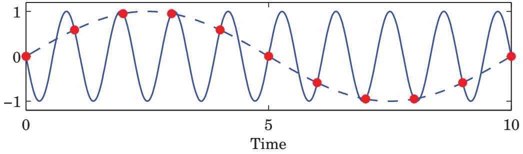
Lecture 11: Implementation Aspects

[IFAC PB Ch 12, RTCS Ch 11]

1. Sampling, aliasing, and choice of sampling interval
2. Computational delay
3. A-D and D-A quantization
4. Computer arithmetic
5. Controller realizations

Sampling and Aliasing

Recall this example from Lecture 6:



Aliasing

Sampling a signal with frequency ω creates new signal components with frequencies

$$\omega_{\text{sampled}} = n\omega_s \pm \omega$$

where $\omega_s = 2\pi/h$ is the sampling frequency and $n \in \mathbb{Z}$

Nyquist frequency:

$$\omega_N = \omega_s/2$$

The *fundamental alias* for a signal with frequency ω is given by

$$\omega_{\text{fundamental}} = |(\omega + \omega_N) \bmod (\omega_s) - \omega_N|$$

(This frequency lies in the interval $0 \leq \omega_{\text{fundamental}} < \omega_N$)

Antialiasing Filter

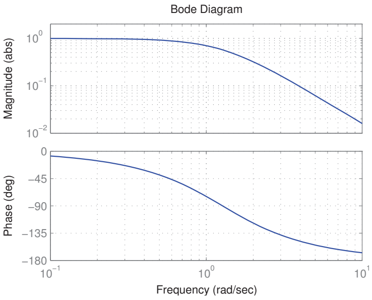
Low-pass filter that eliminates all frequencies above the Nyquist frequency before sampling. **Must contain analog part!** Options:

- Analog filter
  - E.g. 2–6th order Bessel or Butterworth filter
  - Difficult to change sampling interval
- Analog + digital filter
  - Fixed, fast sampling with fixed analog filter
  - Downsampling using digital LP-filter
  - Control algorithm at the lower rate
  - Easier to change sampling interval

Example: Second-Order Bessel Filter

$$G_f(s) = \frac{\omega^2}{(s/\omega_B)^2 + 2\zeta\omega(s/\omega_B) + \omega^2}, \quad \omega = 1.27,\ \zeta = 0.87$$

$\omega_B = 1$:



Antialiasing Filter and Control Design

As a rule of thumb, the cut-off frequency of the filter should be chosen so that

$$|G_f(i\omega_N)| \leq 0.1,$$

meaning that frequencies above the Nyquist frequency are attenuated by at least a factor 10.

Unless extremely fast sampling is used, the filter will affect the phase margin of the system. Include the filter in the process description or approximate it by a delay.

- Digital design: E.g. 2nd order Bessel filter: $\tau \approx 1.3/\omega_B$. If $|G_f(i\omega_N)| = 0.1$ then $\tau \approx 1.5h$
- Analog design + discretization: must sample fast

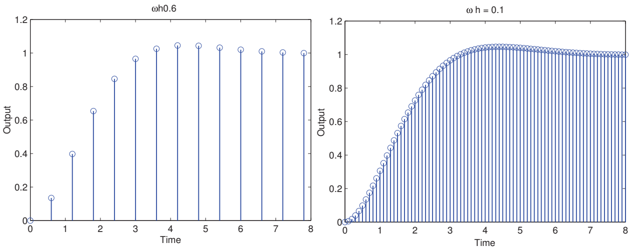
Choice of Sampling Interval – Digital Design

Common rule of thumb:

$$\omega h \approx 0.1 \text{ to } 0.6$$

ω is the desired natural frequency of the closed-loop system

Gives about 4 to 20 samples per rise time



Choice of Sampling Interval – Analog Design

Sampler + ZOH $\approx$ delay of $0.5h \Leftrightarrow e^{-s0.5h}$

Antialiasing filter $\approx$ delay of $1.5h \Leftrightarrow e^{-s1.5h}$

Will affect phase margin (at cross-over frequency ω_c) by

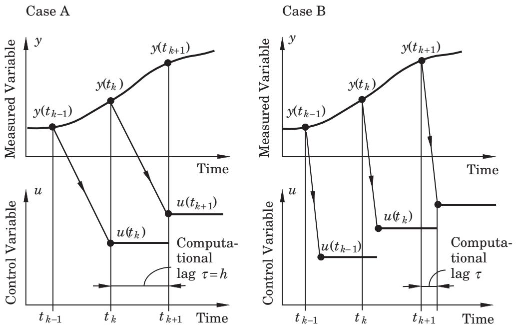
$$\arg e^{-i\omega_c 2h} = -2\omega_c h$$

