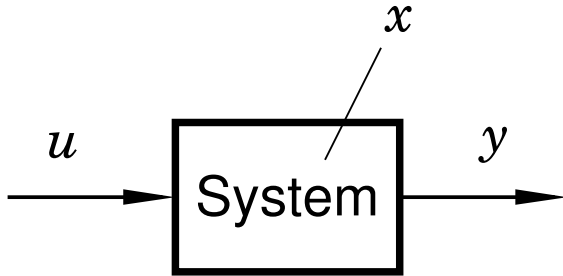



# Lecture 7: Input-Output Models

[IFAC PB pg 23-35]

- Shift operators; the pulse transfer operator
- Z-transform; the pulse transfer function
- System response
- Poles and zeros
- Transformations between system representations

# Linear System Models

	State-space model	Input-output models	
			
		Differential/difference equation	Transfer operator/fcn
<b>CT</b>	$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t)$	$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y$ $= b_1 \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_n u$	$G(p) / G(s)$
<b>DT</b>	$x(k+1) = \Phi x(k) + \Gamma u(k)$ $y(k) = Cx(k)$	$y(k) + a_1 y(k-1) + \dots +$ $a_n y(k-n) = b_1 u(k-1)$ $+ \dots + b_n u(k-n)$	$H(q) / H(z)$

More I-O models: pulse response, step response, frequency function, ...

# Shift Operators

Operators on time series

Assume  $h = 1$  (the *sampling-time convention*)

Time series are doubly infinite sequences:

- $f(k) : k = \dots - 1, 0, 1, \dots$

**Forward shift operator:**

- denoted  $q$
- $qf(k) = f(k + 1)$
- $q^n f(k) = f(k + n)$

# Shift Operators

## Backward shift operator:

- denoted  $q^{-1}$
- $q^{-1} f(k) = f(k - 1)$
- $q^{-n} f(k) = f(k - n)$

# Pulse Transfer Operator

Rewrite the state-space model using the forward shift operator:

$$\begin{aligned}x(k+1) &= qx(k) = \Phi x(k) + \Gamma u(k) \\y(k) &= Cx(k) + Du(k)\end{aligned}$$

Eliminate  $x(k)$ :

$$\begin{aligned}x(k) &= (qI - \Phi)^{-1}\Gamma u(k) \\y(k) &= Cx(k) + Du(k) = C(qI - \Phi)^{-1}\Gamma u(k) + Du(k) \\&= [C(qI - \Phi)^{-1}\Gamma + D] u(k) = H(q)u(k)\end{aligned}$$

$H(q)$  is the *pulse transfer operator* of the system

Describes how the input and output are related.

## Poles and Zeros (SISO case)

The pulse transfer function can be written as a rational function

$$H(q) = \frac{B(q)}{A(q)}$$

$\deg A = n =$  the number of states

$\deg B = n_b \leq n$

$A(q)$  is the characteristic polynomial of  $\Phi$ , i.e.

$$A(q) = \det(qI - \Phi)$$

The *poles* of the system are given by  $A(q) = 0$

The *zeros* of the system are given by  $B(q) = 0$

# Disk Drive Example

Recall the double integrator from the previous lecture:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [1 \quad 0] x$$

Sample with  $h = 1$ :

$$\Phi = e^{Ah} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\Gamma = \int_0^h e^{As} B ds = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

## Disk Drive Example cont.

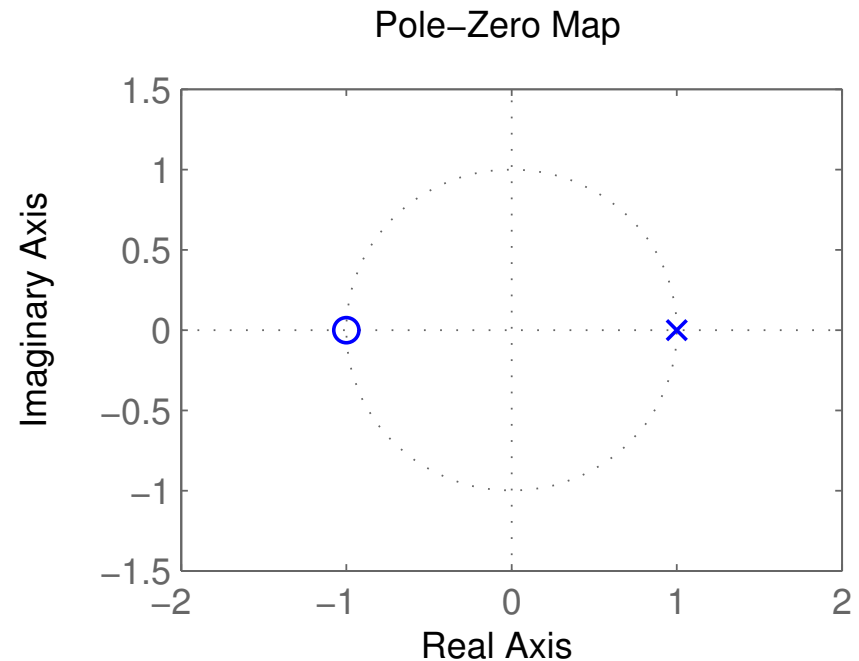
Pulse transfer operator:

$$H(q) = C(qI - \Phi)^{-1}\Gamma + D$$

$$= [1 \quad 0] \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \frac{[1 \quad 0]}{(q-1)^2} \begin{bmatrix} q-1 & 1 \\ 0 & q-1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \frac{0.5(q+1)}{(q-1)^2}$$

Two poles in 1, one zero in  $-1$ .





