Lecture 9: State Feedback and Observers

[IFAC PB Ch 9]

- State Feedback
- Observers
- Disturbance Estimation

Control Design

Many factors to consider, for example:

- Attenuation of load disturbances
- Reduction of the effect of measurement noise
- Command signal following
- Variations and uncertainties in process behavior

Two Classes of Control Problems

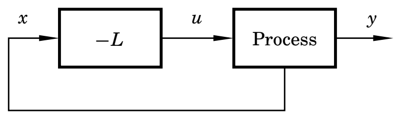
Regulation problems: compromise between rejection of load disturbances and injection of measurement noise

- Lecture 9

Servo problems: make the output respond to command signals in the desired way

- Lecture 10

State Feedback: Problem Formulation



- Discrete-time process model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

- Linear feedback from all state variables

$$u(k) = -Lx(k)$$

- Disturbances modelled by nonzero initial state $x(0) = x_0$
- Goal: Control the state to the origin, using a reasonable control signal

Closed-Loop System

The state equation

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

with the control law

$$u(k) = -Lx(k)$$

gives the closed-loop system

$$x(k+1) = (\Phi - \Gamma L)\, x(k)$$

Pole placement design: Choose L to obtain the desired characteristic equation

$$\det(zI - \Phi + \Gamma L) = 0$$

(Matlab: `place` or `acker`)

Example – Double Integrator

$$x(k+1) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(k)$$

Linear state-feedback controller

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

The closed-loop system becomes

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)x(k) \\ &= \begin{pmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{pmatrix} x(k) \end{aligned}$$

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2 \right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1 \right) = 0$$

Example Cont'd

Characteristic equation

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right) z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = 0$$

Assume desired characteristic equation $z^2 + a_1 z + a_2 = 0$.

Linear equations for l_1 and l_2

$$\frac{l_1 h^2}{2} + l_2 h - 2 = a_1 \qquad \frac{l_1 h^2}{2} - l_2 h + 1 = a_2$$

Example Cont'd

Solution:

$$l_1 = \frac{1}{h^2}(1 + a_1 + a_2)$$
$$l_2 = \frac{1}{2h}(3 + a_1 - a_2)$$

- L depends on h

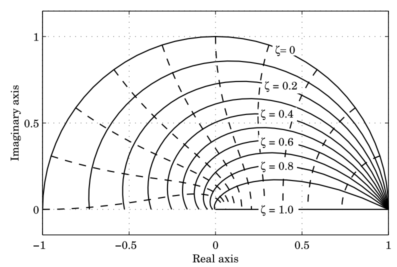
Where to Place the Poles?

Recall from Lecture 7:

Loci of constant ζ (solid) and ωh (dashed) when

$$\frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

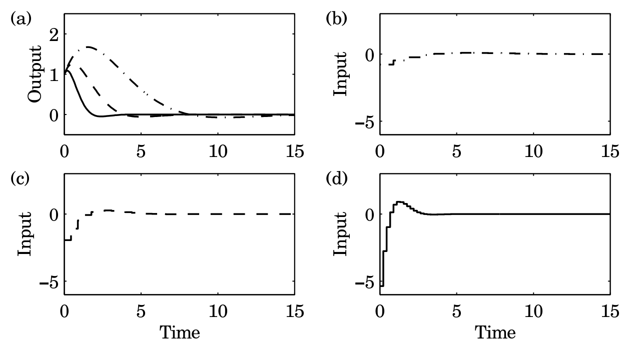
is sampled:



Example – Choice of Design Parameters

Double integrator, $x_0^T = [1 \quad 1]$, $\omega h = 0.44$, $\zeta = 0.707$

(b) $\omega = 0.5$ (dash-dotted), (c) $\omega = 1$ (dashed), (d) $\omega = 2$ (solid)



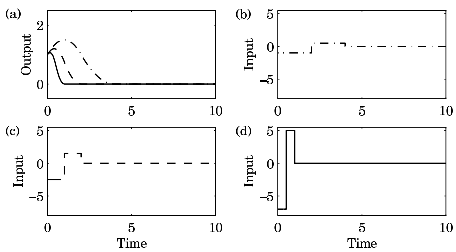
Deadbeat Control — Only in Discrete Time

Choose $P(z) = z^n \Rightarrow h$ only remaining design parameter

Drives all states to zero in at most n steps after an impulse disturbance in the states (can be very aggressive for small h!)