Assume phase margin can be decreased by 5° to 15° ($= 0.087$ to 0.262 rad). Then

$$\omega_c h \approx 0.04 \text{ to } 0.13$$

Computational delay

Problem: $u(k)$ cannot be generated instantaneously at time k when $y(k)$ is sampled. Options:



Case B: Minimizing the computational delay

Controllers with direct term ($D \neq 0$ or $D_c \neq 0$)

A general linear controller in state-space form (including state feedback, observer, reference model, etc.):

$$\begin{aligned} x_c(k+1) &= Fx_c(k) + Gy(k) + G_c u_c(k) \\ u(k) &= Cx_c(k) + Dy(k) + D_c u_c(k) \end{aligned}$$

Do as little as possible between the input and the output:

```
y := adin(1);
uc := adin(2);
/* Calculate Output */
u := u1 + D*y + Dc*uc;
daout(u);
/* Update State */
xc := F*xc + G*y + Gc*uc;
u1 := C*xc;
```

Case A: One sample delay

Controllers without direct term ($D = D_c = 0$)

A general linear controller in state-space form (including state feedback, observer, reference model, etc.):

$$\begin{aligned} x_c(k+1) &= Fx_c(k) + Gy(k) + G_c u_c(k) \\ u(k) &= Cx_c(k) \end{aligned}$$

Wait with outputting the control signal until the beginning of next sample

```
daout(u);
y := adin(1);
uc := adin(2);
/* Update State */
xc := F*xc + G*y + Gc*uc;
u  := C*xc;
```

Finite-Wordlength Implementation

Control analysis and design usually assumes infinite-precision arithmetic, parameters/variables are assumed to be real numbers

Error sources in a digital implementation with finite wordlength:

- Quantization in A-D converters
- Quantization of parameters (controller coefficients)
- Round-off and overflow in addition, subtraction, multiplication, division, function evaluation and other operations
- Quantization in D-A converters

The magnitude of the problems depends on

- The wordlength
- The type of arithmetic used (fixed or floating point)
- The controller realization

A-D and D-A Quantization

A-D and D-A converters often have quite poor resolution, e.g.

- A-D: 10–16 bits
- D-A: 8–12 bits

Quantization is a nonlinear phenomenon; can lead to limit cycles and bias. Analysis approaches (outside scope of this course):

- Nonlinear analysis
  - Describing function approximation
  - Theory of relay oscillations
- Linear analysis
  - Quantization as a stochastic disturbance

Example: Control of the Double Integrator

Process:

$$P(s) = 1/s^2$$

Sampling period:

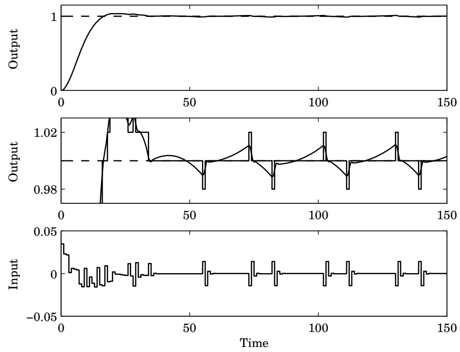
$$h = 1$$

Controller (PID):

$$C(z) = \frac{0.715z^2 - 1.281z + 0.580}{(z-1)(z+0.188)}$$

Simulation with Quantized A-D Converter

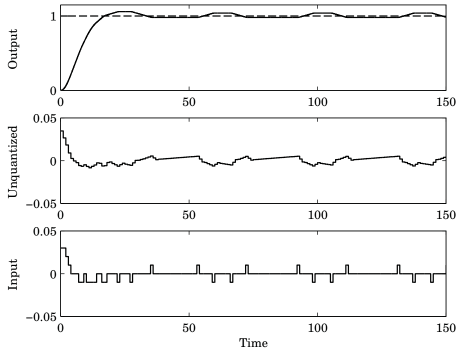
($\delta y = 0.02$)



Limit cycle in process output with period 28 s, ampl. 0.01

Simulation with Quantized D-A Converter

($\delta u = 0.01$)