# From Pulse Transfer Operator to Difference Equation

$$y(k) = H(q)u(k)$$
$$A(q)y(k) = B(q)u(k)$$

$$(q^n + a_1q^{n-1} + \cdots + a_n)y(k) = (b_0q^{n_b} + \cdots + b_{n_b})u(k)$$

which means

$$y(k+n) + a_1y(k+n-1) + \cdots + a_ny(k)$$
$$= b_0u(k+n_b) + \cdots + b_{n_b}u(k)$$

# Difference Equation with Backward Shift

$$\begin{aligned}y(k+n) + a_1y(k+n-1) + \cdots + a_ny(k) \\ = b_0u(k+n_b) + \cdots + b_{n_b}u(k)\end{aligned}$$

can be written as

$$\begin{aligned}y(k) + a_1y(k-1) + \cdots + a_ny(k-n) \\ = b_0u(k-d) + \cdots + b_{n_b}u(k-d-n_b)\end{aligned}$$

where  $d = n - n_b$  is the *pole excess* of the system (the number of pure time delays in the system)

The *reciprocal polynomial*

$$A^*(q) = 1 + a_1q + \cdots + a_nq^n = q^n A(q^{-1})$$

is obtained from the polynomial  $A$  by reversing the order of the coefficients.

Now the system can instead be written as

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k - d)$$

# Difference Equation Example

Using forward shift

$$y(k + 2) + 2y(k + 1) + 3y(k) = 2u(k + 1) + u(k)$$

can be written

$$(q^2 + 2q + 3)y(k) = (2q + 1)u(k)$$

Hence,

$$A(q) = q^2 + 2q + 3$$

$$B(q) = 2q + 1$$

# Difference Equation Example, continued

Using backward shift

$$y(k) + 2y(k-1) + 3y(k-2) = 2u(k-1) + u(k-2)$$

can be written ( $d = 1$ )

$$(1 + 2q^{-1} + 3q^{-2})y(k) = (2 + q^{-1})u(k-1)$$

Hence,

$$A^*(q^{-1}) = 1 + 2q^{-1} + 3q^{-2}$$

$$B^*(q^{-1}) = 2 + q^{-1}$$

# Z-transform

The discrete-time counterpart to the Laplace transform

Defined on semi-infinite time series  $f(k) : k = 0, 1, \dots$

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

$z$  is a complex variable

## Example — Discrete-Time Step Signal

Let  $y(k) = 1$  for  $k \geq 0$ . Then

$$Y(z) = 1 + z^{-1} + z^{-2} + \dots = \frac{z}{z-1}, \quad |z| > 1$$

Application of the following result for power series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x| < 1$$

## Example — Discrete-Time Ramp Signal

Let  $y(k) = k$  for  $k \geq 0$ . Then

$$Y(z) = 0 + z^{-1} + 2z^{-2} + 3z^{-3} \dots = \frac{z}{(z-1)^2}$$

Application of the following result for power series

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \text{ for } |x| < 1$$



# Z-transform Table

Table 2 (pg 26) in IFAC PB (ignore the middle column!)

$f$	$\mathcal{L}f$	$\mathcal{Z}f$
$\delta(k)$ (pulse)	–	1
1 $k \geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z-1}$
$kh$	$\frac{1}{s^2}$	$\frac{hz}{(z-1)^2}$
$\frac{1}{2} (kh)^2$	$\frac{1}{s^3}$	$\frac{h^2 z(z+1)}{2(z-1)^3}$
$e^{-kh/T}$	$\frac{T}{1+sT}$	$\frac{z}{z-e^{-h/T}}$
$1 - e^{-kh/T}$	$\frac{1}{s(1+sT)}$	$\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$
$\sin \omega kh$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z \sin \omega h}{z^2 - 2z \cos \omega h + 1}$

# Some Properties of the Z-transform

$$\mathcal{Z}(\alpha f + \beta g) = \alpha F(z) + \beta G(z)$$

$$\mathcal{Z}(q^{-n} f) = z^{-n} F(z)$$

$$\mathcal{Z}(qf) = z(F(z) - f(0))$$

$$\mathcal{Z}(f * g) = \mathcal{Z} \left\{ \sum_{j=0}^k f(j)g(k-j) \right\} = F(z)G(z)$$

# From State Space to Pulse Transfer Function

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

$$\begin{cases} z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z) \\ Y(z) = CX(z) + DU(z) \end{cases}$$

$$Y(z) = C(zI - \Phi)^{-1}z x(0) + [C(zI - \Phi)^{-1}\Gamma + D]U(z)$$

The rational function  $H(z) = C(zI - \Phi)^{-1}\Gamma + D$  is called the *pulse transfer function* from  $u$  to  $y$ .

It is the Z-transform of the pulse response.

