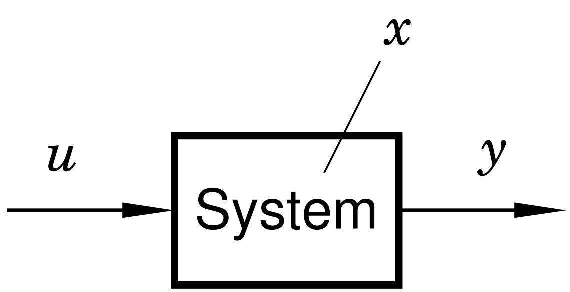
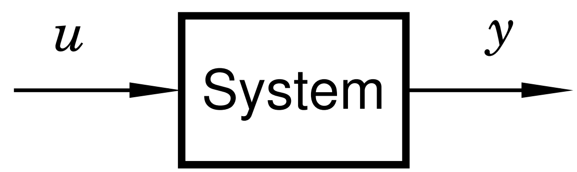
Lecture 7: Input-Output Models

[IFAC PB pg 23-35]

- Shift operators; the pulse transfer operator
- Z-transform; the pulse transfer function
- System response
- Poles and zeros
- Transformations between system representations

Linear System Models

	State-space model	Input-output models	
			
		Differential/difference equation	Transfer operator/fcn
CT	$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t)$	$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y$ $= b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_n u$	$G(p)$ / $G(s)$
DT	$x(k+1) = \Phi x(k) + \Gamma u(k)$ $y(k) = Cx(k)$	$y(k) + a_1 y(k-1) + \cdots +$ $a_n y(k-n) = b_1 u(k-1)$ $+ \cdots + b_n u(k-n)$	$H(q)$ / $H(z)$

More I-O models: pulse response, step response, frequency function, ...

Shift Operators

Operators on time series

Assume $h = 1$ (the *sampling-time convention*)

Time series are doubly infinite sequences:

- $f(k) : k = \ldots - 1, 0, 1, \ldots$

Forward shift operator:

- denoted q
- $qf(k) = f(k + 1)$
- $q^n f(k) = f(k + n)$

Shift Operators

Backward shift operator:

- denoted q^{-1}
- $q^{-1}f(k) = f(k-1)$
- $q^{-n}f(k) = f(k-n)$

Pulse Transfer Operator

Rewrite the state-space model using the forward shift operator:

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k) + Du(k)$$

Eliminate $x(k)$:

$$\begin{aligned} x(k) &= (qI - \Phi)^{-1}\Gamma u(k) \\ y(k) &= Cx(k) + Du(k) = C(qI - \Phi)^{-1}\Gamma u(k) + Du(k) \\ &= \left[C(qI - \Phi)^{-1}\Gamma + D\right] u(k) = H(q)u(k) \end{aligned}$$

$H(q)$ is the *pulse transfer operator* of the system

Describes how the input and output are related.

Poles and Zeros (SISO case)

The pulse transfer function can be written as a rational function

$$H(q) = \frac{B(q)}{A(q)}$$

$\deg A = n$ = the number of states
$\deg B = n_b \leq n$

$A(q)$ is the characteristic polynomial of Φ, i.e.

$$A(q) = \det(qI - \Phi)$$

The *poles* of the system are given by $A(q) = 0$

The *zeros* of the system are given by $B(q) = 0$

Disk Drive Example

Recall the double integrator from the previous lecture:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Sample with $h = 1$:

$$\Phi = e^{Ah} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\Gamma = \int_0^h e^{As} B \, ds = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Disk Drive Example cont.

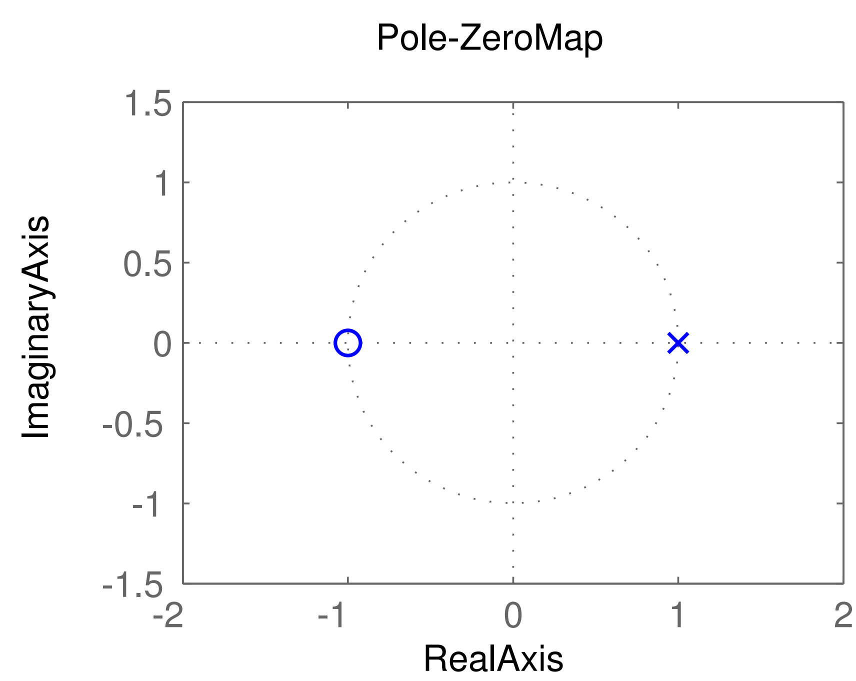
Pulse transfer operator:

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

$$= [1 \quad 0] \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \frac{[1 \quad 0]}{(q-1)^2} \begin{bmatrix} q-1 & 1 \\ 0 & q-1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \frac{0.5(q+1)}{(q-1)^2}$$

Two poles in 1, one zero in -1.