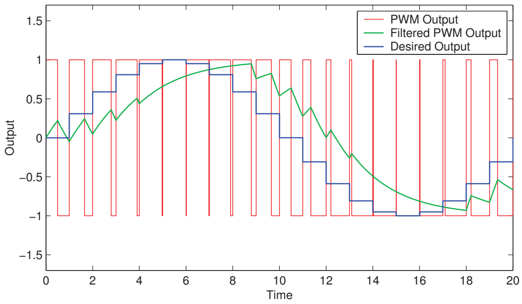


Limit cycle in process input with period 39 s, ampl. 0.01

Pulse-Width Modulation (PWM)

Poor D-A resolution (e.g. 1 bit) can often be handled by fast switching between levels + low-pass filtering

The new control variable is the duty-cycle of the switched signal



Floating-Point Arithmetic

Hardware-supported on modern high-end processors (FPUs)

Number representation:

$$\pm f \times 2^{\pm e}$$

- f: mantissa, significand, fraction
- 2: base
- e: exponent

The binary point is variable (floating) and depends on the value of the exponent

Dynamic range and resolution

Fixed number of significant digits

IEEE 754 Binary Floating-Point Standard

Used by almost all FPUs; implemented in software libraries

Single precision (Java/C `float`):

- 32-bit word divided into 1 sign bit, 8-bit biased exponent, and 23-bit mantissa (≈ 7 decimal digits)
- Range: $2^{-126} - 2^{128}$

Double precision (Java/C `double`):

- 64-bit word divided into 1 sign bit, 11-bit biased exponent, and 52-bit mantissa (≈ 15 decimal digits)
- Range: $2^{-1022} - 2^{1024}$

Supports Inf and NaN

What is the output of this C program?

```
#include <stdio.h>

main() {

  float a[] = { 10000.0, 1.0, 10000.0 };
  float b[] = { 10000.0, 1.0, -10000.0 };
  float sum = 0.0;
  int i;

  for (i=0; i<3; i++)
    sum += a[i]*b[i];

  printf("sum = %f\n", sum);
}
```

Remarks:

- The result depends on the order of the operations
- Finite-wordlength operations are neither associative nor distributive

Arithmetic in Embedded Systems

Small microprocessors used in embedded systems typically do not have hardware support for floating-point arithmetic

Options:

- Software emulation of floating-point arithmetic
  - compiler/library supported
  - large code size, slow
- Fixed-point arithmetic
  - often manual implementation
  - fast and compact

Fixed-Point Arithmetic

Represent all numbers (parameters, variables) using **integers**

Use **binary scaling** to make all numbers fit into one of the integer data types, e.g.

- 8 bits (`char`, `int8_t`): $[-128, 127]$
- 16 bits (`short`, `int16_t`): $[-32768, 32767]$
- 32 bits (`long`, `int32_t`): $[-2147483648, 2147483647]$

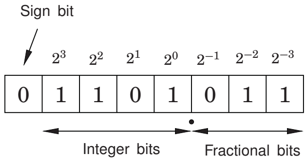
Challenges

- Must select data types to get sufficient numerical precision
- Must know (or estimate) the minimum and maximum value of every variable in order to select appropriate scaling factors
- Must keep track of the scaling factors in all arithmetic operations
- Must handle potential arithmetic overflows

Fixed-Point Representation

In fixed-point representation, a real number x is represented by an integer X with $N = m + n + 1$ bits, where

- N is the wordlength
- m is the number of integer bits (excluding the sign bit)
- n is the number of fractional bits



"Q-format": X is sometimes called a $Qm.n$ or Qn number

Conversion to and from fixed point

Conversion from real to fixed-point number:

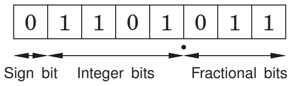
$$X := \text{round}(x \cdot 2^n)$$

Conversion from fixed-point to real number:

$$x := X \cdot 2^{-n}$$

Example: Represent $x = 13.4$ using $Q4.3$ format

$$X = \text{round}(13.4 \cdot 2^3) = 107 \ (= 01101011_2)$$



A Note on Negative Numbers

In almost all CPUs today, negative integers are handled using **two's complement**: A "1" in the sign bit means that 2^N should be subtracted from the stored value

Example ($N = 8$):

Binary representation	Interpretation
00000000	0
00000001	1
⋮	⋮
01111111	127
10000000	-128
10000001	-127
⋮	⋮
1111111	-1

Range vs Resolution for Fixed-Point Numbers

A $Qm.n$ fixed-point number can represent real numbers in the range

$$[-2^m,\ 2^m - 2^{-n}]$$

while the resolution is

$$2^{-n}$$

Fixed range and resolution

- n too small ⇒ poor resolution
- n too large ⇒ risk of overflow

Example: Choose number of integer and fractional bits

We want to store x in a signed 8-bit variable.