## $H(q)$ vs $H(z)$

The pulse transfer operator  $H(q)$  and the pulse transfer function  $H(z)$  are the same rational functions

They have the same poles and zeros

$H(q)$  is used in the time domain ( $q =$  shift operator)

$H(z)$  is used in the Z-domain ( $z =$  complex variable)

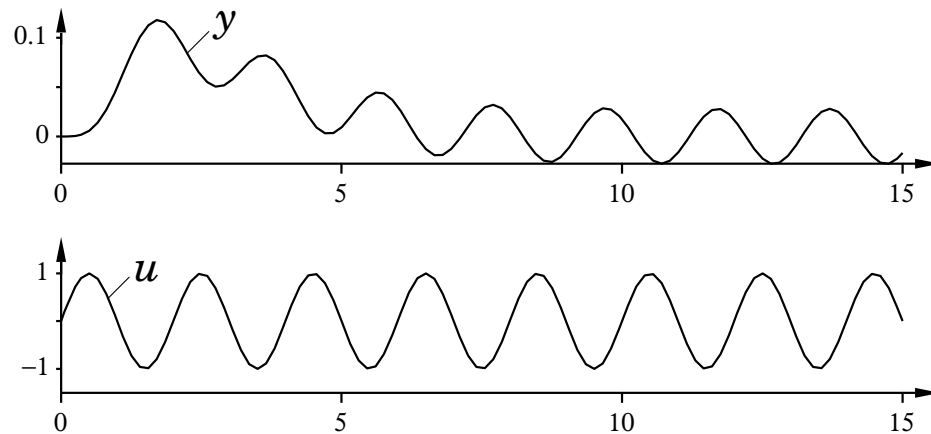
# Calculating System Response Using the Z-transform

1. Find the pulse transfer function  $H(z) = C(zI - \Phi)^{-1}\Gamma + D$
2. Compute the Z-transform of the input:  $U(z) = \mathcal{Z}\{u(k)\}$
3. Compute the Z-transform of the output:

$$Y(z) = C(zI - \Phi)^{-1}z x(0) + H(z)U(z)$$

4. Apply the inverse Z-transform (table) to find the output:  
 $y(k) = \mathcal{Z}^{-1}\{Y(z)\}$

# Frequency Response in Continuous Time

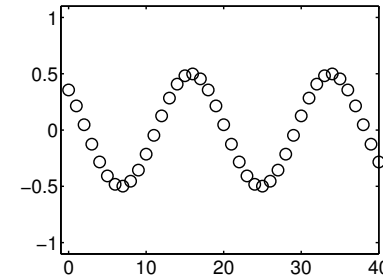
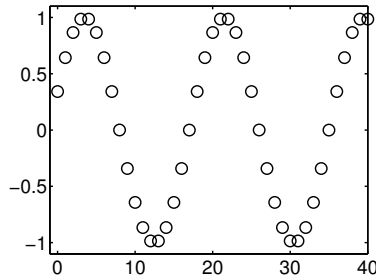


Given a stable system  $G(s)$ , the input  $u(t) = \sin \omega t$  will, after a transient, give the output

$$y(t) = |G(i\omega)| \sin(\omega t + \arg G(i\omega))$$

- The amplitude and phase shift for different frequencies are given by the value of  $G(s)$  along the imaginary axes, i.e.  $G(i\omega)$
- Plotted in Bode and Nyquist diagrams.

# Frequency Response in Discrete Time

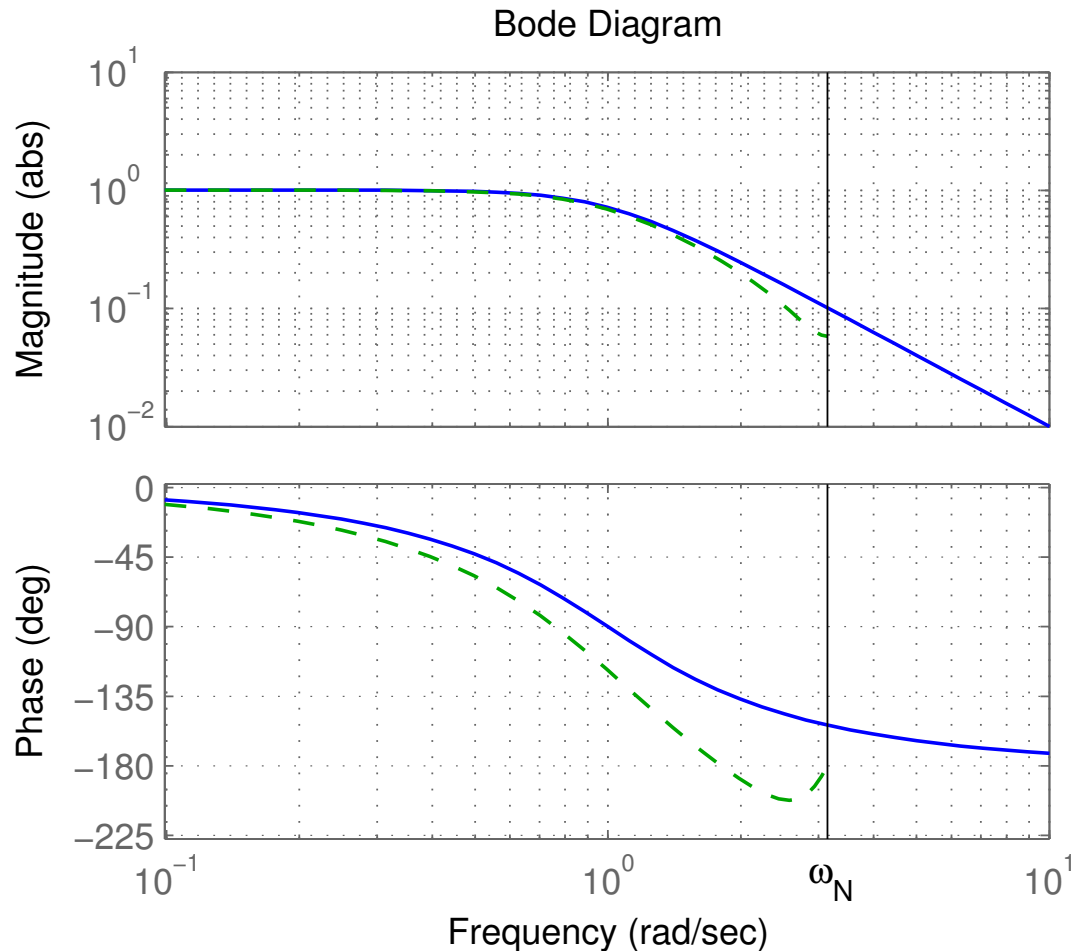


Given a stable system  $H(z)$ , the input  $u(k) = \sin(\omega k)$  will, after a transient, give the output