From Pulse Transfer Operator to Difference Equation

$$y(k) = H(q)u(k)$$
$$A(q)y(k) = B(q)u(k)$$

$$(q^n + a_1 q^{n-1} + \cdots + a_n)y(k) = (b_0 q^{n_b} + \cdots + b_{n_b})u(k)$$

which means

$$\begin{aligned} y(k+n) + a_1 y(k+n-1) + \cdots + a_n y(k) \\ = b_0 u(k+n_b) + \cdots + b_{n_b} u(k) \end{aligned}$$

Difference Equation with Backward Shift

$$\begin{aligned} y(k+n) + a_1 y(k+n-1) + \cdots + a_n y(k) \\ = b_0 u(k+n_b) + \cdots + b_{n_b} u(k) \end{aligned}$$

can be written as

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) \\ = b_0 u(k-d) + \cdots + b_{n_b} u(k-d-n_b) \end{aligned}$$

where $d = n - n_b$ is the *pole excess* of the system.

The *reciprocal polynomial*

$$A^*(q) = 1 + a_1 q + \cdots + a_n q^n = q^n A(q^{-1})$$

is obtained from the polynomial A by reversing the order of the coefficients.

Now the system can instead be written as

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$$

Difference Equation Example

Using forward shift

$$y(k+2) + 2y(k+1) + 3y(k) = 2u(k+1) + u(k)$$

can be written

$$(q^2 + 2q + 3)y(k) = (2q + 1)u(k)$$

Hence,

$$A(q) = q^2 + 2q + 3$$
$$B(q) = 2q + 1$$

Difference Equation Example, continued

Using backward shift

$$y(k) + 2y(k-1) + 3y(k-2) = 2u(k-1) + u(k-2)$$

can be written ($d = 1$)

$$(1 + 2q^{-1} + 3q^{-2})y(k) = (2 + q^{-1})u(k-1)$$

Hence,

$$A^*(q^{-1}) = 1 + 2q^{-1} + 3q^{-2}$$
$$B^*(q^{-1}) = 2 + q^{-1}$$

Z-transform

The discrete-time counterpart to the Laplace transform

Defined on semi-infinite time series $f(k) : k = 0, 1, \ldots$

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

z is a complex variable

Example — Discrete-Time Step Signal

Let $y(k) = 1$ for $k \geq 0$. Then

$$Y(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{z}{z-1}, \qquad |z| > 1$$

Application of the following result for power series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1$$

Example — Discrete-Time Ramp Signal

Let $y(k) = k$ for $k \geq 0$. Then

$$Y(z) = 0 + z^{-1} + 2z^{-2} + 3z^{-3} \cdots = \frac{z}{(z-1)^2}$$

Application of the following result for power series

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \text{ for } |x| < 1$$

Z-transform Table

Table 2 (pg 26) in IFAC PB (ignore the middle column!)

f	$\mathcal{L}f$	$\mathcal{Z}f$
$\delta(k)$ (pulse)	–	1
$1 \quad k \geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z-1}$
kh	$\frac{1}{s^2}$	$\frac{hz}{(z-1)^2}$
$\frac{1}{2}(kh)^2$	$\frac{1}{s^3}$	$\frac{h^2 z(z+1)}{2(z-1)^3}$
$e^{-kh/T}$	$\frac{T}{1+sT}$	$\frac{z}{z-e^{-h/T}}$
$1-e^{-kh/T}$	$\frac{1}{s(1+sT)}$	$\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$
$\sin \omega kh$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega h}{z^2 - 2z\cos \omega h + 1}$

Some Properties of the Z-transform

$$\mathcal{Z}(\alpha f + \beta g) = \alpha F(z) + \beta G(z)$$

$$\mathcal{Z}(q^{-n} f) = z^{-n} F(z)$$

$$\mathcal{Z}(q f) = z(F(z) - f(0))$$

$$\mathcal{Z}(f * g) = \mathcal{Z}\left\{ \sum_{j=0}^{k} f(j) g(k-j) \right\} = F(z) G(z)$$

From State Space to Pulse Transfer Function

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ \quad y(k) = Cx(k) + Du(k) \end{cases}$$

$$\begin{cases} z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z) \\ \quad Y(z) = CX(z) + DU(z) \end{cases}$$

$$Y(z) = C(zI - \Phi)^{-1} z\, x(0) + [C(zI - \Phi)^{-1}\Gamma + D]U(z)$$

The rational function $H(z) = C(zI - \Phi)^{-1}\Gamma + D$ is called the *pulse transfer function* from u to y.

It is the Z-transform of the pulse response.

$H(q)$ vs $H(z)$

The pulse transfer operator $H(q)$ and the pulse transfer function $H(z)$ are the same rational functions

They have the same poles and zeros

$H(q)$ is used in the time domain (q = shift operator)

$H(z)$ is used in the Z-domain (z = complex variable)

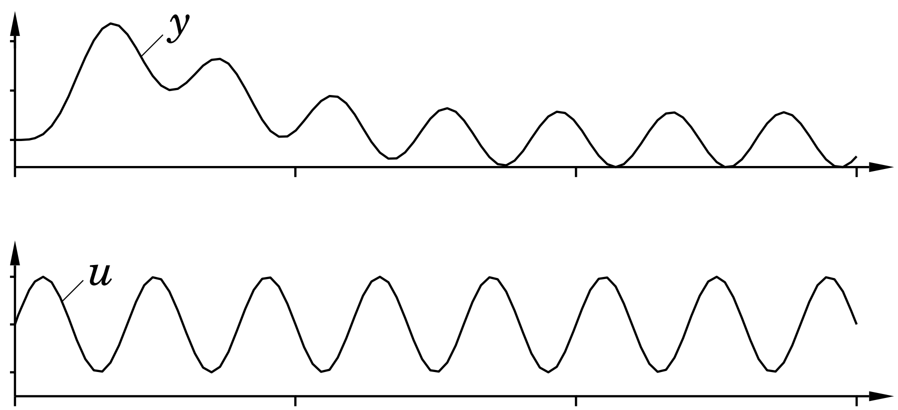
Calculating System Response Using the Z-transform