We know that $-28.3 < x < 17.5$.

We hence need $m = 5$ bits to represent the integer part. ($2^4 = 16 < 28.3 < 32 = 2^5$)

$n = 8 - 1 - m = 2$ bits are left for the fractional part.

x should be stored in $Q5.2$ format

Fixed-Point Addition/Subtraction

Two fixed-point numbers in the same $Qm.n$ format can be added or subtracted directly

The result will have the same number of fractional bits

$$z = x + y \quad \Leftrightarrow \quad Z = X + Y$$

$$z = x - y \quad \Leftrightarrow \quad Z = X - Y$$

- The result will in general require $N+1$ bits; risk of overflow

Example: Addition with Overflow

Two numbers in $Q4.3$ format are added:

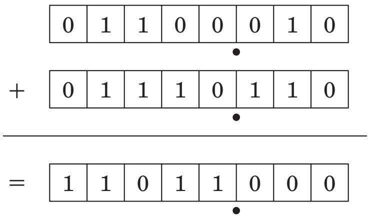
$$x = 12.25 \quad \Rightarrow \quad X = 98$$

$$y = 14.75 \quad \Rightarrow \quad Y = 118$$

$$Z = X + Y = 216$$

This number is however out of range and will be interpreted as

$$216 - 256 = -40 \quad \Rightarrow \quad z = -5.0$$



Fixed-Point Multiplication and Division

If the operands and the result are in the same Q-format, multiplication and division are done as

$$z = x \cdot y \quad \Leftrightarrow \quad Z = (X \cdot Y)/2^n$$

$$z = x/y \quad \Leftrightarrow \quad Z = (X \cdot 2^n)/Y$$

- Double wordlength is needed for the intermediate result
- Division by 2^n is implemented as a right-shift by n bits
- Multiplication by 2^n is implemented as a left-shift by n bits
- The lowest bits in the result are truncated (round-off noise)
- Risk of overflow

Example: Multiplication

Two numbers in $Q5.2$ format are multiplied:

$$x = 6.25 \quad \Rightarrow \quad X = 25$$

$$y = 4.75 \quad \Rightarrow \quad Y = 19$$

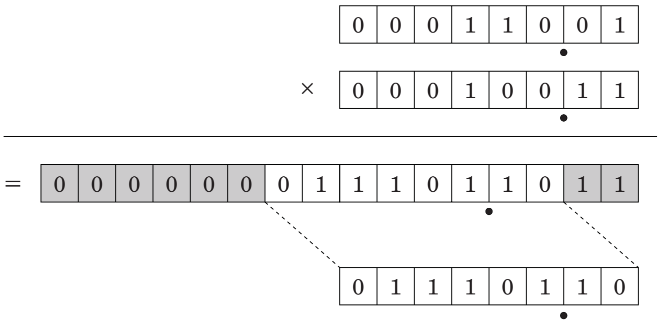
Intermediate result:

$$X \cdot Y = 475$$

Final result:

$$Z = 475/2^2 = 118 \quad \Rightarrow \quad z = 29.5$$

(exact result is 29.6875)



Example: Division

Two numbers in $Q3.4$ format are divided:

$$x = 5.375 \quad \Rightarrow \quad X = 86$$
$$y = 6.0625 \quad \Rightarrow \quad Y = 97$$

Not associative:

$$Z_{bad} = (X/Y) \cdot 2^4 = (86/97) \cdot 2^4 = 0 \cdot 2^4 = 0$$

$$Z_{good} = (X \cdot 2^4)/Y = 1376/97 = 14 \quad \Rightarrow \quad z = 0.875$$

(correct first 6 digits are 0.888531)