$$y(k) = |H(e^{i\omega})| \sin(\omega k + \arg H(e^{i\omega}))$$

- $G(s)$  and the imaginary axis are replaced by  $H(z)$  and the unit circle.
- Only describes what happens at the sampling instants
- The inter-sample behavior is not studied in this course

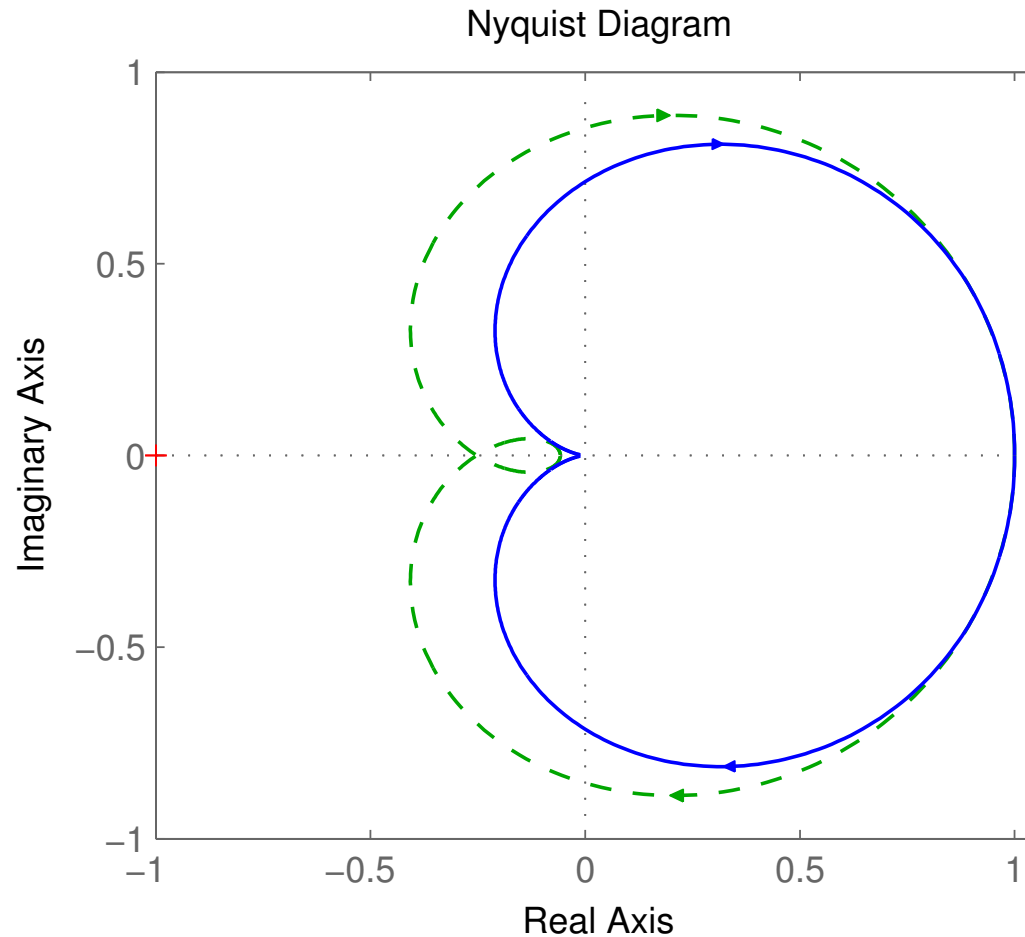
**Bode diagram** for continuous transfer function  $1/(s^2+1.4s+1)$  (solid) and for ZOH-sampled counterpart (dashed, plotted for  $\omega h \in [0, \pi]$ )



For slow signals, the hold circuit is approximately a  $h/2$  delay.  
 For fast signals, the hold circuit destroys the sinusoidal shape.



**Nyquist diagram** for cont. transfer function  $1/(s^2 + 1.4s + 1)$  (solid) and for ZOH-sampled counterpart (dashed, plotted for  $\omega h \in [0, \pi]$ )



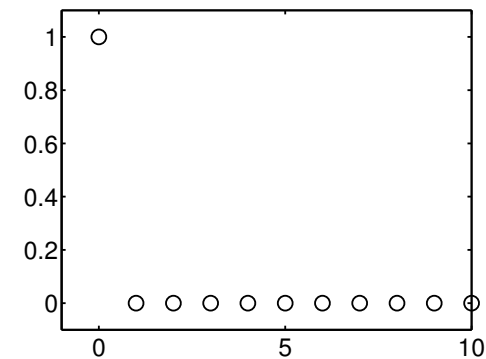
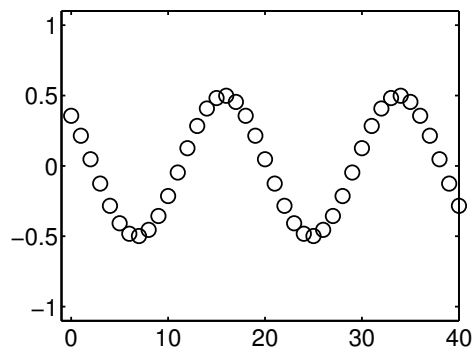
# Interpretation of Poles and Zeros