1. Find the pulse transfer function $H(z) = C(zI - \Phi)^{-1}\Gamma + D$
2. Compute the Z-transform of the input: $U(z) = \mathcal{Z}\{u(k)\}$
3. Compute the Z-transform of the output:

$$Y(z) = C(zI - \Phi)^{-1} z\, x(0) + H(z)U(z)$$

4. Apply the inverse Z-transform (table) to find the output: $y(k) = \mathcal{Z}^{-1}\{Y(z)\}$

Frequency Response in Continuous Time

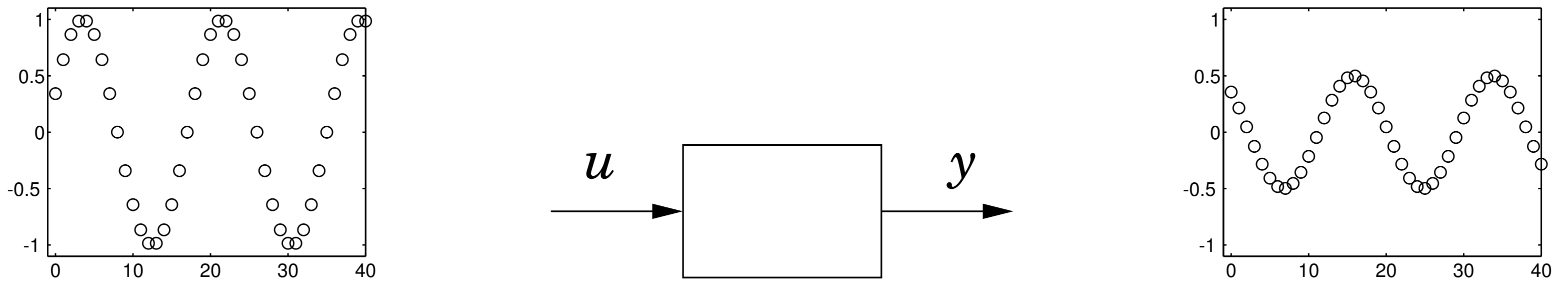


Given a stable system $G(s)$, the input $u(t) = \sin \omega t$ will, after a transient, give the output

$$y(t) = |G(i\omega)| \sin \Big(\omega t + \arg G(i\omega) \Big)$$

- The amplitude and phase shift for different frequencies are given by the value of $G(s)$ along the imaginary axes, i.e. $G(i\omega)$
- Plotted in Bode and Nyquist diagrams.

Frequency Response in Discrete Time

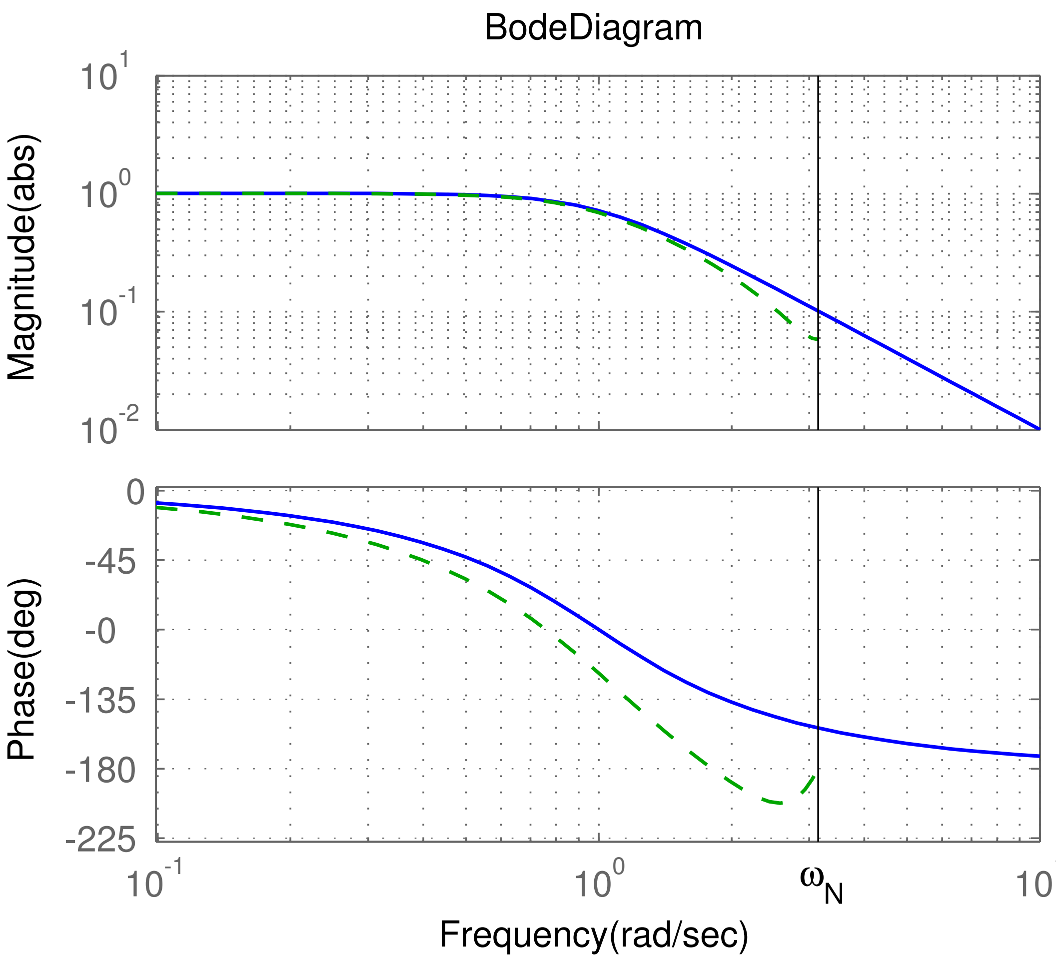


Given a stable system $H(z)$, the input $u(k) = \sin(\omega k)$ will, after a transient, give the output

