k-1 | k-1)$$

- $\Phi - KC\Phi$ can be given arbitrary eigenvalues if $\Phi - KC$ can
- K may be chosen so that some of the states will be observed directly through $y \Rightarrow$ the order of the observer can be reduced
– Reduced order observer or *Luenberger observer*

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Output Feedback

State feedback from observed state:



Controller:

$$\begin{aligned} \hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K(y(k) - C\hat{x}(k)) \\ u(k) &= -L\hat{x}(k) \end{aligned}$$

Transfer function from y to u : $-L(zI - \Phi + \Gamma L + KC)^{-1}K$

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Analysis of the Closed-Loop System

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ \tilde{x}(k+1) &= (\Phi - KC)\tilde{x}(k) \\ u(k) &= -L\hat{x}(k) = -L(x(k) - \tilde{x}(k)) \end{aligned}$$

Eliminate $u(k)$

$$\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix}$$

Separation

$$\begin{aligned} \text{Control poles: } A_c(z) &= \det(zI - \Phi + \Gamma L) \\ \text{Observer poles: } A_o(z) &= \det(zI - \Phi + KC) \end{aligned}$$

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Disturbance Estimation

How to handle disturbances that can not be modeled as impulse disturbances in the process state?

Assume that the process is described by

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu + v \\ y &= Cx\end{aligned}$$

where v is a disturbance modeled as

$$\begin{aligned}\frac{dw}{dt} &= A_w w \\ v &= C_w w\end{aligned}$$

Since disturbances typically have most of their energy at low frequencies, the eigenvalues of A_w are typically in the origin or on the imaginary axis (sinusoidal disturbance)

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Disturbance Estimation

Augment the state vector: $\begin{pmatrix} x \\ w \end{pmatrix}$

Gives the augmented system

$$\begin{aligned}\frac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix} &= \begin{pmatrix} A & C_w \\ 0 & A_w \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}\end{aligned}$$

which is sampled into

$$\begin{aligned}\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} &= \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix}\end{aligned}$$

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Augmented Observer and State Feedback

Augmented observer:

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix} \begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} K \\ K_w \end{pmatrix} \epsilon(k)$$

with $\epsilon(k) = y(k) - C\hat{x}(k)$

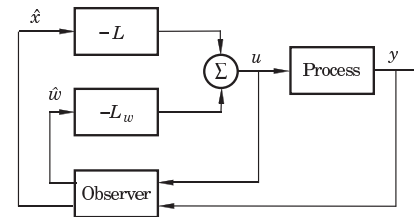
Augmented state feedback control law:

$$u(k) = -L\hat{x}(k) - L_w\hat{w}(k)$$

If possible, select L_w such that $\Phi_{xw} - \Gamma L_w = 0$

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Disturbance Estimation: Block Diagram



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Disturbance Estimation: Closed-Loop System

The closed-loop system can be written

$$\begin{aligned}x(k+1) &= (\Phi - \Gamma L)x(k) + (\Phi_{xw} - \Gamma L_w)w(k) + \Gamma L\hat{x}(k) + \Gamma L_w\hat{w}(k) \\ w(k+1) &= \Phi_w w(k) \\ \hat{x}(k+1) &= (\Phi - KC)\hat{x}(k) + \Phi_{xw}\hat{w}(k) \\ \hat{w}(k+1) &= \Phi_w\hat{w}(k) - K_w C\hat{x}(k)\end{aligned}$$

- L ensures that x goes to zero at the desired rate after a disturbance.
- The gain L_w reduces the effect of the disturbance v on the system by feedforward from the estimated disturbances \hat{w} .
- K and K_w influence the rate at which the estimation errors go to zero.

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Special Case: Constant Input Disturbance

Assume constant disturbance acting on the plant input:

- $v = w$
- $\Phi_w = 1$
- $\Phi_{xw} = \Gamma$

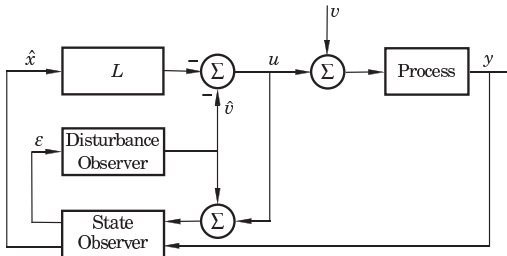
If we choose $L_w = 1$ we will have perfect cancellation of the load disturbance

New controller + estimator

$$\begin{aligned}u(k) &= -L\hat{x}(k) - \hat{v}(k) \\ \hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma(\hat{v}(k) + u(k)) + K\epsilon(k) \\ \hat{v}(k+1) &= \hat{v}(k) + K_w\epsilon(k) \\ \epsilon(k) &= y(k) - C\hat{x}(k)\end{aligned}$$

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Special Case: Block Diagram



The disturbance estimator is integrating the prediction error of the observer.

The overall controller will have integral action (see IFAC PB)

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Example – Design

- Control of double integrator

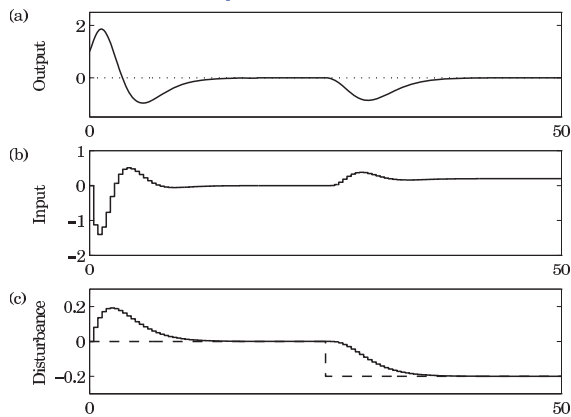
$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

- Sample with $h = 0.44$
- Discrete state feedback designed based on continuous-time specification $\omega = 1$, $\zeta = 0.7$
 - Gives $L = [0.73 \quad 1.21]$
- Extended observer assuming constant input disturbance to obtain integral action; all three poles placed in $z = 0.75$.

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Example – Simulation



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Optimization-Based Design

Pole-placement design:

- L and K derived through pole-placement

In the course Multivariable Control (Flervariabel Reglering), L and K are instead derived through optimization

- LQ (Linear Quadratic) and LQG (Linear Quadratic Gaussian) control
- Short overview in Ch 11 of IFAC PB
- Not part of this course

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Examples in Matlab

```
>> A = [0 1; 0 0];
>> B = [0; 1];
>> h = 0.44;
>> % Sampled system matrices
>> [Phi, Gamma] = c2d(A, B, h)

>> % Desired poles in continuous time
>> omega = 1; zeta = 0.7;
>> pc = roots([1 2*zeta*omega omega^2])
>> % Corresponding desired discrete poles
>> pd = exp(pc*h)

>> % Design state feedback
>> L = place(Phi, Gamma, pd)

>> % Design augmented observer
>> Phie = [Phi Gamma; zeros(1,2) 1];
>> Ce = [C 0];
>> Ke = acker(Phie', Ce', [0.75 0.75 0.75])'
```

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