Finite time as opposed to infinite time in continuous time.

Example: Double integrator, $x_0^T = [1 \quad 1]$



Controllability

The eigenvalues of $\Phi - \Gamma L$ can be assigned to arbitrary positions if and only if the system is *controllable*, i.e. if the controllability matrix

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma & \dots & \Phi^{n-1}\Gamma \end{pmatrix}$$

has full rank.

In practice, moving some eigenvalues could require high gain and lead to bad controllers.

State Feedback in Controllable Form

We previously derived the controllable canonical form

$$x(k+1) = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

In this case, application of the state feedback

$$u = -l_1 x_1 - \dots - l_n x_n$$

changes the coefficients $a_1, \dots, a_n$ to $a_1 + l_1, \dots, a_n + l_n$, so the characteristic polynomial changes to

$$z^n + (a_1 + l_1) z^{n-1} + \dots + (a_{n-1} + l_{n-1}) z + a_n + l_n$$

Design method: Transform to controllable canonical form, apply state feedback, transform the controller back again – Ackermann's formula (see IFAC PB)

State Feedback with Integral Action

Integral action can be introduced by augmenting the plant model with an extra state variable, x_i, that integrates the plant output:

$$x_i(k+1) = x_i(k) + y(k) = x_i(k) + Cx(k)$$

The augmented open-loop system becomes

$$\begin{pmatrix} x(k+1) \\ x_i(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ C & I \end{pmatrix} \begin{pmatrix} x(k) \\ x_i(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k)$$

We can then design a state feedback controller

$$u(k) = -\begin{pmatrix} L & L_i \end{pmatrix} \begin{pmatrix} x(k) \\ x_i(k) \end{pmatrix}$$

using the same techniques as before

(Integral action can also be introduced using a disturbance observer, as we will see later)

Reconstruction

What should you do if you can not measure the full state vector or if you have noisy measurements?

Reconstruction Through Direct Calculations

Basic idea: Reconstruct the state vector through direct calculations using the input and output sequences $y(k)$, $y(k-1)$, …, $u(k)$, $u(k-l)$, … together with the state-space model of the plant.

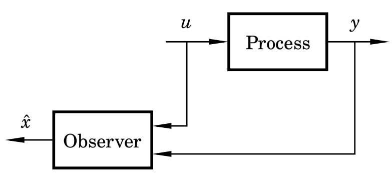
Explained in detail in IFAC PB pg 61–62

Make sure that you understand it (a lot of notation but not difficult!)

Often sensitive to disturbances.

A better alternative is to use the model information explicitly.

Reconstruction Using An Observer



Simulated process model:

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k)$$
$$\hat{y}(k) = C\hat{x}(k)$$

Introduce "feedback" from measured $y(k)$

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K\left(y(k) - C\hat{x}(k)\right)$$

Form the estimation error $\tilde{x} = x - \hat{x}$

$$\begin{aligned} \tilde{x}(k+1) &= \Phi \tilde{x}(k) - KC\tilde{x}(k) \\ &= [\Phi - KC]\tilde{x}(k) \end{aligned}$$

- Any observer poles possible, provided the observability matrix

$$W_o = \begin{pmatrix} C \\ \vdots \\ C\Phi^{n-1} \end{pmatrix}$$

  has full rank
- Choose K to get good convergence
- Trade-off against measurement noise amplification

Deadbeat Observer

A *deadbeat observer* is obtained if the observer gain K is chosen so that the matrix $\Phi - KC$ has all eigenvalues zero.

The observer error goes to zero in finite time (in at most n steps, where n is the order of the system)

Noise sensitive (fast observer dynamics)

Equivalent to reconstruction using direct calculations.

Observer for the Double Integrator

$$\Phi - KC = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-k_1 & h \\ -k_2 & 1 \end{pmatrix}$$

Characteristic equation

$$z^2 - (2-k_1)z + 1 - k_1 + k_2 h = 0$$

Desired characteristic equation:

$$z^2 + p_1 z + p_2 = 0$$

Gives:

$$\begin{aligned} 2 - k_1 &= -p_1 \\ 1 - k_1 + k_2 h &= p_2 \end{aligned}$$

Observer for the Double Integrator cont'd

Solution:

$$\begin{aligned} k_1 &= 2 + p_1 \\ k_2 &= (1 + p_1 + p_2)/h \end{aligned}$$

Assume deadbeat observer ($p_1 = p_2 = 0$)

$$\begin{aligned} k_1 &= 2 \\ k_2 &= 1/h \end{aligned}$$

Resulting observer

$$\begin{aligned} \hat{x}_1(k+1) &= \hat{x}_1(k) + h\hat{x}_2(k) + 2\Big(y(k) - \hat{x}_1(k)\Big) \\ \hat{x}_2(k+1) &= \hat{x}_2(k) + \frac{1}{h}\Big(y(k) - \hat{x}_1(k)\Big) \end{aligned}$$