$$y(k) = |H(e^{i\omega})| \sin\Big(\omega k + \arg H(e^{i\omega})\Big)$$

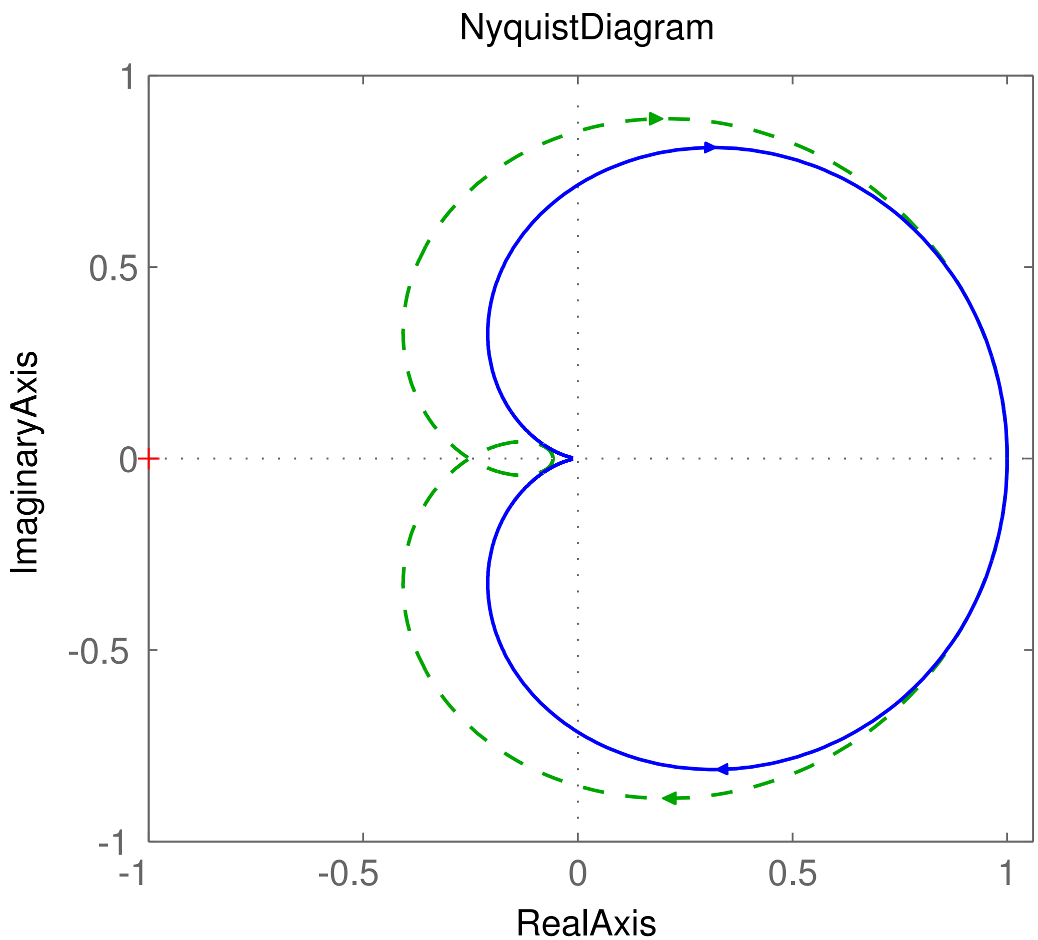
- $G(s)$ and the imaginary axis are replaced by $H(z)$ and the unit circle.
- Only describes what happens at the sampling instants
- The inter-sample behavior is not studied in this course

Bode diagram for continuous transfer function $1/(s^2+1.4s+1)$ (solid) and for ZOH-sampled counterpart (dashed, plotted for $\omega h \in [0, \pi]$)



For slow signals, the hold circuit is approximately a $h/2$ delay.
For fast signals, the hold circuit destroys the sinusoidal shape.

Nyquist diagram for cont. transfer function $1/(s^2 + 1.4s + 1)$ (solid) and for ZOH-sampled counterpart (dashed, plotted for $\omega h \in [0, \pi]$)



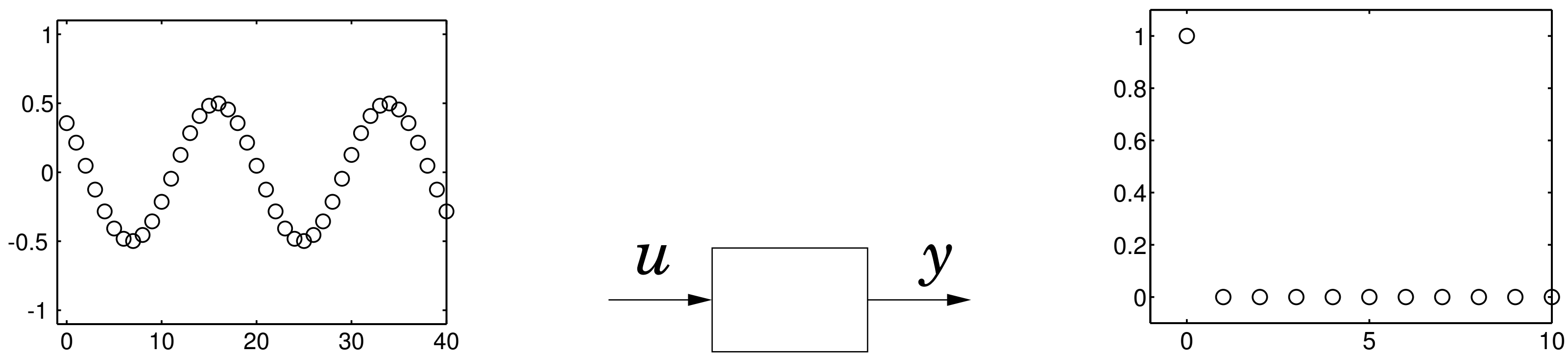
Interpretation of Poles and Zeros

Poles:

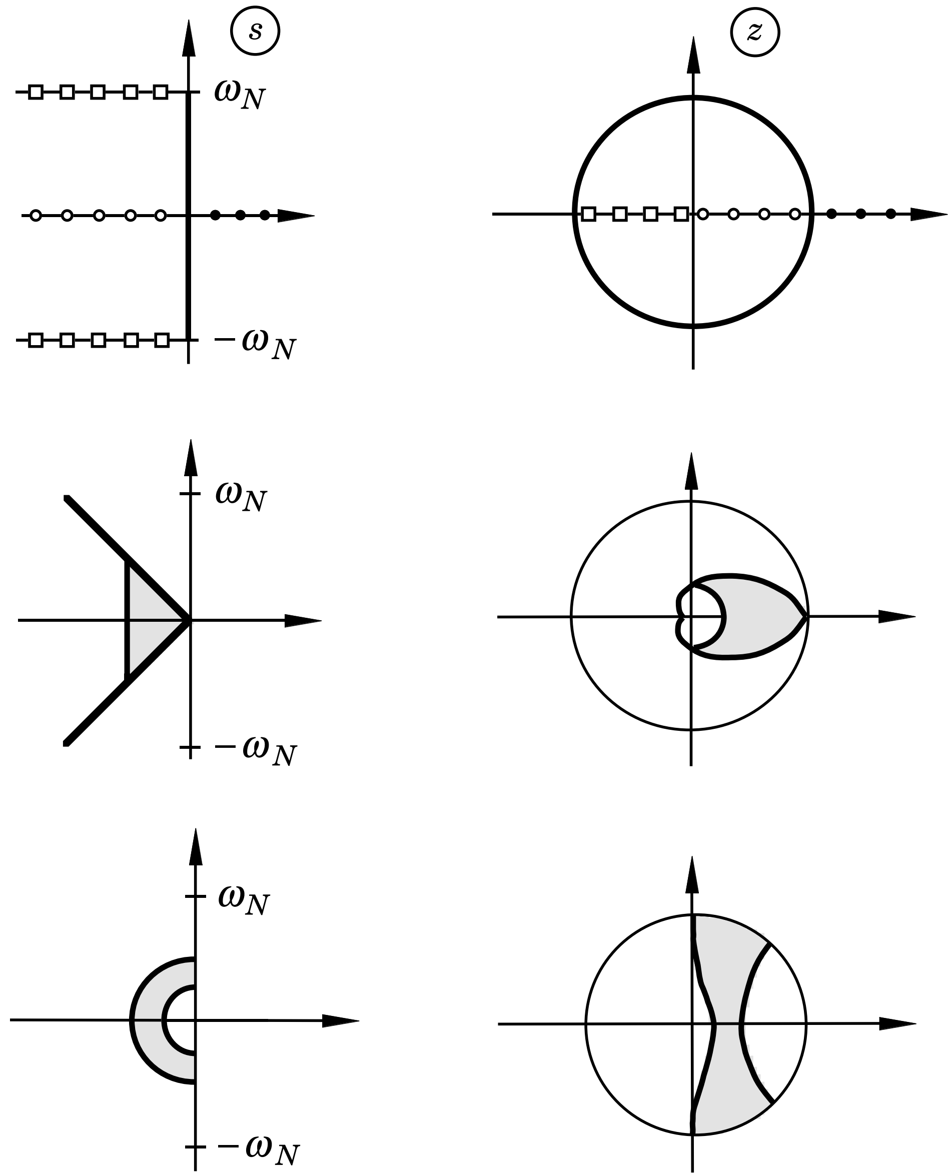
- A pole $z = a$ is associated with the time function $f(k) = a^k$

Zeros:

- A zero $z = a$ implies that the transmission of the input $u(k) = a^k$ is blocked by the system
- Related to how inputs and outputs are coupled to the states