Poles:

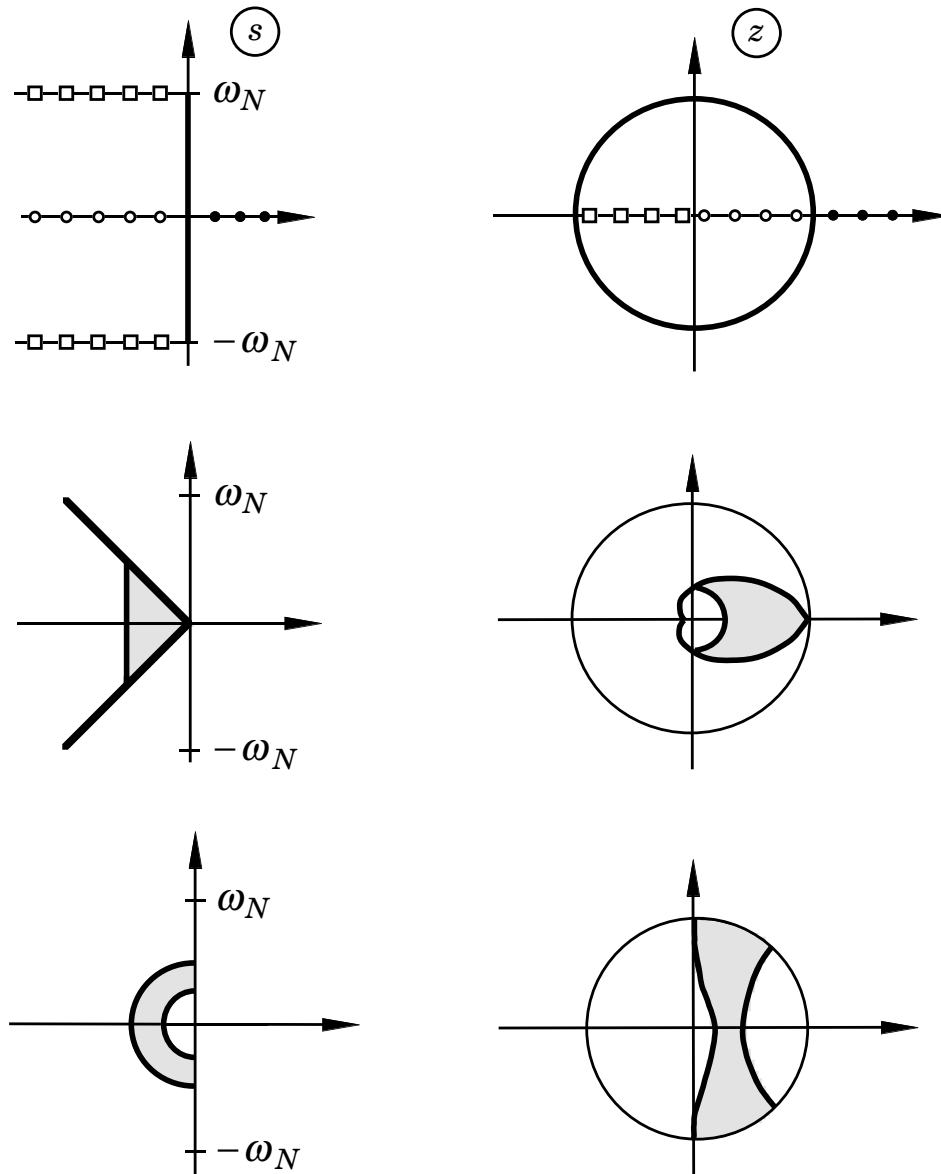
- A pole  $z = a$  is associated with the time function  $f(k) = a^k$

Zeros:

- A zero  $z = a$  implies that the transmission of the input  $u(k) = a^k$  is blocked by the system
- Related to how inputs and outputs are coupled to the states