An Alternative Observer

The observer presented so far has a one sample delay: $\hat{x}(k \mid k-1)$ depends only on measurements up to time $k-1$

Alternative observer with direct term:

$$\begin{aligned} \hat{x}(k \mid k) &= \Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1) \\ &\quad + K\Big[y(k) - C\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big)\Big] \\ &= (I - KC)\Big(\Phi\hat{x}(k-1 \mid k-1) + \Gamma u(k-1)\Big) + Ky(k) \end{aligned}$$

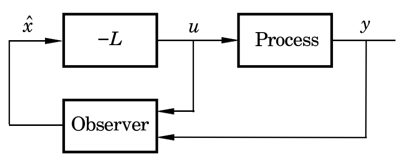
Reconstruction error:

$$\tilde{x}(k \mid k) = x(k) - \hat{x}(k \mid k) = (\Phi - KC\Phi)\,\tilde{x}(k-1 \mid k-1)$$

- $\Phi - KC\Phi$ can be given arbitrary eigenvalues if $\Phi - KC$ can
- K may be chosen so that some of the states will be observed directly through $y \Rightarrow$ the order of the observer can be reduced
  - Reduced order observer or *Luenberger observer*

Output Feedback

State feedback from observed state:



Controller:

$$\begin{aligned} \hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K(y(k) - C\hat{x}(k)) \\ u(k) &= -L\hat{x}(k) \end{aligned}$$

Transfer function from y to u: $\quad -L(zI - \Phi + \Gamma L + KC)^{-1}K$

Analysis of the Closed-Loop System

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ \tilde{x}(k+1) &= (\Phi - KC)\tilde{x}(k) \\ u(k) &= -L\hat{x}(k) = -L(x(k) - \tilde{x}(k)) \end{aligned}$$

Eliminate $u(k)$

$$\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - \Gamma L & \Gamma L \\ 0 & \Phi - KC \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix}$$

Separation

$$\begin{aligned} &\text{Control poles:} \quad A_c(z) = \det(zI - \Phi + \Gamma L) \\ &\text{Observer poles:} \quad A_o(z) = \det(zI - \Phi + KC) \end{aligned}$$

Disturbance Estimation

How to handle disturbances that can not be modeled as impulse disturbances in the process state?

Assume that the process is described by

$$\frac{dx}{dt} = Ax + Bu + v$$
$$y = Cx$$

where v is a disturbance modeled as

$$\frac{dw}{dt} = A_w w$$
$$v = C_w w$$

Since disturbances typically have most of their energy at low frequencies, the eigenvalues of A_w are typically in the origin or on the imaginary axis (sinusoidal disturbance)

Disturbance Estimation

Augment the state vector: $\begin{pmatrix} x \\ w \end{pmatrix}$

Gives the augmented system

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} A & C_w \\ 0 & A_w \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}$$

which is sampled into

$$\begin{pmatrix} x(k+1) \\ w(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix}\begin{pmatrix} x(k) \\ w(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k)$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x(k) \\ w(k) \end{pmatrix}$$

Augmented Observer and State Feedback

Augmented observer:

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{pmatrix}\begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} K \\ K_w \end{pmatrix} \epsilon(k)$$

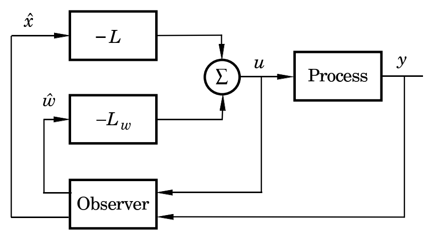
with $\epsilon(k) = y(k) - C\hat{x}(k)$

Augmented state feedback control law:

$$u(k) = -L\hat{x}(k) - L_w\hat{w}(k)$$

If possible, select L_w such that $\Phi_{xw} - \Gamma L_w = 0$

Disturbance Estimation: Block Diagram



Disturbance Estimation: Closed-Loop System

The closed-loop system can be written

$$x(k+1) = (\Phi - \Gamma L)x(k) + (\Phi_{xw} - \Gamma L_w)w + \Gamma L\tilde{x}(k) + \Gamma L_w\tilde{w}$$
$$w(k+1) = \Phi_w w(k)$$
$$\tilde{x}(k+1) = (\Phi - KC)\tilde{x}(k) + \Phi_{xw}\tilde{w}(k)$$
$$\tilde{w}(k+1) = \Phi_w\tilde{w}(k) - K_w C\tilde{x}(k)$$