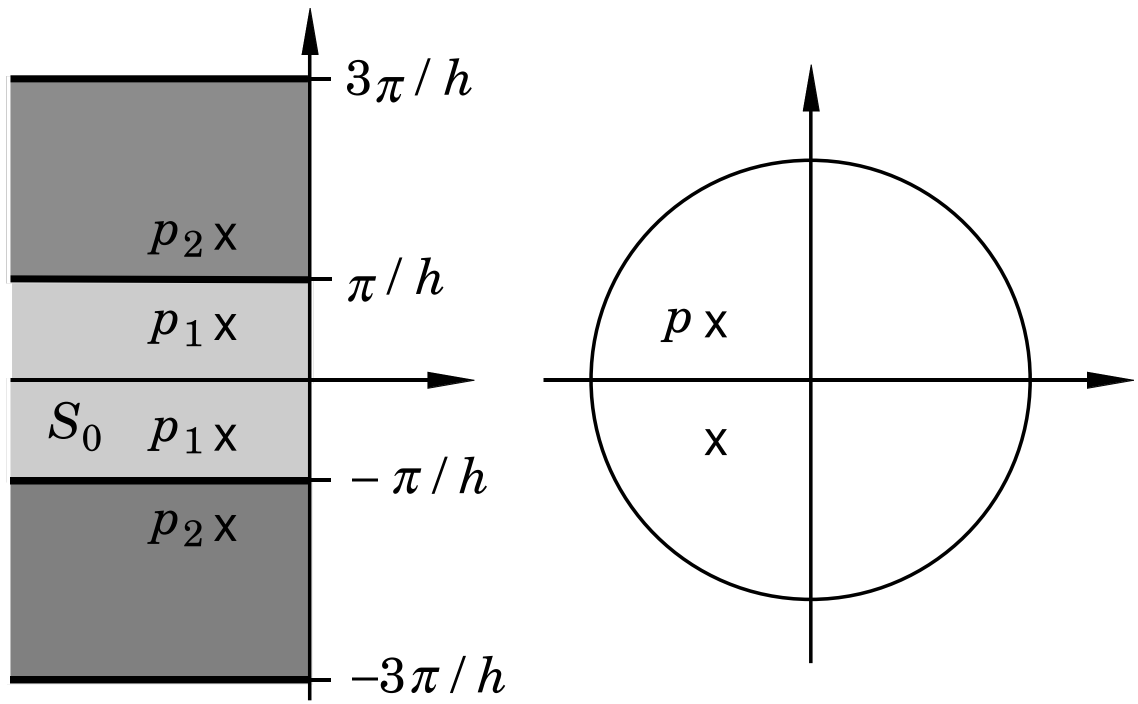


Transformation of Poles via Sampling: $z_i = e^{s_i h}$



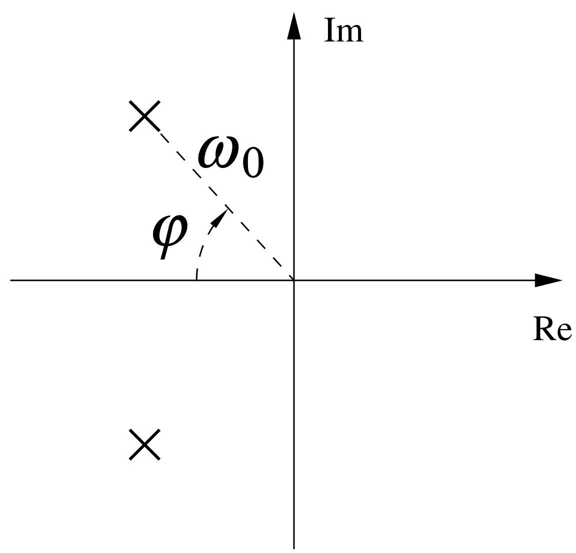
New Evidence of the Alias Problem

Several points in the s-plane are mapped into the same point in the z-plane. The map is not bijective



Sampling of a Second Order System

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad |\zeta| < 1$$



- Larger $\omega_0 \Rightarrow$ faster system response
- Smaller $\varphi \Rightarrow$ larger damping (relative damping $\zeta = \cos\varphi$). (Common design choice: $\zeta = \cos 45^\circ \approx 0.7$)