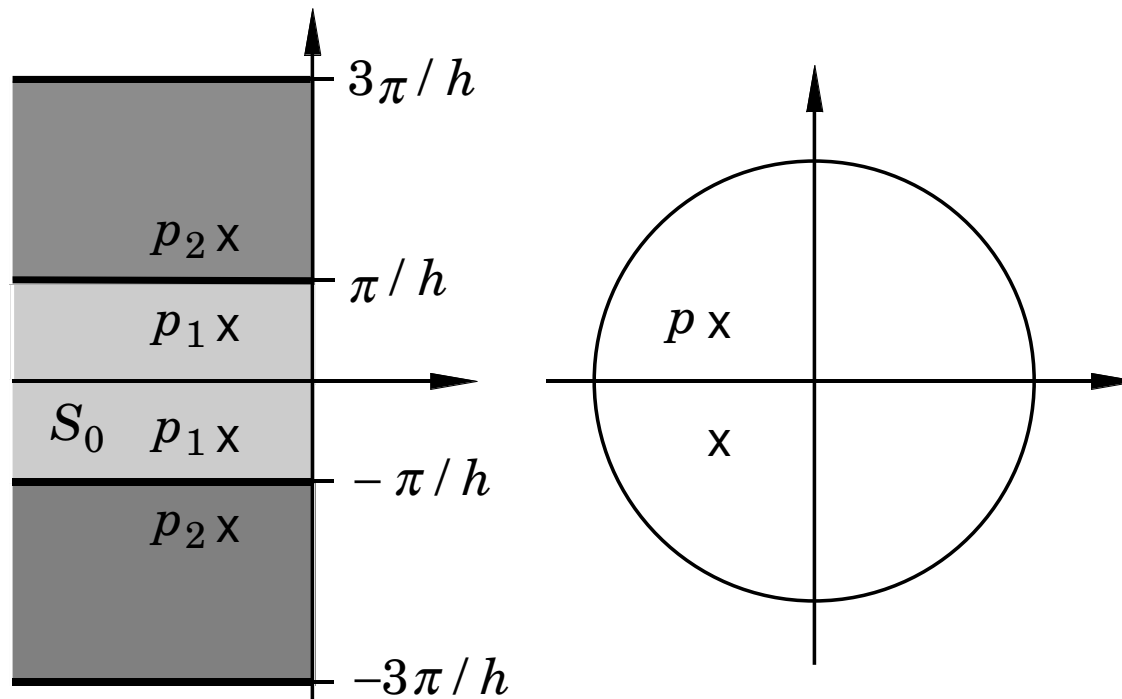


# Transformation of Poles via Sampling: $z_i = e^{s_i h}$



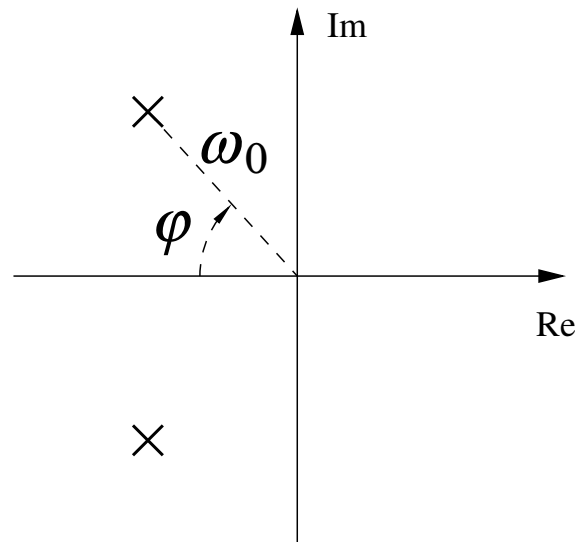
# New Evidence of the Alias Problem

Several points in the  $s$ -plane are mapped into the same point in the  $z$ -plane. The map is not bijective



# Sampling of a Second Order System

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}, \quad |\zeta| < 1$$



- Larger  $\omega_0 \Rightarrow$  faster system response
- Smaller  $\varphi \Rightarrow$  larger damping (relative damping  $\zeta = \cos \varphi$ ).  
(Common design choice:  $\zeta = \cos 45^\circ \approx 0.7$ )

# Sampling of a Second Order System

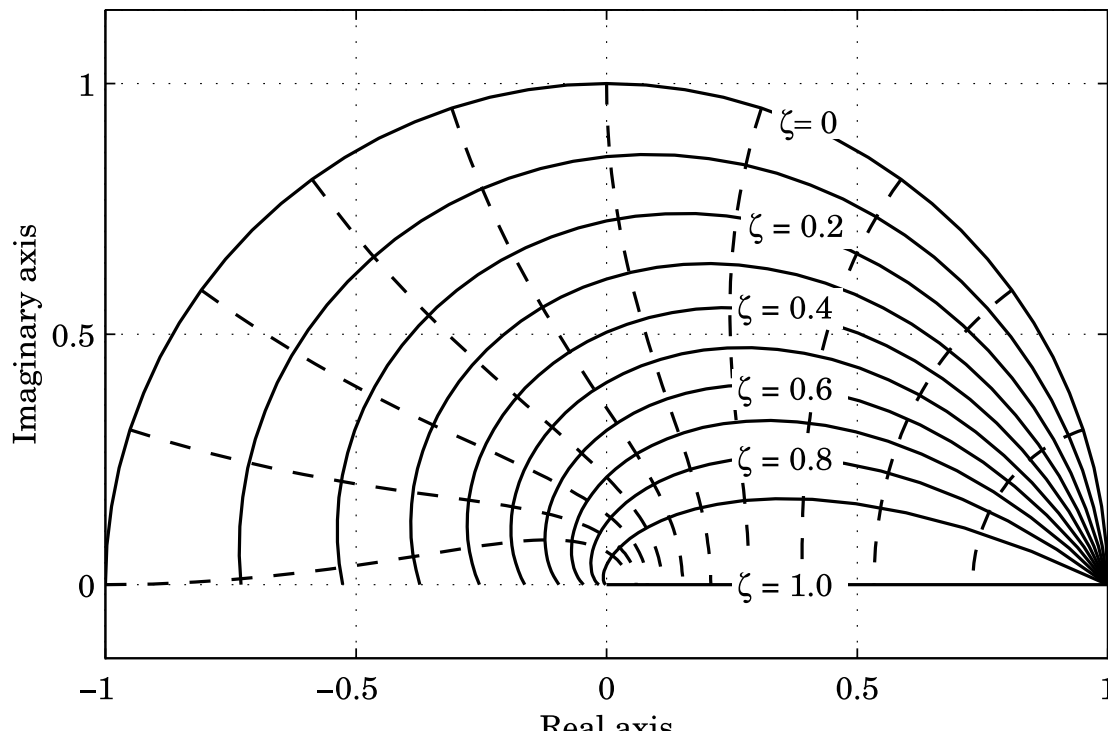
The poles of the sampled system are given by