Multiplication of Operands with Different Q-format

In general, multiplication of two fixed-point numbers $Qm_1.n_1$ and $Qm_2.n_2$ gives an intermediate result in the format

$$Qm_1{+}m_2.n_1{+}n_2$$

which may then be right-shifted $n_1{+}n_2{-}n_3$ steps and stored in the format

$$Qm_3.n_3$$

Common case: $n_2 = n_3 = 0$ (one real operand, one integer operand, and integer result). Then

$$Z = (X \cdot Y)/2^{n_1}$$

Implementation of Multiplication in C

Assume $Q4.3$ operands and result

```
#include <inttypes.h>     /* define int8_t, etc. (Linux only)      */
#define n 3               /* number of fractional bits             */
int8_t X, Y, Z;           /* Q4.3 operands and result              */
int16_t temp;             /* Q9.6 intermediate result              */
...
temp = (int16_t)X * Y;    /* cast operands to 16 bits and multiply */
temp = temp >> n;         /* divide by 2^n                         */
Z = temp;                 /* truncate and assign result            */
```

Implementation of Multiplication in C with Rounding and Saturation

```
#include <inttypes.h>     /* defines int8_t, etc. (Linux only)     */
#define n 3               /* number of fractional bits             */
int8_t X, Y, Z;           /* Q4.3 operands and result              */
int16_t temp;             /* Q9.6 intermediate result              */
...
temp = (int16_t)X * Y;    /* cast operands to 16 bits and multiply */
temp = temp + (1 << n-1); /* add 1/2 to give correct rounding      */
temp = temp >> n;         /* divide by 2^n                         */
if (temp > INT8_MAX)      /* saturate the result before assignment */
   Z = INT8_MAX;
else if (temp < INT8_MIN)
   Z = INT8_MIN;
else
   Z = temp;
```

Implementation of Division in C with Rounding

```
#include <inttypes.h>     /* define int8_t, etc. (Linux only)      */
#define n 3               /* number of fractional bits             */
int8_t X, Y, Z;           /* Q4.3 operands and result              */
int16_t temp;             /* Q9.6 intermediate result              */
...
temp = (int16_t)X << n;   /* cast operand to 16 bits and shift     */
temp = temp + (Y >> 1);   /* Add Y/2 to give correct rounding      */
temp = temp / Y;          /* Perform the division (expensive!)     */
Z = temp;                 /* Truncate and assign result            */
```

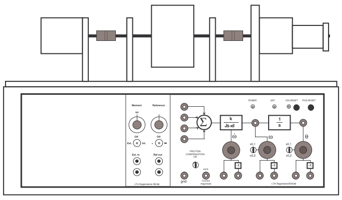
Atmel mega8/16 instruction set

Mnemonic	Description	# clock cycles
ADD	Add two registers	1
SUB	Subtract two registers	1
MULS	Multiply signed	2
ASR	Arithmetic shift right (1 step)	1
LSL	Logical shift left (1 step)	1

- No division instruction; implemented in math library using expensive division algorithm

Laboratory Exercise 3

- Control of a rotating DC servo using the ATmega16



- Velocity control (PI controller)
- Position control (state feedback from extended observer)
- Floating-point and fixed-point implementations
- Measurement of code size and execution time

Example Evaluation: Measurements

Floating-point implementation using `float`s:

- Velocity control: 950 μs
- Position control: 1220 μs
- Total code size: 13708 bytes

Fixed-point implementation using 16-bit integers:

- Velocity control: 130 μs
- Position control: 270 μs
- Total code size: 3748 bytes

The measured times include 115 μs A-D conversion. This gives a 25–50 times actual speedup for fixed point math compared to floating point. The floating point math library takes about 10K (out of 16K available!)

Controller Realizations

A linear controller

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

can be realized in a number of different ways with equivalent input-output behavior, e.g.

- Direct form
- Companion (canonical) form
- Series (cascade) or parallel form

Direct Form

The input-output form can be directly implemented as

$$u(k) = \sum_{i=0}^{n} b_i y(k-i) - \sum_{i=1}^{n} a_i u(k-i)$$

- Nonminimal (all old inputs and outputs are used as states)
- Very sensitive to roundoff in coefficients
- Avoid!

Companion Forms

E.g. controllable or observable canonical form

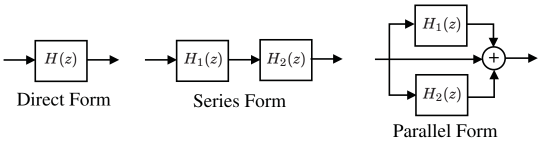
$$x(k+1) = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} y(k)$$

$$u(k) = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} x(k)$$

- Same problem as for the Direct form
- Very sensitive to roundoff in coefficients
- Avoid!