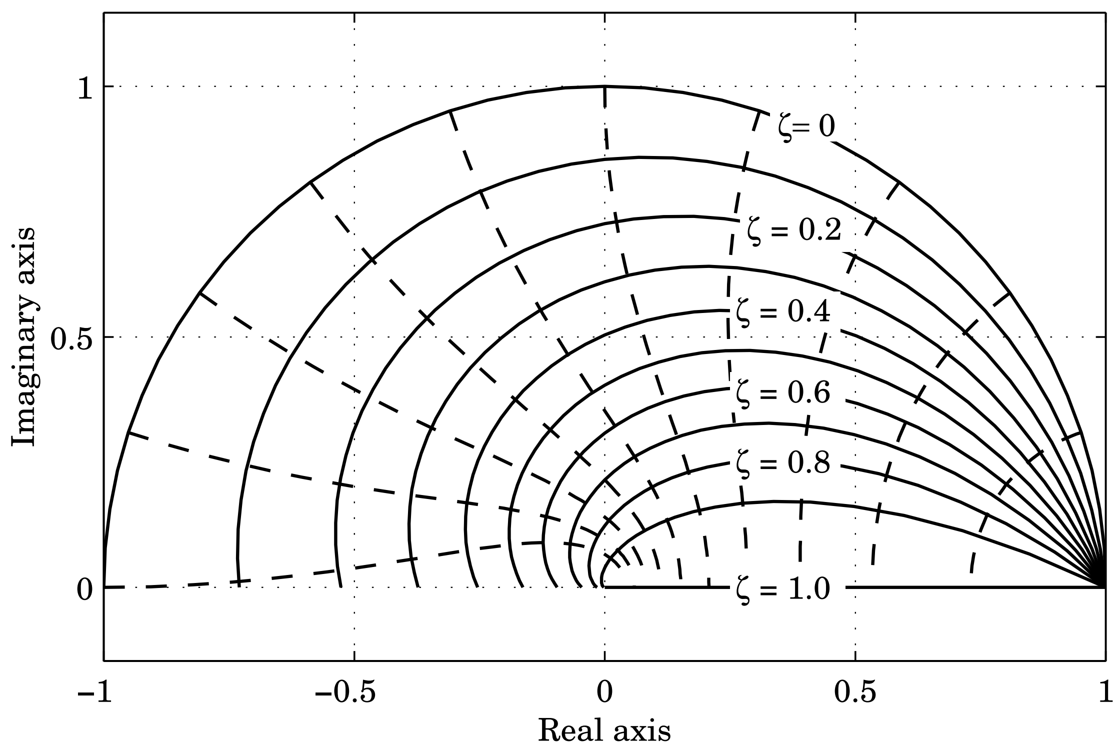
Sampling of a Second Order System

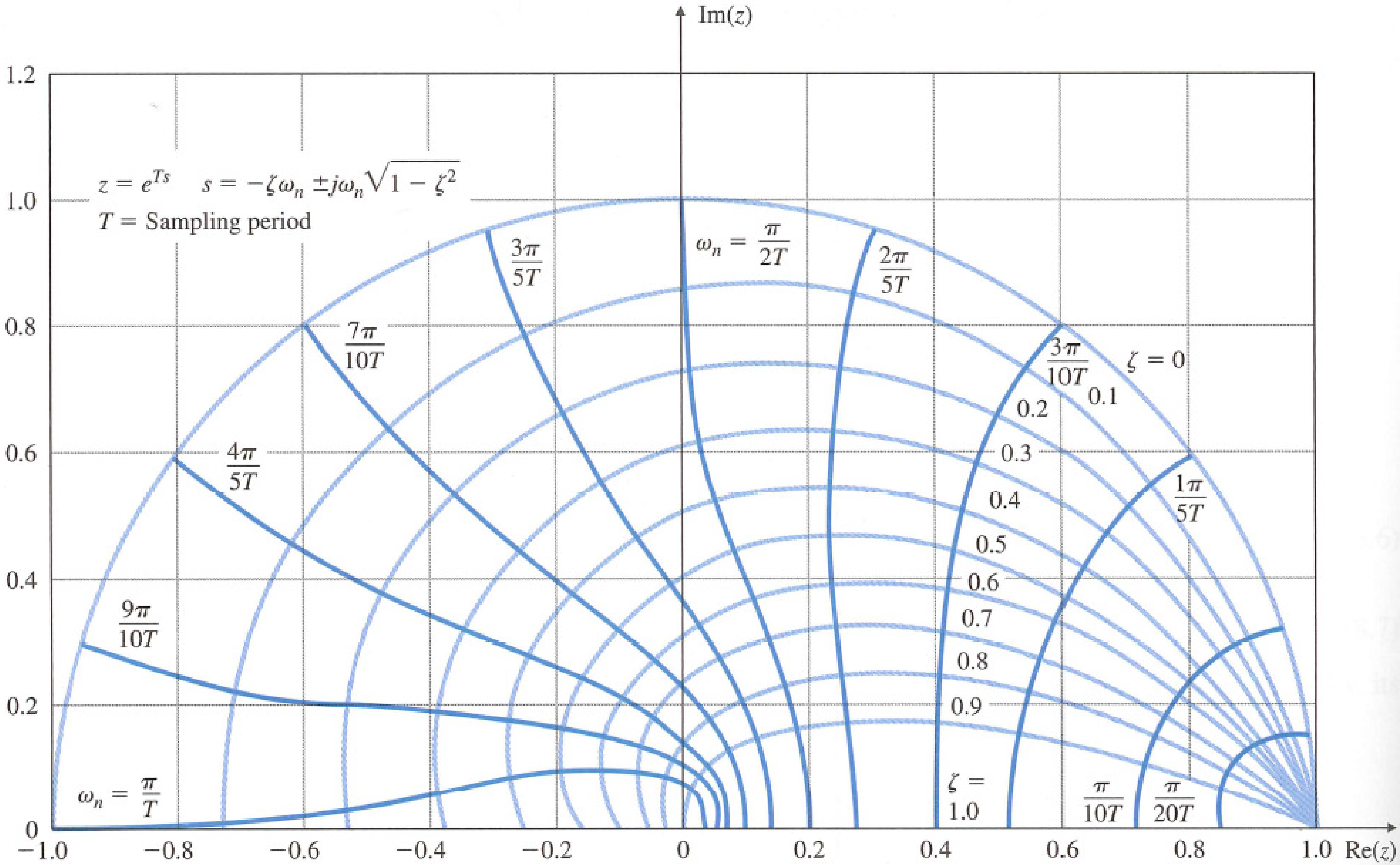
The poles of the sampled system are given by