- L ensures that x goes to zero at the desired rate after a disturbance.
- The gain L_w reduces the effect of the disturbance v on the system by feedforward from the estimated disturbances $\hat{w}$.
- K and K_w influence the rate at which the estimation errors go to zero.

Special Case: Constant Input Disturbance

Assume constant disturbance acting on the plant input:

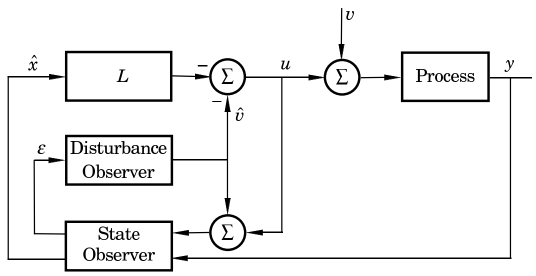
- $v = w$
- $\Phi_w = 1$
- $\Phi_{xw} = \Gamma$

If we choose $L_w = 1$ we will have perfect cancellation of the load disturbance

New controller + estimator

$$u(k) = -L\hat{x}(k) - \hat{v}(k)$$
$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma\Big(\hat{v}(k) + u(k)\Big) + K\epsilon(k)$$
$$\hat{v}(k+1) = \hat{v}(k) + K_w\epsilon(k)$$
$$\epsilon(k) = y(k) - C\hat{x}(k)$$

Special Case: Block Diagram



The disturbance estimator is integrating the prediction error of the observer.

The overall controller will have integral action (see IFAC PB)

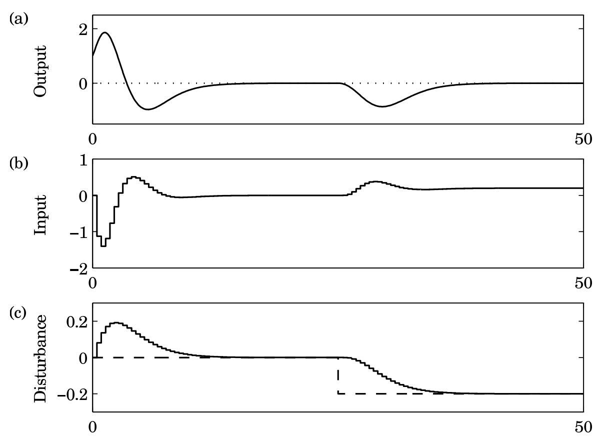
Example – Design

- Control of double integrator

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

- Sample with $h = 0.44$
- Discrete state feedback designed based on continuous-time specification $\omega = 1$, $\zeta = 0.7$
  - Gives $L = [0.73 \quad 1.21]$
- Extended observer assuming constant input disturbance to obtain integral action; all three poles placed in $z = 0.75$.

Example – Simulation



Optimization-Based Design

Pole-placement design:

- L and K derived through pole-placement

In the course Multivariable Control (Flervariabel Reglering), L and K are instead derived through optimization

- LQ (Linear Quadratic) and LQG (Linear Quadratic Gaussian) control
- Short overview in Ch 11 of IFAC PB
- Not part of this course

Examples in Matlab

```
>> A = [0 1; 0 0];
>> B = [0; 1];
>> h = 0.44;
>> % Sampled system matrices
>> [Phi,Gamma] = c2d(A,B,h)

>> % Desired poles in continuous time
>> omega = 1; zeta = 0.7;
>> pc = roots([1 2*zeta*omega omega^2])
>> % Corresponding desired discrete poles
>> pd = exp(pc*h)

>> % Design state feedback
>> L = place(Phi,Gamma,pd)

>> % Design augmented observer
>> Phie = [Phi Gamma; zeros(1,2) 1];
>> Ce = [C 0];
>> Ke = acker(Phie',Ce',[0.75 0.75 0.75])'
```