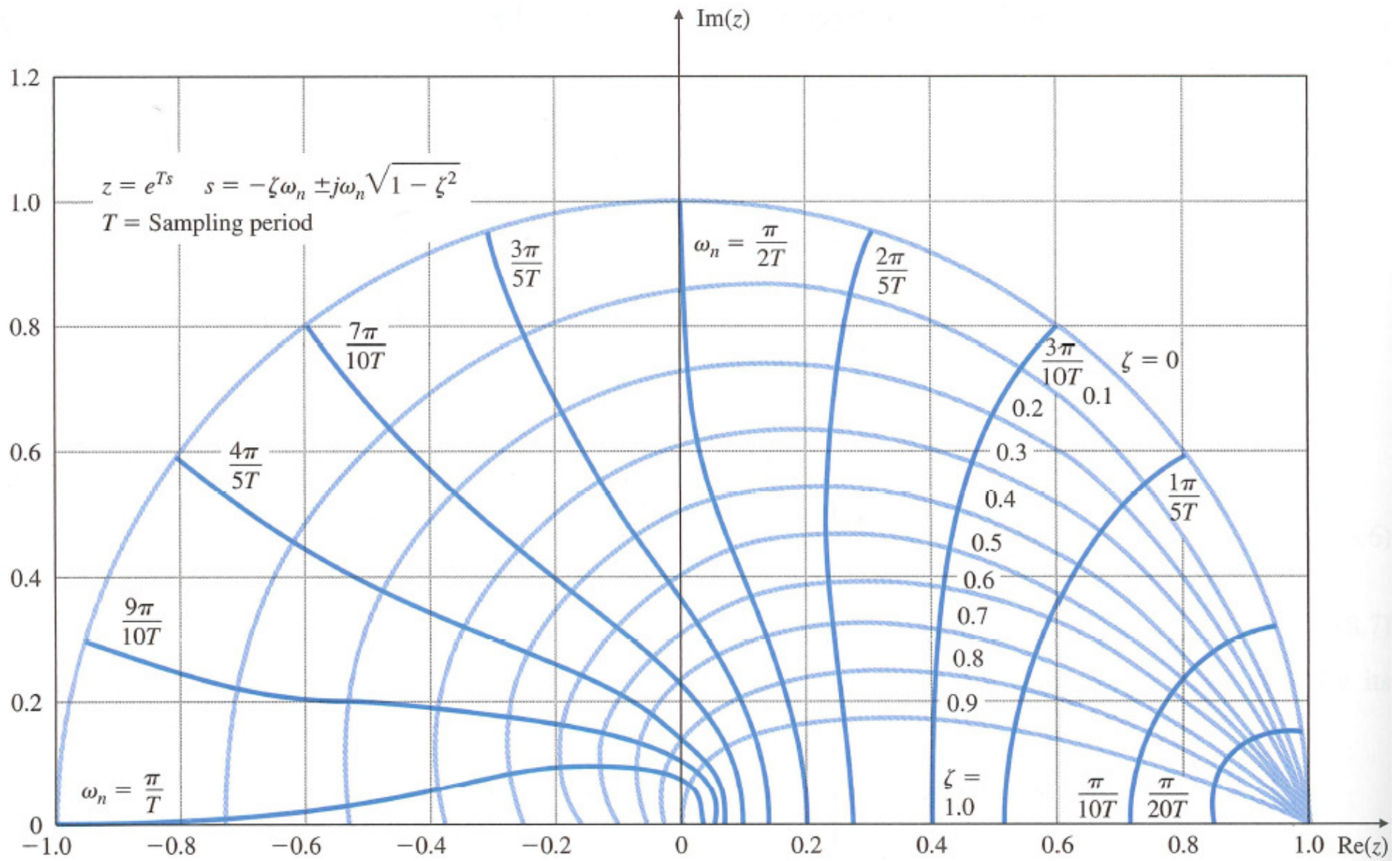
$$z^2 + a_1z + a_2 = 0$$

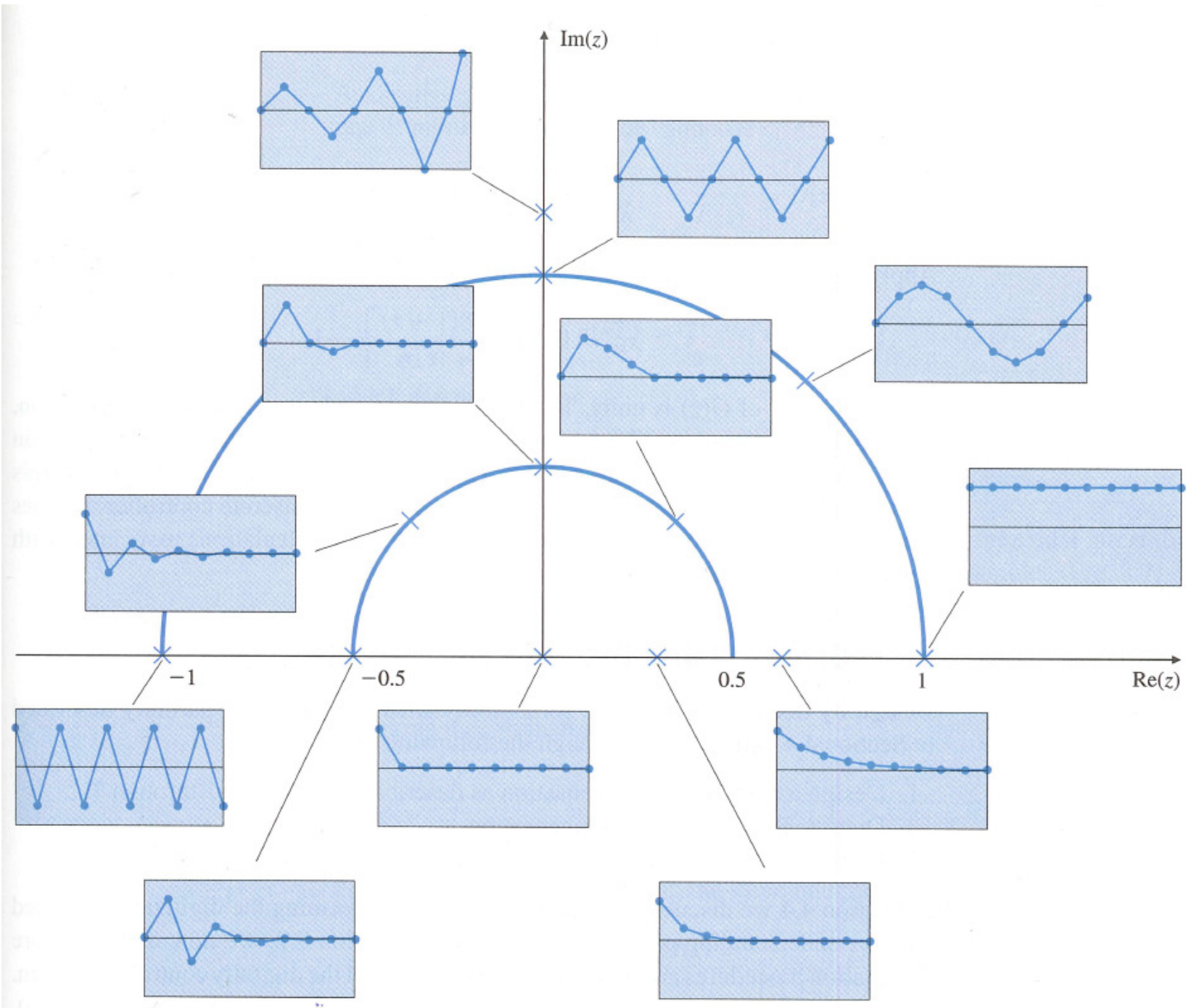
where

$$a_1 = -2e^{-\zeta\omega_0h} \cos\left(\sqrt{1-\zeta^2}\omega_0h\right)$$

$$a_2 = e^{-2\zeta\omega_0h}$$









# Transformation of Zeros via Sampling

- More complicated than for poles
- Extra zeros may appear in the sampled system
- There can be zeros outside the unit circle (non-minimum phase) even if the continuous system has all the zeros in the left half plane
- For short sampling periods

$$z_i \approx e^{s_i h}$$

# Calculation of $H(z)$ Given $G(s)$

Three approaches:

1. Make state-space realization of  $G(s)$ . Sample to get  $\Phi$  and  $\Gamma$ . Then  $H(z) = C(zI - \Phi)^{-1}\Gamma + D$ .
2. Directly using the formula

$$\begin{aligned} H_{zoh}(z) &= \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} ds \\ &= \sum_{s=s_i} \frac{1}{z - e^{sh}} \text{Res} \left\{ \frac{e^{sh} - 1}{s} G(s) \right\} \end{aligned}$$

- $s_i$  are the poles of  $G(s)$  and Res denotes the residue.
- outside the scope of the course

### 3. Use Table 3 (pg 28) in IFAC PB

$G(s)$	$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$	
$\frac{1}{s}$	$\frac{h}{z-1}$	
$\frac{1}{s^2}$	$\frac{h^2(z+1)}{2(z-1)^2}$	
$e^{-sh}$	$z^{-1}$	