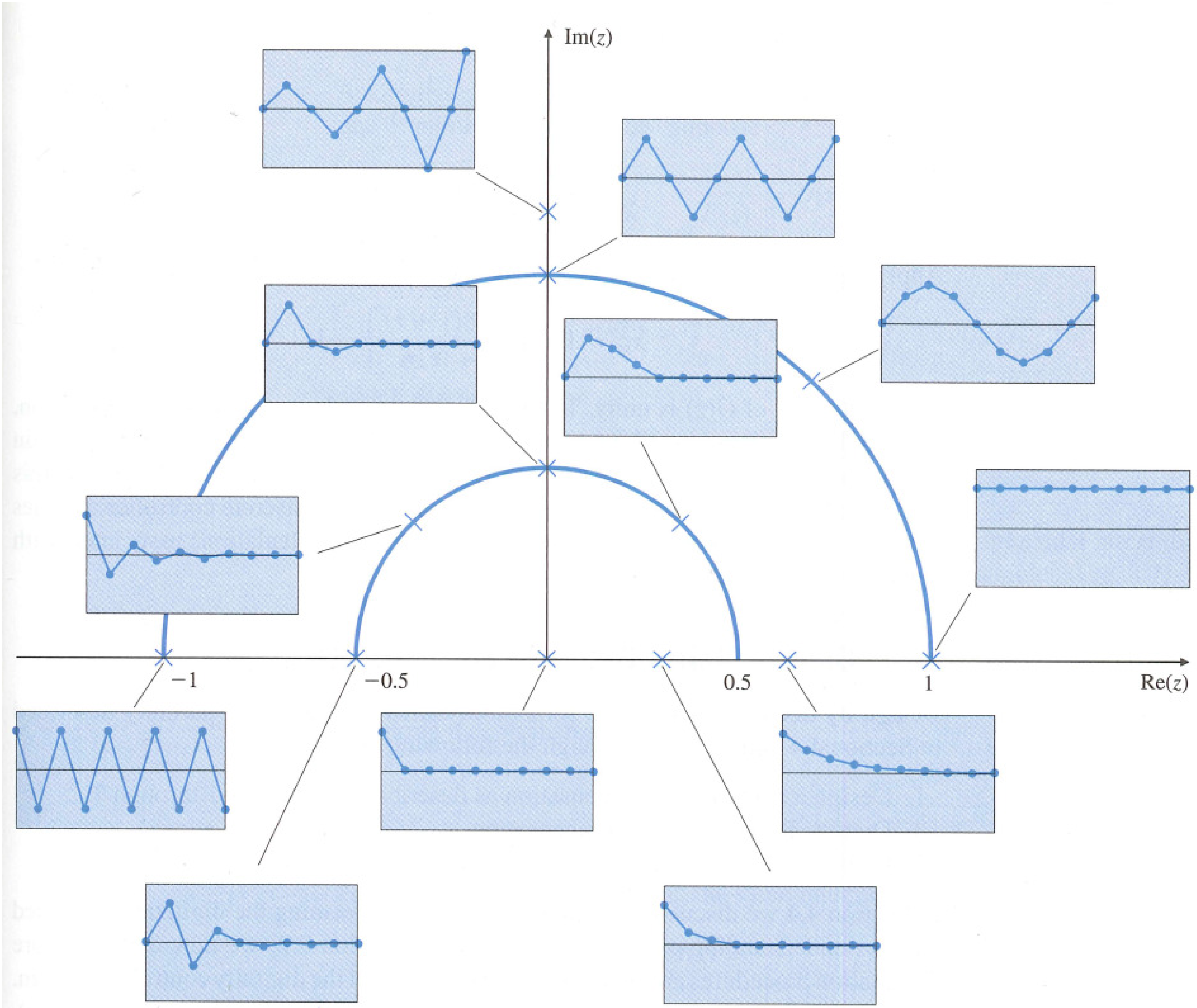
$$z^2 + a_1 z + a_2 = 0$$

where

$$a_1 = -2e^{-\zeta\omega_0 h}\cos\left(\sqrt{1-\zeta^2}\,\omega_0 h\right)$$
$$a_2 = e^{-2\zeta\omega_0 h}$$




Im(z)
Re(z)
$z = e^{Ts}$ $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
T = Sampling period
$\omega_n = \frac{\pi}{2T}$
$\frac{3\pi}{5T}$
$\frac{2\pi}{5T}$
$\frac{7\pi}{10T}$
$\frac{3\pi}{10T}$
$\zeta = 0$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\frac{4\pi}{5T}$
$\frac{1\pi}{5T}$
$\frac{9\pi}{10T}$
$\omega_n = \frac{\pi}{T}$
$\zeta =$ 1.0
$\frac{\pi}{10T}$
$\frac{\pi}{20T}$
1.2
1.0
0.8
0.6
0.4
0.2
0
−1.0
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1.0


Im(z)
Re(z)
−1
−0.5
0.5
1

Transformation of Zeros via Sampling

- More complicated than for poles
- Extra zeros may appear in the sampled system
- There can be zeros outside the unit circle (non-minimum phase) even if the continuous system has all the zeros in the left half plane
- For short sampling periods

$$z_i \approx e^{s_i h}$$

Calculation of $H(z)$ Given $G(s)$

Three approaches:

1. Make state-space realization of $G(s)$. Sample to get Φ and Γ. Then $H(z) = C(zI - \Phi)^{-1}\Gamma + D$.

2. Directly using the formula

$$
\begin{aligned}
H_{zoh}(z) &= \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} \, ds \\
&= \sum_{s=s_i} \frac{1}{z - e^{sh}} \operatorname{Res} \left\{ \frac{e^{sh} - 1}{s} G(s) \right\}
\end{aligned}
$$

- s_i are the poles of $G(s)$ and Res denotes the residue.
- outside the scope of the course

3. Use Table 3 (pg 28) in IFAC PB

$G(s)$	$H(z) = \dfrac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$	
$\dfrac{1}{s}$	$\dfrac{h}{z-1}$	
$\dfrac{1}{s^2}$	$\dfrac{h^2(z+1)}{2(z-1)^2}$	
e^{-sh}	z^{-1}	
$\dfrac{a}{s+a}$	$\dfrac{1-\exp(-ah)}{z-\exp(-ah)}$	
$\dfrac{a}{s(s+a)}$	$b_1 = \dfrac{1}{a}(ah - 1 + e^{-ah})$ $a_1 = -(1 + e^{-ah})$	$b_2 = \dfrac{1}{a}(1 - e^{-ah} - ahe^{-ah})$ $a_2 = e^{-ah}$
$\dfrac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$

Calculation of $H(z)$ Given $G(s)$

Example: For $G(s) = e^{-\tau s}/s^2$, the previous lecture gave

$$x(kh+h) = \Phi x(kh) + \Gamma_1 u(kh-h) + \Gamma_0 u(kh)$$

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} \tau\left(h - \dfrac{\tau}{2}\right) \\ \tau \end{pmatrix} \qquad \Gamma_0 = \begin{pmatrix} \dfrac{(h-\tau)^2}{2} \\ h-\tau \end{pmatrix}$$

With $h = 1$ and $\tau = 0.5$, this gives

$$H(z) = C(zI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 z^{-1}) = \frac{0.125(z^2+6z+1)}{z(z^2-2z+1)}$$

Order: 3
Poles: 0, 1, and 1
Zeros: $-3 \pm \sqrt{8}$

Examples in Matlab

```
>> Phi = [0.5 -0.2; 0 0];
>> Gamma = [2; 1];
>> C = [1 0];
>> D = 0;
>> h = 1;
>> H = ss(Phi, Gamma, C, D, h);
>> zpk(H)

>> % From cont-time transfer function to discrete-time
>> % pulse transfer function
>> s = zpk('s');
>> G = 1/s^3;
>> H = c2d(G,h)

>> % Another way
>> G = tf([1],[1 3 2 0]);
>> G = ss(G);
>> H = c2d(G,h);
>> tf(H)
```