Better: Series and Parallel Forms

Divide the transfer function of the controller into a number of first- or second-order subsystems:



Direct Form

Series Form

Parallel Form

- Try to balance the gain such that each subsystem has about the same amplification

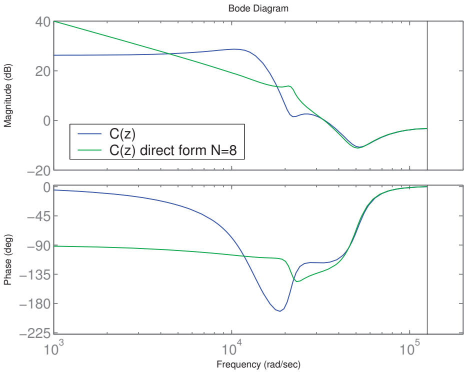
Example: Series and Parallel Forms

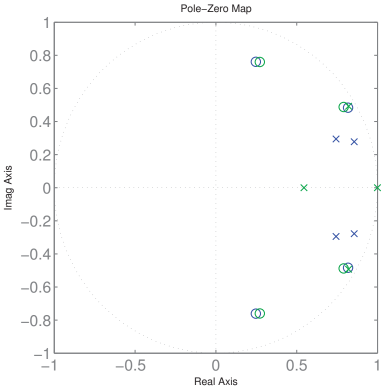
$$C(z) = \frac{z^4 - 2.13z^3 + 2.351z^2 - 1.493z + 0.5776}{z^4 - 3.2z^3 + 3.997z^2 - 2.301z + 0.5184} \qquad \text{(Direct)}$$

$$= \left(\frac{z^2 - 1.635z + 0.9025}{z^2 - 1.712z + 0.81}\right)\left(\frac{z^2 - 0.4944z + 0.64}{z^2 - 1.488z + 0.64}\right) \qquad \text{(Series)}$$

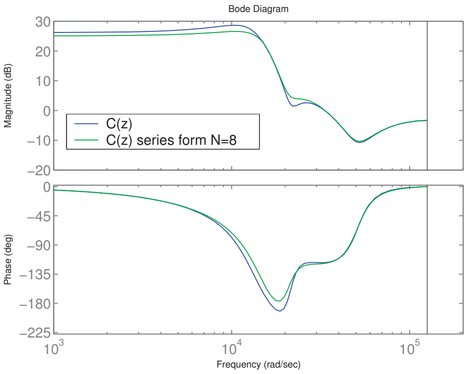
$$= 1 + \frac{-5.396z + 6.302}{z^2 - 1.712z + 0.81} + \frac{6.466z - 4.907}{z^2 - 1.488z + 0.64} \qquad \text{(Parallel)}$$

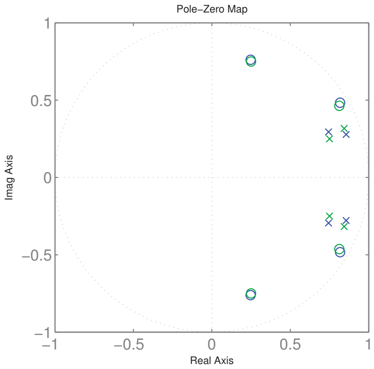
Direct form with quantized coefficients ($N = 8$, $n = 4$):





Series form with quantized coefficients ($N = 8$, $n = 4$):





Jackson's Rules for Series Realizations

How to pair and order the poles and zeros?

Jackson's rules (1970):

- Pair the pole closest to the unit circle with its closest zero. Repeat until all poles and zeros are taken.
- Order the filters in increasing or decreasing order based on the poles closeness to the unit circle.

This will push down high internal resonance peaks.

Short Sampling Interval Modification

In the state update equation

$$x(k+1) = \Phi x(k) + \Gamma y(k)$$

the system matrix Φ will be close to I if h is small. Round-off errors in the coefficients of Φ can have drastic effects.

Better: use the modified equation

$$x(k+1) = x(k) + (\Phi - I)x(k) + \Gamma y(k)$$

- Both $\Phi - I$ and Γ are roughly proportional to h
  - Less round-off noise in the calculations
- Also known as the δ-form

Short Sampling Interval and Integral Action

Fast sampling and slow integral action can give roundoff problems:

$$I(k+1) = I(k) + \underbrace{e(k) \cdot h/T_i}_{\approx 0}$$

Possible solutions:

- Use a dedicated high-resolution variable (e.g. 32 bits) for the I-part
- Update the I-part at a slower rate

(This is a general problem for filters with very different time constants)