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{z - \exp(-ah)}$	
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a} (ah - 1 + e^{-ah})$ $a_1 = -(1 + e^{-ah})$	$b_2 = \frac{1}{a} (1 - e^{-ah} - ahe^{-ah})$ $a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$

## Calculation of $H(z)$ Given $G(s)$

**Example:** For  $G(s) = e^{-\tau s}/s^2$ , the previous lecture gave

$$x(kh + h) = \Phi x(kh) + \Gamma_1 u(kh - h) + \Gamma_0 u(kh)$$

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} \tau \left( h - \frac{\tau}{2} \right) \\ \tau \end{pmatrix} \quad \Gamma_0 = \begin{pmatrix} \frac{(h - \tau)^2}{2} \\ h - \tau \end{pmatrix}$$

With  $h = 1$  and  $\tau = 0.5$ , this gives

$$H(z) = C(zI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 z^{-1}) = \frac{0.125(z^2 + 6z + 1)}{z(z^2 - 2z + 1)}$$

Order: 3

Poles: 0, 1, and 1

Zeros:  $-3 \pm \sqrt{8}$

# Examples in Matlab

```
>> Phi = [0.5 -0.2; 0 0];
>> Gamma = [2; 1];
>> C = [1 0];
>> D = 0;
>> h = 1;
>> H = ss(Phi, Gamma, C, D, h);
>> zpk(H)

>> % From cont-time transfer function to discrete-time
>> % pulse transfer function
>> s = zpk('s');
>> G = 1/s^3;
>> H = c2d(G,h)

>> % Another way
>> G = tf([1],[1 3 2 0]);
>> G = ss(G);
>> H = c2d(G,h);
>> tf(H)
```